

COMPOSITION AND WEIGHTED COMPOSITION OPERATORS ACTING ON SEQUENCE SPACES DEFINED BY MODULUS FUNCTIONS

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(Received on: 04-01-11; Accepted on: 18-01-11)

ABSTRACT

In this paper we study composition operators and weighted composition operators on sequence spaces defined by modulus functions.

Keywords: Composition operator, Weighted composition operator, Fredholm operator, Closed range and Invertible operator.

Mathematics subject classification: Primary 47B20, Secondary 47B38.

1. INTRODUCTION AND PRELIMINARIES:

A modulus function is a function $f : [0, \infty) \rightarrow [0, \infty)$ such that

- (i) $f(x) = 0$ if and only if $x = 0$.
- (ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$.
- (iii) f is increasing.
- (iv) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example

take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If

$f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Let f be a modulus function and $A = (a_{nk})$ be a

non-negative matrix such that $\sup_n \sum_{k=1}^{\infty} a_{nk}$ is finite. If we

denote by C , the space of all sequence $x = \{x_k\}$, then by

$W_0(A, f)$, we mean the class of all sequence $x \in C$ such that $\lim_n \sum_{k=1}^{\infty} a_{n,k} f(|x_k|) = 0$. The class $W_0(A, f)$ is a linear

space over the complex field C , For every $x \in W_0(A, f)$,

we define $\|x\|_{A,f} = \sup_n \sum_{k=1}^{\infty} a_{n,k} f(|x_k|)$. Bhardwaj and Singh[1]

proved that $\|\cdot\|_{A,f}$ is a paranorm on $W_0(A, f)$ and

$(W_0(A, f), \|\cdot\|_{A,f})$ is a complete linear topological space. If

$v : N \rightarrow N$ and $u : N \rightarrow C$ be two mappings. Then a bounded linear transformation $m_{u,v} : W_0(A, f) \rightarrow W_0(A, f)$

defined by $(m_{u,v}f)(x) = u(x)f(v(x))$ is called a weighted composition operator induced by (u, v) . If we take $u(x) = 1$,

the constant one function, we write $m_{u,v}$ as T_v and call it a

composition operator. In case $v(x) = x$, for every x we write

$m_{u,v}$ as m_u and call it multiplication operator on $W_0(A, f)$

induced by u .

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H.Takagi and K.Yokouchi [9] initiated the study of multiplication and composition operators between L^p – spaces, whereas the study of weighted composition operators on some function spaces are considered by Carlson([2],[3]), Jamison and Rajagopalan [4], Kamowitz[5], Komal, B . S., Raj, Kuldip and Gupta Sunil[6], Takagi[8]. For more details see Singh and Manhas [7].

In this paper we plan to study composition and weighted composition operators acting on sequence spaces defined by modulus functions.

2. COMPOSITION OPERATORS ACTING ON SEQUENCE SPACES DEFINED BY MODULUS FUNCTION:

In this section we discuss composition operators acting on sequence spaces defined by modulus functions.

Theorem: 2.1 Let $T_v : W_0(A, f) \rightarrow W_0(A, f)$ be a linear transformation. Then T_v is bounded if there exists $M > 0$ such that $\sum_{m \in v^{-1}(k)} a_{nm} \leq M a_{nk}$ for all $k \in N$ and $n \in N$.

Proof: Suppose that the condition of the theorem is true. if $x \in W_0(A, f)$, then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{m \in v^{-1}(k)} a_{n,m} f(|x_k|) \leq M \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} f(|x_k|) = 0$$

Which shows that $T_v x \in W_0(A, f)$, Further

$$\begin{aligned} \|T_v x\|_{A,f} &= \sup_n \left\{ \sum_{k=1}^{\infty} a_{nk} f(|x_0 v(k)|) \right\} \\ &\leq M \sup_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|) \\ &= M \|x\|_{A,f} \end{aligned} \tag{1}$$

The continuity of T_v at origin follows from the inequality (i).

Since T_v is linear, so it is continuous everywhere.

Theorem: 2.2 If T_v is bounded, then there exists $M > 0$ such that

$$\sup_n \sum_{m \in v^{-1}(k)} a_{nm} \leq M \sup_n a_{nk} \tag{2}$$

Proof: If the condition (ii) is not satisfied then for every positive integer $k > 0$, there exists positive integer p_k and n_k such that

$$\sup_n \sum_{m \in v^{-1}(k)} a_{nm} > k \sup_n a_{nk} \text{ Let } x^{p_k} \in W_0(A, f) \text{ be such}$$

that

$$x^{p_k}(s) = \begin{cases} f^{-1} \left(\frac{1}{\sup_n \sum_{m \in v^{-1}(k)} a_{nm}} \right) & \text{if } s = p_k \\ 0, & \text{elsewhere} \end{cases}$$

Then

$$\begin{aligned} \|x^{p_k}\|_{A,f} &= \sup_n \left\{ \sum_{m=1}^{\infty} a_{nm} f(|x^{p_k}(m)|) \right\} \\ &= \sup_n \left\{ \sum_{m=1}^{\infty} a_{n,p_k} \frac{1}{\sup_n \sum_{m \in v^{-1}(p_k)} a_{nm}} \right\} \\ &= \frac{1}{k} \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

And

$$\begin{aligned} \|T_v x^{p_k}\|_{A,f} &= \sup_n \left\{ \sum_{m=1}^{\infty} a_{nm} f(|x^{p_k}(v(m))|) \right\} \\ &= 1. \end{aligned}$$

This contradicts the continuity of T_v . Hence the condition (2) must be true.

Theorem: 2.3 Let $T_v \in B(W_0(A, f))$. Then T_v has closed range if there exists $\delta > 0$ such that

$$\sum_{m \in v^{-1}(k)} a_{nm} \geq \delta a_{nk} \tag{3}$$

for every $k \in N$ and $n \in N$.

Proof: We first assume that the condition (3) is true and then show that T_v has closed range. Let $x \in \text{ran} T_v$ and let $\{x^n\}$ be

a sequence in $W_0(A, f)$ such that $T_\nu x^n \rightarrow x$. Then for every $\varepsilon > 0$ there

exists positive integer n_0 such that $\|T_\nu x^n - T_\nu x^m\|_{A,f} < \varepsilon$ for

all $n, m \geq n_0$ or equivalently,

$$\begin{aligned} \varepsilon &> \sup \left\{ \sum_{k=1}^{\infty} \sum_{p \in v^{-1}(k)} a_{np} f(|x_n(p) - x_m(p)|) \right\} \\ &\geq \delta \left| \sup_n \left\{ \sum_{k=1}^{\infty} a_{nk} f(|x^n(p) - x^m(p)|) \right\} \right| \\ &= \delta \|x^n - x^m\|_{A,f} \end{aligned} \tag{4}$$

for all $n, m \geq n_0$. From (4) it follows that $\{x_n\}$ is a Cauchy sequence in $W_0(A, f)$. In view of completeness of $W_0(A, f)$ there exists $y \in W_0(A, f)$ such that $x^n \rightarrow y$. From the continuity of T_ν , $T_\nu x^n \rightarrow T_\nu y$. Hence $x = T_\nu y$. So that $x \in \text{ran} T_\nu$. Hence $\text{ran} T_\nu$ is closed.

Theorem: 4 Let $T_\nu x \in B(W_0(A, f))$. Then T_ν is invertible

if

- (i) ν is invertible
- (ii) $\sup_n a_{\nu(n)} \leq M \sup_n a_{nm}$ for every $n, m \in N$ where

M is a constant such that $M > 0$

Proof. Suppose T_ν is an invertible operator. We show that ν is invertible. If, $k \notin \nu(N)$ then $T_\nu e_k = 0$,

where e_k is defined as

$$e_k(s) = \begin{cases} f^{-1}\left(\frac{1}{\sup a_{nk}}\right) & \text{if } s=k \\ 0, & \text{elsewhere} \end{cases}$$

So that T_ν has non-trivial kernel. Hence ν must be surjective. Next, if ν is not injective, then $\nu(n_1) = \nu(n_2)$ for two distinct positive integers n_1 and n_2 so that n_1 does not belong to the

range of T_ν . Hence ν must be injective. Let w be the inverse of ν . Clearly T_w is the inverse of T_ν . Since T_ν is continuous, in view of theorem (2.2) we have

$$\sup_n \sum_{m \in v^{-1}(k)} a_{nm} \leq M \sup_n a_{nk} \text{ for every } k \in N.$$

Hence $a_{\nu(k)} \leq M a_{nk}$ for every k .

Theorem: 2.5 Let $T_\nu \in B(W_0(A, f))$. Then T_ν is an isometry if $\sum_{m \in v^{-1}(k)} a_{nm} = a_{nk}$ for every $k, n \in N$.

Proof: If the condition of the theorem is satisfied, then for every $x \in W_0(A, f)$, we have

$$\begin{aligned} \|T_\nu x\|_{A,f} &= \sup_n \left\{ \sum_{k=1}^{\infty} \sum_{m \in v^{-1}(k)} a_{nm} f(|x_k|) \right\} \\ &= \sup_n \left\{ \sum_{m=1}^{\infty} a_{nk} f(|x^{pk}(m)|) \right\} \\ &= \|x\|_{A,f} \end{aligned}$$

This proves the theorem.

Theorem: 2.6 Let $T_\nu : W_0(A, f) \rightarrow W_0(A, g)$ be linear transformation. Then T_ν is bounded if there exists $M > 0$ such that

$$\sum_{m \in v^{-1}(k)} a_{nm} g(y) \leq M a_{nk} f(y)$$

for all $k, n \in N$ and $y \in R^+$. (5)

Proof: Suppose the condition (5) is true. Then for $x \in W_0(A, f)$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{m \in v^{-1}(k)} a_{nm} g(|x_k|) &\leq \lim_{n \rightarrow \infty} M \sum_{k=1}^{\infty} a_{nk} f(|x_k|) \\ &= 0 \end{aligned}$$

Which proves that $T_\nu x \in B(W_0(A, g))$ Further,

$$\|T_\nu x\|_{A,g} = \sup_n \left\{ \sum_{k=1}^{\infty} a_{nk} g(|x(\nu(x))|) \right\}$$

$$= M \sup_n \left\{ \sum_{k=1}^{\infty} a_{nk} f(|x_k|) \right\}$$

$$= \|x\|_{A,f}.$$

This shows that T_v is continuous at the origin. From linearity of T_v it is continuous everywhere.

Theorem: 2.7 Let $T_v \in B(W_0(A, f), W_0(A, g))$ Then T_v has closed range if there exists $\delta > 0$ such that

$$\sum_{m \in v^{-1}(k)} a_{nm} g(y) \geq \delta a_{nk} f(y) \text{ for all } k, n \in N. \text{ and } y \in R^+.$$

Proof: Assume that the condition of the theorem is satisfied. We show that T_v has closed range.

Let $x \in \text{ran} T_v$. Then there exists a sequence $\{x_n\} \subset W_0(A, f)$ such that $T_v x_n \rightarrow x$. So for every $\varepsilon > 0$ there exists a positive integer n_0 such that $\|T_v x_n - T_v x_m\|_{A,g} < \varepsilon$ for all $n, m \geq n_0$ or equivalently,

$$\varepsilon > \sup \left\{ \sum_{k=1}^{\infty} \sum_{p \in v^{-1}(k)} a_{np} g(|x_n(p) - x_m(p)|) \right\}$$

$$= \delta \|x_n - x_m\|_{A,f}$$

It follows that $\{x_n\}$ is a cauchy sequence in $W_0(A, f)$. In view of completeness of $W_0(A, f)$, there exists $y \in W_0(A, f)$ such that $x_n \rightarrow y$. From the continuity of T_v we have $T_v x_n \rightarrow T_v y$ Thus $x = T_v y$ Hence $x \in \text{ran} T_v$. So that $\text{ran} T_v$ is closed.

3. WEIGHTED COMPOSITION OPERATORS ACTING BETWEEN SEQUENCE SPACES DEFINED BY MODULUS FUNCTION

In this section we discuss weighted composition operators acting between sequence spaces defined by modulus functions.

Theorem: 3.1 Let $m_{u,v} : W_0(A, f) \rightarrow W_0(A, g)$ be a linear transformation. Then $m_{u,v}$ is bounded if

there exists $M > 0$ such that

$$\sum_{m \in v^{-1}(p)} a_{nm} g(|u(p)|) \leq M a_{np} f(y) \tag{6}$$

for all $p, n \in N$ and $y \in R^+$.

Proof: Suppose that the condition (6) is true. If $x \in W_0(A, f)$ then

$$\lim_{n \rightarrow \infty} \sum_{p=1}^{\infty} \sum_{m \in v^{-1}(p)} a_{nm} g(|x_p|) \leq M \lim_{n \rightarrow \infty} \sum_{p=1}^{\infty} a_{np} f(|x_p|) = 0,$$

Which shows that $m_{u,v} x \in W_0(A, g)$. Further,

$$\|m_{u,v} x\|_{A,g} = \sup_n \left\{ \sum_{p=1}^{\infty} a_{np} g(|u(p)x \circ v(p)|) \right\}$$

$$\leq M \sup_n \sum_{p=1}^{\infty} a_{np} f(|x_p|)$$

$$= M \geq \|x\|_{A,f} \tag{7}$$

The continuity of $m_{u,v}$ at the origin follows from the inequality (7). Since $m_{u,v}$ is linear so it is continuous everywhere.

Corollary: 3.2. If $m_{u,v}$ is bounded, then there exists $M > 0$ such that

$$\sup_n \sum_{m \in v^{-1}(p)} a_{nm} g(|u(p)y|) \leq M \sup_n a_{np} f(y)$$

for all $p, n \in N$ and $y \in R^+$.

Theorem: 3.3 Let $m_{u,v} \in B(W_0(A, f), W_0(A, g))$. Then $m_{u,v}$ has closed range if there exists $\delta > 0$ such that

$$\sum a_{nm} g(|u(k)y|) \geq \delta a_{nk} f(y) \text{ for all } k, n \in N \text{ and } y \in R^+.$$

Proof: Assume that the condition of the theorem is satisfied. We show that $m_{u,v}$ has closed range.

Let $x \in \text{ran} m_{u,v}$. Then there exists a sequence $\{x_n\} \subset W_0(A, f)$ such that $m_{u,v} x_n \rightarrow x$ So for every

$\varepsilon > 0$ there exists n_0 such that $\|m_{u,v}x_n - m_{u,v}x_m\|_{A,g} < \varepsilon$

for all $n, m \geq n_0$ or equivalently,

$$\begin{aligned} \varepsilon &> \sup \left\{ \sum_{k=1}^{\infty} \sum_{p \in v^{-1}(k)} a_{np} g(|u(p)(x_n(p) - x_m(p))|) \right\} \\ &\geq \delta \sup_n \left\{ \sum_{k=1}^{\infty} a_{nk} f(|x_n(p) - x_m(p)|) \right\} \\ &= \delta \|x_n - x_m\|_{A,f}. \end{aligned}$$

It follows that $\{x_n\}$ is a Cauchy sequence in $W_0(A, f)$. In view of completeness of $W_0(A, f)$ there exists $y \in W_0(A, f)$ such that $x_n \rightarrow y$. From the continuity $m_{u,v}$ we have $m_{u,v}x_n \rightarrow m_{u,v}y$. Hence $x = (m_{u,v}y) \in \text{ran } m_{u,v}$. So that $\text{ran } m_{u,v}$ is closed.

Corollary: 3.4 Let $m_{u,v} \in B(W_0(A, f), W_0(A, g))$ be such that $m_{u,v}$ has closed range if there exists $\delta > 0$ such that

$$\sup_n \sum_{m \in v^{-1}(p)} a_{nm} g(|u(k)y|) \geq \delta \sup_n a_{nk} f(y) \quad \text{for all } k, n \in N$$

and $y \in R^+$.

REFERENCES:

[1] Bhardwaj, Vinod K. and Singh, Niranjana, On some sequence spaces defined by a modulus, Indian J.Pure Appl.Math. 30(8)(1999), 809-817.
 [2] Carlson, J. W., Weighted Composition operators on l^2 , Ph.D. thesis, Purdue Univ. (1985).
 [3] Carlson, J. W., Hyponormal and quasinormal weighted composition operators on l^2 , Rocky Mountain Journal of Mathematics, 20 (1990), 399-407.
 [4] Jamison, J. E. and Rajagopalan, M., Weighted composition Operators on $C(X, E)$, Journal of operator Theory, 19(1988), 307-317.
 [5] Kamowitz, H., Compact weighted endomorphism of $C(X)$ Proc. Amer. Math. Soc. 83(1981), 517-521.

[6] Komal, B. S., Raj, Kuldip and Gupta, Sunil, On operators of weighted substitution on the generalised spaces of entire functions-I math today Vol.XV (1997), 3-10.
 [7] Singh, R. K. and Manhas, J. S., Composition operators on function spaces, North-Holland, 1993.
 [8] Takagi, H., Fredholm Weighted composition operators, Integ.Eqns. Oper.Theory, 16(1993), 267-276.
 [9] Takagi, H. and Yokouchi, K., Multiplication and composition operators between L_p -spaces, Contemporary Math., 232(1999), 321-338.