International Journal of Mathematical Archive-3(5), 2012, 2065-2069
Available online through www.ijma.info ISSN 2229 – 5046

ON EINSTEIN-SASAKIAN PROJECTIVE RECURRENT SPACES

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(Received on: 03-05-12; Revised & Accepted on: 27-05-12)

ABSTRACT

In the present paper, we define and study Einstein-Sasakian Projective recurrent spaces and establish several theorems. We investigate the necessary and sufficient condition for an Einstein-Projective (E-P) space to be Sasakian recurrent also Einstein- Sasakian (E-S) space is an Einstein-Projective (E-P) space, iff the scalar curvature is equal to zero.

Key words: *Einstein, Kaehlerian, Sasakian space, Conharmonic and Recurrent.*

Subject classification: *53C20, 53C25.*

1. INTRODUCTION

An n −dimensional Sasakian space S_n (or, normal contact metric space) is an odd dimensional Riemannian space, which admits a unit killing vector field η^i satisfying [4]:

$$\eta_{,i,j} = \eta_i g_{ik} - \eta_k g_{ij}, \tag{1.1}$$

where the comma (,) followed by an index denotes the operator of covariant differentiation w.r.t. the metric tensor g_{ij} of the Riemannian space. Also we have the relation:

$$F_i^h F_h^i = -\delta_i^j \tag{1.2}$$

The Riemannian curvature tensor field R_{ijk}^h is given by

$$R_{ijk}^h = \partial_j \left\{ {h \atop i \quad k} \right\} - \partial_k \left\{ {h \atop i \quad j} \right\} + \left\{ {m \atop i \quad k} \right\} \left\{ {h \atop m \quad j} \right\} - \left\{ {m \atop i \quad j} \right\} \left\{ {h \atop m \quad k} \right\}, \tag{1.3}$$

where $\partial_j \equiv \frac{\partial}{\partial x^j}$.

The Ricci-tensor and scalar curvature are respectively given by $R_{ij} = R_{ijk}^k$ and $R = R_{ij} g^{ij}$.

It is well known that these tensors satisfy the following identities (Tachibana [3]):

$$F_i^a R_a^j = R_i^a F_a^j \tag{1.4}$$

and

$$F_i^a R_{aj} = -R_{ai} F_j^a.$$

In view of (1.1), the relation (1.2) gives

$$F_i^a R_a^b F_b^j = -R_i^j$$

Thus, both S_n and K_n are Riemannian spaces, satisfying all the properties of a Riemannian space.

The Sasakian Conharmonic curvature tensor T_{ijk}^h and the holomorphic Sasakian projective curvature tensor P_{ijk}^h are respectively given by [3]:

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$$T^h_{ijk} = R^h_{ijk} + \frac{1}{n+4}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik}R^h_j - g_{jk}R^h_i + S_{ik}F^h_j - S_{jk}F^h_i + F_{ik}S^h_j - F_{jk}S^h_i + \quad + 2S_{ij}F^h_k + 2F_{ij}S^h_k\right) \quad (1.5)$$

and

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right), \quad (1.6)$$

where

$$S_{ij} = F^a_i R_{aj}.$$

Let us suppose that a Sasakian space is an Einstein one, then Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij}, \qquad R_{,a} = 0 \quad (1.7)$$

from which, we have

$$R_{ij,a} = 0, \quad S_{ij,a} = 0 \text{ and } S_{ij} = \frac{R}{n} F_{ij}. \quad (1.8)$$

If a Sasakian space is an Einstein one, then the Sasakian Conharmonic curvature tensor and Sasakian projective curvature tensor respectively reduce to the forms:

$$E^h_{ijk} = R^h_{ijk} + \frac{2}{n(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \quad (1.9)$$

and

$$*P^h_{ijk} = R^h_{ijk} + \frac{R}{n+2}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \quad (1.10)$$

In view of equations (1.9) and (1.10), we have

$$E^h_{ijk} = *P^h_{ijk} + \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \quad (1.11)$$

Now, we shall use the following definition:

Definition (1.1): a Sasakian space is said to be recurrent if, we have [1]:

$$R^h{}_{ijk,a} - \lambda_a R^h_{ijk} = 0, \quad (1.12)$$

for some non-zero recurrence vector λ_a and is called Sasakian Ricci-recurrent, if it satisfies the relation:

$$R_{ij,a} - \lambda_a R_{ij} = 0. \quad (1.13)$$

Multiplying the above equation by g^{ij}, we get

$$R_{,a} - \lambda_a R = 0. \quad (1.14)$$

Remark (1.1): From (1.13), it follows that every Sasakian recurrent space is Sasakian Ricci-recurrent, but the converse is not necessarily true.

Definition (1.2): a Sasakian space satisfying the relation [6]:

$$E^h{}_{ijk,a} - \lambda_a E^h_{ijk} = 0, \quad (1.15)$$

where λ_a is a non-zero recurrence vector, will be called an Einstein-Sasakian conharmonic recurrent, or briefly an $E - S^* - space$.

Definition (1.3): a Sasakian space is called a space of constant holomorphic sectional curvature tensor [3], if the tensor U^h_{ijk} defined by

$$U^h_{ijk} = R^h_{ijk} + \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \quad (1.16)$$

Vanishes identically.

Remark (1.2): In an Einstein-Sasakian space, the curvature tensor P^h_{ijk} coincides with constant holomorphic sectional curvature.

2. EINSTEIN-SASAKIAN PROJECTIVE RECURRENT SPACES

Definition (2.1): A Sasakian space satisfying the relation:

$$*P^h_{ijk,a} - \lambda_a * P^h_{ijk} = 0, \tag{2.1}$$

for some non-zero recurrence vector λ_a and is called an Einstein-Sasakian projective recurrent space, or briefly an $E - P^* - space$.

We, now, have the following theorem:

Theorem (2.1): A necessary and sufficient condition for an $E - P$ space to be a Sasakian recurrent space is that the scalar curvature be equal to zero.

Proof: Suppose that an $E - P^*$space is Sasakian recurrent. Making use of equations (1.7), (1.8) and (1.10) in (2.1), we get

$$R^h_{ijk,a} = \lambda_a \left[R^h_{ijk} + \frac{n}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right)\right]. \tag{2.2}$$

Since an $E - P^*$ space is Sasakian recurrent, equation (2.2) reduces to

$$\frac{R}{n(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) = 0, \tag{2.3}$$

which gives $R = o$, i.e., scalar curvature is zero.

Conversely, if an $E - P^*$ space satisfies $R = 0$, then (2.2) reduces to

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0,$$

which shows that the space is Sasakian recurrent.

This completes the proof of the theorem.

Similarly, in view of Theorem (2.1) and equations (1.7), (1.8) and (1.11), we can prove the following theorem:

Theorem (2.2): A necessary and sufficient condition for an $E - S *$ space to be a Sasakian recurrent is that the scalar curvature be equal to zero.

Theorem (2.3): An $E - P *$ space is an $E - S *$ space, iff the scalar curvature R is equal to zero.

Proof: Suppose that an $E - P *$ space is an $E - S *$ space. Differentiating (1.11) covariantly with respect to x^n and using (1.7), we get

$$E^h_{ijk,a} = * P^h_{ijk,a} + \frac{R_{,a}}{(n+2)(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right). \tag{2.4}$$

Multiplying (1.11) by λ_a and subtracting the resulting equation from (2.4), we have

$$E^h_{ijk,a} - \lambda_a E^h_{ijk} = * P^h_{ijk,a} - \lambda_a * P^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{(n+2)(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \tag{2.5}$$

Now, making use of the above supposition and (1.7), equation (2.5) reduces to

$$\frac{\lambda_a R}{(n+2)(n+4)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) = 0, \text{ which implies } R = 0.$$

Conversely, let us suppose that in an $E - P *$ space, the scalar curvature $R = 0$.

Hence equation (2.5) reduces to

$$E^h_{ijk,a} - \lambda_a E^h_{ijk} = 0,$$

which shows that the space is an $E - S *$ space.

This completes the proof of the theorem.

Theorem (2.4): If a Sasakian space satisfies any two of the following properties:
(i). the space is an $E-S*$ space,
(ii). the space is an $E-P*$ space,
(iii). the scalar curvature is equal to zero, then it must satisfy the third also.

Proof: An $E-S*$ space and an $E-P*$ space are characterized respectively by equations (1.15) and (2.1).

The statement of the above theorem follows in view of equations (1.15), (2.1) and (2.5).

We have the following Lemmas from Walkar [7] and Yano & Bochner [9].

Lemma (2.1): The curvature tensor R_{hijk} satisfies the identity:

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{lmhi,jk} + R_{lmhi,kj} = 0, \tag{2.6}$$

where $R_{hijk,l,m} \overset{\text{def}}{=} R_{hijk,lm}$.

Lemma (2.2): If $a_{\alpha\beta}, b_r$ are quantities satisfying:

$$a_{\alpha\beta} = a_{\beta\alpha} \text{ and } a_{\alpha\beta} b_\gamma + a_{\beta\gamma} b_\alpha + a_{\gamma\alpha} b_\beta = 0, \tag{2.7}$$

for $\alpha, \beta, \gamma = 1,2, \dots\dots\dots, N$, then either all the $a_{\alpha\beta}$ are zero, or all the b_γ are zero.

Making use of the above Lemmas, we shall prove the following theorems:

Theorem (2.5): In an $E-P^*$ space, either recurrence vector is gradient, or the space is of constant holomorphic sectional curvature.

Proof: Differentiating (2.2) covariantly with respect to x^b and using equations (1.4), (1.7), (1.8) and (1.10), we get

$$R^h_{ijk,ab} = (\lambda_{a,b} + \lambda_a \lambda_b) * P^h_{ijk},$$

where $R^h_{ijk,a,b} \overset{\text{def}}{=} R^h_{ijk,ab}$. (2.8)

Multiplying (2.8) by g_{hi}, we have

$$R_{hijk,ab} = (\lambda_{a,b} + \lambda_a \lambda_b) * P_{ijkl}. \tag{2.9}$$

From (2.9) and the identity (2.6), we get

$$\lambda_{ab} * P_{ijkh} + \lambda_{ij} * P_{khab} + \lambda_{kh} * P_{abij} = 0, \tag{2.10}$$

where $\lambda_{ab} \overset{\text{def}}{=} \lambda_{b,\ a} - \lambda_{a,\ b}$.

Equation (2.10) is of the form (2.7), since $*P_{ijkh} = *P_{khij}$. Thus, from Lemma (2.2), we prove the Theorem (2.5).

Now, from (1.8) and the Bianchi identity

$$R^h_{ijk,a} + R^h_{jka,i} + R^h_{kai,j} = 0 \tag{2.11}$$

we have

$$R^h_{ijk,l} = 0.$$

Thus, contracting (2.2) with respect to h and a, we get

$$\lambda_a R^l_{ijk} + \frac{R}{n(n+2)}\left(\lambda_j g_{ik} - \lambda_i g_{jk} + \lambda_l F_{ik} F^l_j - \lambda_l F_{jk} F^l_i + 2\lambda_l F_{ij} F^l_k\right) = 0, \tag{2.12}$$

Furthermore, sustituting (2.2) in (2.11) and then transvecting with λ^a, we get

$$\begin{aligned}
\lambda^l \lambda_l R^h_{ijk} &+ \lambda^l \lambda_l \frac{R}{n(n+2)}\left(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k\right) \\
&+ \lambda_i \left\{\lambda^l R^h_{jik} + \frac{R}{n(n+2)}\left(\lambda^h g_{jk} - \lambda_k \delta^h_j + \lambda^a F_{jk} - \lambda^a F_{kj} - \lambda^a F_{ak} F_{jh} + 2\lambda^a F_{ja} F^h_k\right)\right\} \\
&+ \lambda_j \left\{\lambda^l R^h_{ijk} + \frac{R}{n(n+2)}\left(\lambda_k \delta^h_i - \lambda^h g_{ik} + \lambda^a F_{ak} F^h_i - \lambda^a F_{ik} F^h_a + 2\lambda^a F_{ai} F^h_k\right)\right\} = 0.
\end{aligned} \tag{2.13}$$

Which, in view of (2.12), gives

$$\lambda^l \lambda_l \left\{R^h_{ijk} + \frac{R}{n(n+2)}\left(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}\,F^h_k\right)\right\} = 0. \tag{2.14}$$

Since λ_a is a non-zero vector, in view of Definition (1.3), we have the following:

Theorem (2.6): An $E - P^*$ space is the space of constant holomorphic sectional curvature [8].

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Source of support: Nil, Conflict of interest: None Declared