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A THEOREM RELAED TO KAZAKOV'S FORMULA

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ABSTRACT

In this paper I have find a theorem related to Kazakov's Formula. I also find applications of the theorem as proposition and remark.

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1. INTRODUCTION:

I will express the correlation kernel of the deformed Laguerre ensemble $K_N(u,v;y)$ as a double integral over some contours in the complex plane. From the unitary invariance of the Gaussian law, we know that the correlation function depends only on H^*H through its empirical spectral measure. "Kazakov's formula," which was first used in [1], is the trick to explicitly bring out the spectral measure $\frac{1}{N}\sum_{i=1}^{N}\delta_{yi}$.

$$Let\ s = 2t = \frac{4\sigma_1^2 a^2}{N}.$$

2. MAIN RESULTS:

Theorem: 1 The correlation kernel of the deformed Languerre ensemble is given by

$$(a)\qquad K_N(u,v;y) - \frac{e^{\frac{v-u}{s}}}{i\pi s^2}\int_\Gamma\int_\gamma \exp\left(\frac{z^2-w^2}{s}\right)J_v\left(2\frac{zu^{\frac{1}{2}}}{s}\right)J_v\left(2\frac{wv^{\frac{1}{2}}}{s}\right)$$

$$\times\prod_{i=1}^{N}\frac{w^2+y_i}{z^2+y_i}\left(\frac{w}{z}\right)^v\frac{-2zw}{w^2-z^2}dzdw,$$

where $\gamma = \mathbb{R}^+$ and Γ is a contour encircling the $i\sqrt{y_j}, j = 1,\dots N(but\ not\ the$ $-i\sqrt{y_j},\ 1,\dots N)$ not crossing γ.

Proof of Theorem: 1 We can first rewrite, using Cramer's formula,

$$(b)\qquad K_N^T(u,v) = \sum_{j=1}^{N}p_t(y_j,u)\frac{\det A_j(v)}{\det A}$$

where $A_{i,j} = p_{t+T}(y_j, z_i)$ and A_j (v) is the matrix obtained from A by replacing the column j by $(p_T(v, z_1).....p_T(v,z_N))^T$. This can also be written, by multilinearity of the determinant, as

$$(c)\qquad K_N^T(u,v;y) = \left(\frac{u}{v}\right)^{\frac{v}{2}}\sum_{j=1}^{N}p_t(y_j,u)\left(\frac{y_j}{u}\right)^{\frac{v}{2}}\frac{\det B(v)}{\det B}$$

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where

$$B_{i,j} = I_v\left(\frac{\sqrt{y_i z_j}}{T+t}\right)\exp\left(-\frac{y_i + z_j}{2(T+t)}\right)$$

and B(v) is obtained from B by replacing T+t with T and y_j with v.

The next step will be achieved in the following proposition. In this proposition we rewrite the ratio of determinants in (c) and then let T grow infinity to obtain an expression for the correlation kernel of the deformed Laguerre ensemble.

Proposition: 2

$K_N(u,v;y)=$

$$(d)\qquad \sum_{j=1}^{N}\frac{2}{s^2}e^{\left(\frac{v-u}{s}\right)}\left(\frac{u}{v}\right)^{\frac{v}{2}}e^{i\frac{v\pi}{2}}\exp(-y_j)\,I_v\left(\frac{2\sqrt{y_j}\,u^{\frac{1}{2}}}{s}\right)$$
$$\times\int_{\mathbb{R}^+}exp\left(-\frac{w^2}{s}\right)J_v\left(\frac{2v^{\frac{1}{2}}w}{s}\right)\prod_{i\neq j}\frac{-w^2-y_i}{y_j-y_i}\left(\frac{iw}{\sqrt{y_j}}\right)^v wdw.$$

Remark: 3 After wards we will not consider $\left(\frac{u}{v}\right)^{\frac{v}{2}}$ any more since it will not play a role in the asymptotic

$$\det\left(K_N(x_i,x_j;y)\right)_{i,j=1}^{m} = det\left(\left(\frac{x_j}{x_i}\right)^{\frac{v}{2}}K_N(x_i,x_j;y)\right)_{i,j=1}^{m}$$

Proof of Proposition: 2 Because the two matrices under consideration in (c) differ by the j^{th} column only, we will find an integral transform expressing this column in B(v) in terms of that of B. This will make use of some kind of time inversion for the semigroup with transition density $p_t(y,x)$. Eventually we will let $T\to\infty$ to obtain the correlation kernel of the deformed Laguerre ensemble.

Lemma: 4

$$(e)\qquad \frac{1}{p^2}I_v\left(\frac{\sqrt{v}\sqrt{z}}{T}\right)\exp\left(\frac{-v(T+t)}{2tT}\right)\exp\left(\frac{-zt}{2(T+t)^2}\right)=$$
$$\frac{1}{t}\int_{i\mathbb{R}^-}\exp\left(\frac{x^2(T+t)}{2tT}\right)I_v\left(\frac{(T+t)\sqrt{vx}}{tT}\right)I_v\left(\frac{\sqrt{z}\,x}{T+t}\right)xdx$$

Proof: We start from formula [2, p. 108], valid for any a, b:

$$\int_{\mathbb{R}^+}\exp(-p^2x^2)xJ_v(ax)J_v(bx)dx = \frac{1}{2p^2}I_v\left(\frac{ab}{2p^2}\right)\exp\left(\frac{-a^2-b^2}{4p^2}\right).$$

The left-hand side can be rewritten as

$$(f) \qquad \frac{1}{p^2}\int_{\mathbb{R}^+} \exp(-x^2) J_v\left(\frac{ax}{p}\right) J_v\left(\frac{bx}{p}\right) x dx.$$

We first make the change of variables x = iy, obtaining that (f) can be rewritten as

$$(f) \qquad = \frac{1}{p^2}\int_{i\mathbb{R}^-} \exp(y^2) J_v\left(\frac{aiy}{p}\right) J_v\left(\frac{biy}{p}\right) y dy$$
$$= e^{(vi\pi)} \frac{1}{p^2}\int_{i\mathbb{R}^-} \exp(y^2) I_v\left(\frac{ay}{p}\right) I_v\left(\frac{by}{p}\right) y dy$$

Where we have used in the last equality that $I_v(z) = J_v(iz) exp^{\frac{vi\pi}{2}}$.

For $p = \sqrt{t/(T(t+T))}$ and making the change of variables $y = x/\sqrt{2t}, we$ obtain

$$(g) \qquad \int_{i\mathbb{R}^-} e^{y^2} I_v\left(\frac{ay}{p}\right) I_v\left(\frac{by}{p}\right) y dy =$$
$$\frac{1}{2t}\int_{i\mathbb{R}^-} e^{\frac{x^2}{2t}} I_v\left(\frac{ax\sqrt{T(t+T)}}{\sqrt{2}t}\right) I_v\left(\frac{bx\sqrt{T(t+T)}}{\sqrt{2}t}\right) x dx.$$

Setting then

$$a = \frac{T+t}{T} \; \frac{\sqrt{2}\sqrt{v}}{\sqrt{T(t+T)}}, \qquad\qquad b = \frac{T}{T+t} \; \frac{\sqrt{2t}\sqrt{z}}{T\sqrt{T(t+T)}}$$

in (g) we obtain

$$(h) \quad , \quad g = \frac{1}{2t}\int_{i\mathbb{R}^-} \exp\left(\frac{x^2(T+t)}{2tT}\right) I_v\left(\frac{(T+t)\sqrt{v}x}{tT}\right) I_v\left(\frac{\sqrt{z}\,x}{T+t}\right) x dx$$

and the left-hand side is then given by

$$(I) \qquad \frac{1}{2p^2} I_v\left(\frac{\sqrt{v}\sqrt{z}}{T}\right) \exp\left(\frac{-v(T+t)}{2tT}\right) \exp\left(\frac{-zt}{2(T+t)^2}\right)$$

which finishes the proof of the lemma.

We come back to the proof of proposition 2. Then developing the determinant along the j[th] column, we obtain the representation

$$\frac{detB(v)}{\det B} = (-1)^v \int_{i\mathbb{R}^-} \frac{1}{t} \exp\left(\frac{u^2(T+t)}{2tT}\right) I_v\left(\frac{\sqrt{vu(T+t)}}{tT}\right) \frac{detB(u)}{\det B} u du,$$

where the matrix $\bar{B}(u)$ has been obtained from B by changing y_j to u. We can now pass to the limit $T \to \infty$, thanks to the dominated convergence theorem and to the fact (proven in [3] that $\prod_{i<j}(x_i - x_j)$ is a minimal harmonic function for squared Bessel processes on the Weyl chamber W={x_1 <...<x_N}. We obtain that

$$\lim_{T\to\infty}\frac{det\tilde{B}(u)}{\det B}=\prod_{i\neq j}\frac{u^2-y_i}{y_j-y_i}$$

This then gives that

$$\frac{detB(v)}{\det B}=\frac{(-1)^v}{t}\int_{i\mathbb{R}^-}\exp\left(\frac{u^2}{2t}\right)I_v\left(\frac{u\sqrt{v}}{t}\right)\prod_{i\neq j}\frac{u^2-y_i}{y_j-y_i}\left(\frac{u}{\sqrt{y_j}}\right)^v udu.$$

We then change $u \to iw$ using that $I_v(z) = J_v(iz)exp^{\frac{vi\pi}{2}}$ and then change t to $\frac{s}{2}$; we thus obtain the result.

CONCLUSION:

We can now turn to the proof of Theorem 1. Then sum over y_j occurring in Proposition 2 can be written as a residue integral. This is Kazakov's formula [4], which seems to have been used first by Brezin and Hikami [5]. We eventually make the change of variable $z \mapsto iz$.

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