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ON HARMONIC UNIFORMLY STARLIKE FUNCTIONS WITH OTHER POINTS DEFINED BY AN INTEGRAL OPERATOR

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ABSTRACT

T*he authors apply integral operator on* $(z-\omega)^p+\sum_{k=2}^{\infty}a_k(z-\omega)^k$ *to define a certain class of harmonic functions. They obtained coefficient inequalities, extreme points and distortion bounds for the functions in this class.*

Keywords*: Harmonic function, uniformly harmonic starlike and integral operator.*

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1. INTRODUCTION

For an arbitrary fixed positive ω in U, we denote by $H_p(\omega)$ the set of all harmonic multivalent functions $f=h+\overline{g}$ which are sense-preserving in the open unit disk $U=\{z:|z|<1\}$ where h and g are of the form

$$h(z)=(z-\omega)^p+\sum_{k=2}^{\infty}a_k(z-\omega)^k, \qquad g(z)=\sum_{k=2}^{\infty}b_k(z-\omega)^k, |b_k|<1$$
(1)

The Integral operator I^n was introduced by Salagean [7]. For fixed positive integer n and for $f=h+\overline{g}$ given by (1) we have

(i) $I^0 f(z)=f(z)$

(ii) $I^1 f(z)=If(z)=\int_0^z f(t)t^{-1}dt$

(iii) $I^n f(z)=I(I^{n-1}f(z)), n\in N=1,2,3...and \quad f\in A$

$$I^n f(z)=I^n h(z)+(-1)^n\overline{I^n g(z)}$$

Recently, Oladipo and Breaz [6], Clunie and Sheil-Small [2], Ahuja and Jahangiri [1], El-Ashwah and Aouf [4], Cotirla [3], Guney and Sakar [5] extended the operator studied in [6] and good results were obtained.

Here

$$I_{\omega,p}^n(\lambda,l)f(z)=I_{\omega,p}^n h(z)+(-1)^n\overline{I_{\omega,p}^n(\lambda,l)g(Z)} \tag{2}$$

where

$$I_{\omega,p}^n(\lambda,l)h(z)=(z-\omega)^p+\sum_{k=p+1}^{\infty}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)^n a_k(z-\omega)^k, \ and$$

$$I_{\omega,p}^n(\lambda,l)g(z)=\sum_{k=p+1}^{\infty}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)^n b_k(z-\omega)^k, |b_k|<1.$$

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For $0 \le \alpha < 1, n \in N, \lambda \ge 0, l \ge 0, s \ge 0, \theta \in R$.

For the purpose of our result, the following definition shall be necessary.

Definition A: Let $R^{\omega}_{s,q}(\lambda, l, \alpha, n, p)$ the family of harmonic function f of the form (2) such that

$$\operatorname{Re}\left\{(1+se^{i\theta})\frac{I^{n}_{\omega,p}(\lambda,l)f(z)}{I^{n+q}_{\omega,p}(\lambda,l)f(z)} - se^{i\theta}\right\} \ge \alpha. \tag{3}$$

For particular cases s and q, especially for $s = 0, p = 1, \lambda = 1, l = 0$ and $q = 1, \omega = 0$, we can write

$$R_{0,1}(1,1,0,\alpha,n) = R(\alpha,n)$$

which is the class studied by Cotirla in [3].

Definition B: Let denote the subclass $\overline{R^{\omega}_{s,q}}(\lambda, l, \alpha, n, p)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $R^{\omega}_{s,q}(\lambda, l, \alpha, n, p)$ so that h and g_n are of the form

$$I^{n}_{\omega,p}(\lambda,l)h(z) = (z-\omega)^p - \sum_{k=p+1}^{\infty}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)^n |a_k|(z-\omega)^k, \tag{4}$$

and

$$I^{n}_{\omega,p}(\lambda,l)g(z) = \sum_{k=p+1}^{\infty}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)^n |b_k|(z-\omega)^k, |b_k| < 1.$$

In this work, we investigate the coefficient bounds, extreme points, distortion bounds and the convexity properties of the functions are also obtained.

2. MAIN RESULT

We prove the following sufficient conditions for harmonic functions in $R_{s,q}(\lambda, l, \alpha, n, p)$.

Theorem 1: Let $f = h + \overline{g}$ be such that h and g are given by (1). If

$$\sum_{k=2}^{\infty}\frac{[\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)]}{1-\alpha}|a_k| + \sum_{k=1}^{\infty}\frac{[\gamma^n(1+s) - (-1)^q\gamma^{n+q}(\alpha+s)]}{1-\alpha}|b_k| \le 1. \tag{5}$$

where $\gamma = \left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)$ then $f \in R^{\omega}_{s,q}(\lambda, l, \alpha, n, p)$.

Proof: Suppose that (5) holds. Using the fact that $\operatorname{Re}\omega \ge \rho$ if and only if $|1-\rho+\omega| \ge |1-\rho+\omega|$, it suffices to show that

$$\left|(1-\alpha)I^{n+q}_{\omega}(\lambda,l)f(z) + I^{n+q}_{\omega}(\lambda,l)f(z)(1+se^{i\theta}) - se^{i\theta}I^{n+q}_{\omega}(\lambda,l)f(z)\right|$$
$$-\left|(1-\alpha)I^{n+q}_{\omega}(\lambda,l)f(z) - I^{n+q}_{\omega}(\lambda,l)f(z)(1+se^{i\theta}) + se^{i\theta}I^{n+q}_{\omega}(\lambda,l)f(z)\right| \ge 0, \tag{6}$$

substituting for $I_{\omega}^{n}(\lambda,l)f(z)$ and $I_{\omega}^{n+q}(\lambda,l)f(z)$ in (6) yields

$$\left\| \begin{array}{l} (1-\alpha-se^{i\theta})\left\{(z-\omega)^p+\sum_{k=2}^{\infty}\gamma^{n+q}a_k+(-1)^{n+q}\sum_{k=1}^{\infty}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)^{n+q}\overline{b_k(z-\omega)^k}\right\} \\ +(1+se^{i\theta})\left\{(z-\omega)^p+\sum_{k=2}^{\infty}\gamma^{n+q}a_k+(-1)^{n}\sum_{k=1}^{\infty}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)^{n+q}\overline{b_k(z-\omega)^k}\right\} \end{array} \right|$$

$$-\left\| \begin{array}{l} (1-\alpha-se^{i\theta})\left\{(z-\omega)^p+\sum_{k=2}^{\infty}\gamma^{n+q}a_k+(-1)^{n+q}\sum_{k=p+1}^{\infty}\gamma^{n+q}\overline{b_k(z-\omega)^k}\right\} \\ +(1+se^{i\theta})\left\{(z-\omega)^p+\sum_{k=1}^{\infty}\gamma^{n+q}a_k+(-1)^{n}\sum_{k=p+1}^{\infty}\gamma^{n+q}\overline{b_k(z-\omega)^k}\right\} \end{array} \right|$$

$$=\left\| \begin{array}{l} (2-\alpha)(z-\omega)^p+\sum_{k=2}^{\infty}\gamma^{n}(1+se^{i\theta}+\gamma^q(1-\alpha-se^{i\theta}))a_k(z-\omega)^k \\ +(-1)^n\sum_{k=1}^{\infty}\gamma^{n}(1+se^{i\theta}+(-1)^q\gamma^q(1-\alpha-se^{i\theta}))\overline{b_k(z-\omega)^k} \end{array} \right|$$

$$-\left\| \begin{array}{l} \alpha(z-\omega)^p+\sum_{k=2}^{\infty}\gamma^{n}(-1-se^{i\theta}+\gamma^q(1+\alpha+se^{i\theta}))a_k(z-\omega)^k \\ -(-1)^n\sum_{k=1}^{\infty}\gamma^{n}(1+se^{i\theta}-(-1)^q\gamma^q(1+\_\alpha+se^{i\theta}))\overline{b_k(z-\omega)^k} \end{array} \right|$$

$$\geq 2(1-\alpha)|(z-\omega)|^p-2\sum_{k=2}^{\infty}\gamma^n[1+s)-\gamma^q(\alpha+s)]|a_k||(z-\omega)|^k-2\sum_{k=1}^{\infty}\gamma^n[1+s)-(-1)^q\gamma^q(\alpha+s)]|b_k||(z-\omega)|^k$$

$$=2\,1-\alpha()|(z-\omega)|^p\left\{`-\sum_{k=2}^{\infty}\frac{\gamma^n[1+s)-\gamma^q(\alpha+s)]}{1-\alpha}|a_k||(z-\omega)|^{k-p}-2\sum_{k=1}^{\infty}\frac{\gamma^n[1+s)-(-1)^q\gamma^q(\alpha+s)]}{1-\alpha}|b_k||(z-\omega)|^{k-p}\right\}.$$

This last expression is non-negative by hypothesis, and so the proof is complete.

The functions

$$I_{s,q}^{n}(\lambda,l)f(z)=(z-\omega)^p+\sum_{k=2}^{\infty}\frac{1-\alpha}{[\gamma^n(1+r)-\gamma^{n+q}(\alpha+r)]}x_k(z-\omega)^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{[\gamma^n(1+r)-(1-)^q\gamma^{n+q}(\alpha+r)]}\overline{y_k(z-\omega)^k}. \quad (7)$$

shows that the coefficient bound given by (5) is sharp where $n\in N, r\geq 0, q\in N, \lambda\geq 0, l\geq 0$ and

$$\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$$

Theorem 2: Let $f_n=h+\overline{g}_n$ be given by (4). Then $f_n\in\overline{R_{s,q}^{\omega}}(\lambda,l,\alpha,n,p)$. If and only if

$$\sum_{k=2}^{\infty}\frac{[\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)]}{1-\alpha}|a_k|+\sum_{k=2}^{\infty}\frac{[\gamma^n(1+s)-(-1)^q\gamma^{n+q}(\alpha+s)]}{1-\alpha}|b_k|\leq 1, \quad (8)$$

where $\gamma=\left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)$ where $\alpha_1=1,0\leq\alpha<1, n\in N, \lambda\geq 0, l\geq 0, s\geq 0, \theta\in R$.

Proof: Since $\overline{R_{s,q}^{\omega}}(\lambda,l,\alpha,n,p)\subset R_{s,q}^{\omega}(\lambda,l,\alpha,n,p)$ we only need to prove the 'only if' part of the theorem. For function f_n of the form (4), we note that condition

$$\operatorname{Re}\left\{(1+se^{i\theta})\frac{I_{\omega}^{n}(\lambda,l)f_n(z)}{I_{\omega}^{n+q}(\lambda,l)f_n(z)}-se^{i\theta}\right\}\geq\alpha$$

is equivalent to

$$\operatorname{Re}\left\{\frac{(1+se^{i\theta})I_{\omega}^{n}(\lambda,l)f_n(z)-I_{\omega}^{n+q}(\lambda,l)f_n(z)(se^{i\theta}+\alpha)}{I_{\omega}^{n+q}(\lambda,l)f_n(z)}\right\}$$

$$=\operatorname{Re}\left\{\begin{array}{l}\left\{\dfrac{(1+se^{i\theta})\left[(z-\omega)^p-\sum_{k=2}^{\infty}\gamma^n|a_k|(z-\omega)^k+(-1)^{2n-1}\sum_{k=1}^{\infty}\gamma^n|a_k|(z-\omega)^k\right]}{(z-\omega)^p-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^k+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}}\right.\\ -\dfrac{(\alpha+se^{i\theta})\left[(z-\omega)^p-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^k+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}\right]}{(z-\omega)^p-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^k+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}}\end{array}\right\}$$

$$=\operatorname{Re}\left\{\begin{array}{l}\left\{\dfrac{(1-\alpha)(z-\omega)^p-\sum_{k=2}^{\infty}\left[\gamma^n(1+se^{i\theta})-\gamma^{n+q}(\alpha+se^{i\theta})\right]a_k|(z-\omega)^k}{(z-\omega)^p-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^k+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}}\right.\\ +\dfrac{(-1)^{2n-1}\sum_{k=2}^{\infty}\left[\gamma^n(1+se^{i\theta})-(-1)^q\gamma^{n+q}(\alpha+se^{i\theta})\right]b_k|\overline{(z-\omega)^k}}{(z-\omega)^p-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^k+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}}\end{array}\right\}$$

$$=\operatorname{Re}\left\{\begin{array}{l}\left\{\dfrac{(1-\alpha)-\sum_{k=2}^{\infty}\left[\gamma^n(1+se^{i\theta})-\gamma^{n+q}(\alpha+se^{i\theta})\right]|a_k|(z-\omega)^{k-p}}{1-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^{k-p}+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}}\right.\\ +\dfrac{\frac{(z-\omega)}{(z-\omega)^p}(-1)^{2n-1}\sum_{k=2}^{\infty}\left[\gamma^n(1+se^{i\theta})-\gamma^{n+q}(\alpha+se^{i\theta})\right]|b_k|\overline{(z-\omega)^k}}{1-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^{k-p}+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|\overline{(z-\omega)^k}}\end{array}\right\}\geq 0. \tag{9}$$

Upon choosing the value of $(z-\omega)$ on the positive real axis and using $\operatorname{Re}(-e^{i\theta})\geq-|e^{i\theta}|=-1$ where $0\leq(z-\omega)=(r+d)<1$, the above inequalities reduces to

$$=\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)\right]|a_k|(r+d)^{k-p}}{1-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(z-\omega)^{k-p}+(-1)^{2n+q-1}\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|(r+d)^{k-p}}$$
$$-\frac{\sum_{k=2}^{\infty}\left[\gamma^n(1+se^{i\theta})-\gamma^{n+q}(\alpha+se^{i\theta})\right]|b_k|(r+d)^{k-p}}{1-\sum_{k=2}^{\infty}\gamma^{n+q}|a_k|(r+d)^{k-p}-(-1)^q\sum_{k=1}^{\infty}\gamma^{n+q}|b_k|(r+d)^{k-p}}\geq 0. \tag{10}$$

If the condition (8) does not hold, then the numerator (10) is negative for sufficiently close to 1.

Thus there exist $z_0-\omega=r_0+d$ in (0, 1) for which the quotient in (10) is negative. This contradicts the condition for $\overline{R_{s,q}^{\omega}}(\lambda,l,\alpha,n,p)$. So the proof complete.

Convex hull of $\overline{R^{\omega}_{s,q}}(\lambda, l, \alpha, n, p)$.

Theorem 3: Let f_n be given by (2) if and only if

$$f_n(z) = \sum_{k=1}^{\infty}\left[X_k h_k(z) + Y_k g_{nk}(z)\right]$$

where

$$h(z) = (z-\omega)^p \quad h_k(z) = (z-\omega)^p - \frac{1-\alpha}{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}(z-\omega)^k \quad (k = 2,3,...)$$

and $\gamma = \left(\dfrac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right)$

$$g_{n,k}(z) = (z-\omega)^p + (-1)^{n-1} - \frac{1-\alpha}{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}\overline{(z-\omega)^k} \qquad (k = 1,2,3,...)$$

$$\gamma = \left(\frac{1+\lambda(p-1)+l}{1+\lambda(k-1)+l}\right) \; X_k \ge 0, Y_k \ge 0, \sum_{k=1}^{\infty}(X_k + Y_k) = 1.$$

In particular, the extreme points of $R^{\omega}_{s,q}(\lambda, l, \alpha, n)$ are h_k and $g_{n,k}$.

Proof: For function f_n of the form (2), we have

$$f_n(z) = \sum_{k=1}^{\infty}\left[X_k h_k(z) + Y_k g_{nk}(z)\right]$$

$$= \sum_{k=1}^{\infty}\left[X_k + Y_k\right](z-\omega)^p - \sum_{k=2}^{\infty}\frac{1-\alpha}{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}X_k(z-\omega)^k$$

$$+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}\overline{Y_k(z-\omega)^k},$$

Then

$$\sum_{k=2}^{\infty}\frac{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}{1-\alpha}\frac{1-\alpha}{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}X_k(z-\omega)^k + \sum_{k=1}^{\infty}\frac{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}{1-\alpha}\frac{1-\alpha}{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}\overline{Y_k(z-\omega)^k}$$

$$= \sum_{k=1}^{\infty}X_k + \sum_{k=2}^{\infty}Y_k = 1 - X_1 \le 1.$$

and so $f_n \in \overline{R^{\omega}_{s,q}}(\lambda, l, \alpha, n, p)$.

Conversely, suppose $f_n \in \overline{R^{\omega}_{s,q}}(\lambda, l, \alpha, n, p)$. Letting

$$X_1 = 1 - \sum_{k=1}^{\infty}X_k + \sum_{k=2}^{\infty}Y_k,$$

$$X_k = \frac{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}{1-\alpha}|a_k| \qquad (k = 2,3...)$$

and

$$Y_k = \frac{\gamma^n(1+s) - (-1)^q\gamma^{n+q}(\alpha+s)}{1-\alpha}|a_k|. \qquad (k = 1,2,3...)$$

We obtained the required representation, since

$$f_n(z)=(z-\omega)^p-\sum_{k=1}^{\infty}|a_k|(z-\omega)^k+(-1)^{n-1}\sum_{k=1}^{\infty}|b_k|\overline{(z-\omega)^k}$$
$$=(z-\omega)^p-\sum_{k=2}^{\infty}\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}X_k(z-\omega)^k+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}Y_k\overline{(z-\omega)^k}$$
$$=(z-\omega)^p-\sum_{k=2}^{\infty}(z-\omega)^p-h_k(z)]X_k-\sum_{k=1}^{\infty}(z-\omega)^p g_{nk}(z)]Y_k$$
$$=\left[1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k\right](z-\omega)^p+\sum_{k=2}^{\infty}X_k h_k(z)-\sum_{k=1}^{\infty}Y_k g_{nk}(z)$$
$$=\sum_{k=2}^{\infty}[X_k h_k(z)+Y_k gnk(z)].$$

Distortion bounds for functions in $\overline{R_{s,q}^{\omega}}(\lambda,l,\alpha,n,p)$ is our next result.

Theorem 4: Let $f_n\in\overline{R_{s,q}^{\omega}}(\lambda,l,\alpha,n,p)$. Then for $|z-\omega|=r+d<1,$ we have

$$|f_n(z)|\le(1+b_1)(r+d)^p+\left[\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}-\frac{(1+s)-(-1)^q(\alpha+s)}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}|b_1|\right](r+d)^2$$

and

$$|f_n(z)|\le(1+b_1)(r+d)^p-\left[\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}-\frac{(1+s)-(-1)^q(\alpha+s)}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}|b_1|\right](r+d)^2$$

where q is a an odd positive integer.

Proof: We prove the right hand inequality. Let $f_n\in\overline{R_{s,q}^{\omega}}(\lambda,l,\alpha,n,p)$. Taking the absolute value of f_n,… we obtain

$$|f_n(z)|\le(1+b_1)(r+d)^p+\sum_{k=2}^{\infty}(|a_k|+|b_k|)(r+d)^k$$
$$\le(1+b_1)(r+d)^p+\sum_{k=2}^{\infty}(|a_k|+|b_k|)(r+d)^2$$
$$\le(1+b_1)(r+d)^p+\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}-\sum_{k=2}^{\infty}\frac{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}{1-\alpha}(|a_k|+|b_1|)(r+d)^2$$
$$\le(1+b_1)(r+d)^p+\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}-\sum_{k=2}^{\infty}\left[\frac{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}{1-\alpha}|a_k|+\frac{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}{1-\alpha}|b_k|\right](r+d)^2$$
$$\le(1+b_1)(r+d)^p+\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}\left[1-\frac{(1+s)-(1-)^q(\alpha+s)}{1-\alpha}|b_1|\right](r+d)^2$$
$$\le(1+b_1)(r+d)^p+\left[\frac{1-\alpha}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}-\frac{(1+s)-(1-)^q(\alpha+s)}{\gamma^n(1+s)-\gamma^{n+q}(\alpha+s)}|b_1|\right](r+d)^2$$

which completes the proof.

Convex combination for the function $\overline{R_{s,q}^{\omega}}(\lambda, l, \alpha, n, p)$:

Theorem 5: The family $\overline{R_{s,q}^{\omega}}(\lambda, l, \alpha, n, p)$ is closed under convex combination.

Proof: For i=1, 2,…, suppose that $f_n^i \in \overline{R_{s,q}^{\omega}}(\lambda, l, \alpha, n, p)$, where

$$f_n^i(z) = (z-\omega)^p - \sum_{k=2}^{\infty} \left|a_k^i\right| (z-\omega)^k + (-1)^{n-1} \sum_{k=1}^{\infty} \left|b_k^i\right| \overline{(z-\omega)^k}$$

then by Theorem 2,

$$\sum_{k=2}^{\infty} \frac{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}{1-\alpha} \left|a_k^i\right| + \sum_{k=1}^{\infty} \frac{\gamma^n(1+s) - (-1)^q \gamma^{n+q}(\alpha+s)}{1-\alpha} \left|b_k^i\right| \le 2(x) \text{ for } \sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1,$$ the

convex combination of f_n^i may be written as

$$\sum_{i=1}^{\infty} t_i f_n^i(z) = (z-\omega)^p - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|a_k^i\right| (z-\omega)^k \right) + (-1)^{n-1} \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|b_k^i\right| \overline{(z-\omega)^k} \right).$$

Then by (x)

$$\sum_{k=2}^{\infty} \frac{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i \left|a_k^i\right| \right) + \sum_{k=1}^{\infty} \frac{\gamma^n(1+s) - (-1)^q \gamma^{n+q}(\alpha+s)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i \left|b_k^i\right| \right)$$

$$\sum_{i=1}^{\infty} t_i \left[\sum_{k=2}^{\infty} \frac{\gamma^n(1+s) - \gamma^{n+q}(\alpha+s)}{1-\alpha} \left|a_k^i\right| + \sum_{k=1}^{\infty} \frac{\gamma^n(1+s) - (-1)^q \gamma^{n+q}(\alpha+s)}{1-\alpha} \left|b_k^i\right| \right] \le 2 \sum_{i=1}^{\infty} t_i = 2$$

and therefore $\sum_{i=1}^{\infty} t_i f_n^i(z) \in \overline{R_{s,q}^{\omega}}(\lambda, l, \alpha, n, p)$.

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