



PARTIAL BALANCED INCOMPLETE BLOCK DESIGNS ARISING FROM SOME MINIMAL DOMINATING SETS OF SRNT GRAPHS

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(Received on: 05-01-11; Accepted on: 18-01-11)

ABSTRACT

In this paper we prove that the set of all minimal dominating sets which induced from the neighbourhoods $N(v)$ of Clebsch graph with parameters $(16, 5, 0, 2)$ are blocks of partial balanced incomplete block designs, and we generalize this result for minimal dominating sets $N(v)$ for all strongly regular graphs without triangles.

Keywords: Clebsch graph, Strongly regular graph without triangles, Minimal dominating set, Partial balanced incomplete block design.

Mathematics Subject Classification (2000): 05C15.

1. INTRODUCTION:

The strongly regular graphs and its relation with partial balanced incomplete block designs (PBIBD's) were studied in [1], [2] and [3] it was shown that the strongly regular graphs are emerged from PBIBD with two association schemes. In this paper, we consider different way to establish a link between PBIBD and strongly regular graph through the number of some minimal dominating sets, in the folded 5-cube $(16, 5, 0, 2)$ graph and we generalize this result to all SRNT graphs.

We refer to [4], [6] and [8] for the necessary background about strongly regular graphs, dominating set, and PBIBD's.

2. DEFINITIONS AND NOTATIONS:

Definition: 2.1

A strongly regular graph with no triangles (SRNT graph) G with the parameters $(n, k, 0, \mu)$ is k -regular graph with n vertices such that for any two adjacent vertices have no common neighbours, and any two non-adjacent vertices have μ common neighbours.

Definition: 2.2

Given v objects a relation satisfying the following conditions is said to be an association scheme with m classes:

- (i) any two objects are either first associates, second associates, ..., m^{th} associates, the relation of association being symmetric.

(ii) each object has n_i i^{th} associates ($i = 1, 2$).

(iii) if two objects α and β are i^{th} associates, then the number of objects common to the j^{th} associates of the first and k^{th} associates of the second is p_{jk}^i and is independent of the pair of i^{th} associates α and β . Also $p_{jk}^i = p_{kj}^i$.

If we have association scheme for the v objects we can define a PBIBD as the following definition.

Definition: 2.3

The PBIB design is arrangement of v objects into b sets of size k where $k < v$ such that:

- (i) every object is contained in exactly r blocks.
- (ii) each block contains k distinct objects.
- (iii) any two objects which are i^{th} associates occur together in exactly λ_i blocks.

The numbers v, b, r, k, λ_i ($i = 1, 2, \dots, m$) are called the parameters of PBIBD with m association.

Definition: 2.4

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset D of $V(G)$ is said to be dominating set if every vertex not in D , is adjacent to at least one vertex in D . The domination number $\gamma(G)$ of a graph G is defined to be the minimum cardinalities taken over all dominating sets of G , and the set D in G is said to be minimum dominating set if $|D| = \gamma(G)$.

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Definition: 2.5

A dominating set D is called minimal dominating set of a graph G if no proper subset of D is dominating set.

3. RESULTS:

Proposition: 3.1

The minimal dominating sets which induced from the neighbourhood of vertices of Clebsch graph in Figure. 1 are blocks of PBIB design.

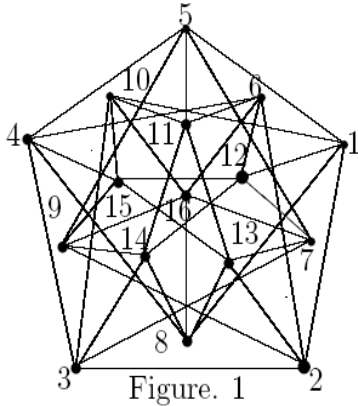


Figure. 1

Proof: Each vertex of the Clebsch graph belongs to exactly 5 minimal dominating sets $N(v)$, and the list of minimal dominating sets given below:

$\{2, 5, 8, 10, 12\} \{1, 3, 6, 9, 13\}$
 $\{2, 4, 7, 10, 14\} \{3, 5, 6, 8, 15\}$
 $\{1, 4, 7, 9, 11\} \{2, 4, 11, 12, 16\}$
 $\{3, 5, 12, 13, 16\} \{1, 4, 13, 14, 16\}$
 $\{2, 5, 14, 15, 16\} \{1, 3, 11, 15, 16\}$
 $\{5, 6, 10, 13, 14\},$
 $\{1, 6, 7, 14, 15\} \{2, 7, 8, 11, 15\}$
 $\{3, 8, 9, 11, 12\} \{4, 9, 10, 12, 13\}$
 $\{6, 7, 8, 9, 10\}$

By consider the minimal dominating sets above as blocks, and the two association scheme can be defined as the two elements α and β are 1st associates if they are adjacent vertices in G and they are second associates otherwise. Thus we have

Second Associates	First Associates	Elements
3,4,6,7,9,11,13,14,15,16	2,8,12,10,5	1
4,5,7,8,10,11,12,14,15,16	1,3,6,9,13	2
1,5,6,8,9,11,12,13,15,16	2,4,7,10,14	3
1,2,7,9,10,11,12,13,14,16	3,5,6,8,15	4

2,3,6,8,10,12,13,14,15,16	1,4,7,9,11	5
1,3,5,7,8,9,10,13,14,15	2,4,11,12,16	6
1,2,4,6,8,9,10,11,14,15	3,5,12,13,16	7
2,3,5,6,7,9,10,11,12,15	1,4,13,14,16	8
1,3,4,6,7,8,10,11,12,13	2,5,14,15,16	9
2,4,5,6,7,8,9,12,13,14	1,3,11,15,16	10
1,2,3,4,7,8,9,12,15,16	5,6,10,13,14	11
2,3,4,5,8,9,10,11,13,16	1,6,7,14,15	12
1,3,4,5,6,9,10,12,14,16	2,7,8,11,15	13
1,2,4,5,6,7,10,13,15,16	3,8,9,11,12	14
1,2,3,5,6,7,8,11,14,16	4,9,10,12,13	15
1,2,3,4,5,11,12,13,14,15	6,7,8,9,10	16

From the above association it is easy to verify that the minimal dominating sets of Clebsch graph which induced from the neighbourhood of the vertices forms PBIBD with the parameters

$$v = 16, b = 5, r = 5, k = 5, \lambda_1 = 0 \text{ and } \lambda_2 = 2,$$

$$P_1 = \begin{bmatrix} P_{11}^1 & P_{12}^1 \\ P_{21}^1 & P_{22}^1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}, \text{ and}$$

$$P_2 = \begin{bmatrix} P_{11}^2 & P_{12}^2 \\ P_{21}^2 & P_{22}^2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}.$$

Thus the sets of minimal dominating sets $N(v)$ of Clebsch graph are the blocks of PBIB design with the above parameters.

Theorem: 3.2

Let G be SRNT graph with the parameters $(n, k, 0, \mu)$ and let v be any vertex of G then $N(v)$ is minimal dominating set of G .

Proof: Let u be any vertex in $V(G) - N(v)$. Then we have two cases, either $u = v$ or u not adjacent to v . Case 1. If $u = v$ then u has k adjacent vertices in $N(v)$.

Case 2. If u not adjacent to v , then from the definition of SRNT graph u adjacent to μ vertices in $N(v)$. Hence $N(v)$ is dominating set. Now we want to prove that $N(v)$ is minimal dominating. Suppose $N(v)$ is not dominating set of G then there is vertex $w \in N(v)$ such that

$N(v) - w$ is dominating set, hence w adjacent to at least one vertex in $N(v)$, and this is contradiction with the definition of SRNT graph. Therefore $N(v)$ is minimal dominating set of G .

Theorem: 3.3

The set of minimal dominating set $N(v)$ in SRNT graph G with the parameters $(n, k, 0, \mu)$, where v any vertex of G are blocks of PBIB design.

Proof: We can defined the PBIB design as following:

Point set is the vertices of G , and the **block set** is the set of minimal dominating sets $N(v)$, where v any vertex in G , and for any points α and β are 1^{st} associates if they are adjacent in G and 2^{nd} associates otherwise and it is clear that each vertex of the SRNT graph belong to exactly k minimal dominating sets, so the parameters of PBIB design are $v', r', k', \lambda_1, \lambda_2$ where $v = b = n$, $r' = k' = k$, $\lambda_1 = 0$, $\lambda_2 = \mu$.

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