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AN INTEGRAL INEQUALITY FOR POLYNOMIALS

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ABSTRACT

In this paper, a compact generalization of certain known Lp inequalities for polynomials is obtained, which refine some results due to De-Bruijn, Boas and Rahman and others.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let P_n (z) denote the space of all complex polynomials $P(z)=\sum_{j=1}^{n} a_j z^j$ of degree n. For P∈P_n, define

$$\|P(z)\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \quad p\geq 1$$

and

$$\|P(z)\|_\infty := \underset{|z|=1}{Max}|P(z)|.$$

If P∈P_n, then according to a famous result known as Bernstein's inequality (for reference see [13, 16, 18]),

$$\|P'(z)\|_\infty \leq n\|P(z)\|_\infty, \tag{1}$$

whereas concerning the maximum modulus of P (z) on the circle |z|=R>1, we have

$$\|P(Rz)\|_\infty \leq R^n\|P(z)\|_\infty \tag{2}$$

Inequality (2) is a simple deduction from maximum modulus principle (see [13, p.442] or [14, Vol I, p.137]).

Inequalities (1) and (2) can be obtained by letting $p\to\infty$ in the inequalities

$$\|P'(z)\|_p \leq n\|P(z)\|_p, \quad p\geq 1 \tag{3}$$

and

$$\|P(Rz)\|_p \leq R^n\|P(z)\|_p, \; R>1, \; p>0 \tag{4}$$

respectively. Inequality (3) was found by Zygmund [19], whereas inequality (4) is a simple consequence of a result of Hardy [10] (see also [16, Th.5.5]). Arestov [2] proved that (3) remains true for 0<p<1 as well.

If we restrict ourselves to the class of polynomials P∈P_n, having no zero in |z|<1, then the inequalities (1) and (2) can be sharpened. In fact, if P(z) $\neq 0$ in |z| < 1, then (1) and (2) can be respectively replaced by

$$\|P'(z)\|_\infty \leq \frac{n}{2}\|P(z)\|_\infty \tag{5}$$

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and

$$\|P(Rz)\|_\infty \le \frac{R^n+1}{2}\|P(z)\|_\infty,\ \text{R>1}. \tag{6}$$

Inequality (5) was conjectured by Erdös and later verified by Lax [12]. Ankiny and Rivilin [1] used inequality (5) to prove inequality (6).

Both the inequalities (5) and (6) can be obtained by letting $p\to\infty$ in the inequalities

$$\|P'(z)\|_p \le n\frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p\ge 0, \tag{7}$$

and

$$\|P(Rz)\|_p \le \frac{\|R^n z+1\|_p}{\|1+z\|_p}\|P(z)\|_p,\ R>1,\ p>0. \tag{8}$$

Inequality (7) is due to De Bruijn [9] for $p\ge 1$. Rahman and Schmeisser [15] extended it for $0<p<1$, whereas the inequality (8) was proved by Boas and Rahman [8] for $p\ge 1$ and later extended for $0<p<1$ by Rahman and Schmeisser [15].

Aziz and Dawood [3] refined both the inequalities (5) and (6) by showing that if $P\in P_n$, and P (z) does not vanish in $|z|<1$ and $m=\min_{|z|=1}|P(z)|$, then

$$\|P'(z)\|_\infty \le \frac{n}{2}\{\|P(z)\|_\infty - m\}, \tag{9}$$

and

$$\|P(Rz)\|_\infty \le \frac{R^n+1}{2}\{\|P(z)\|_\infty - m\}. \tag{10}$$

As a compact generalization of the inequalities (3), (4), (5), (6), recently Aziz and Rather [7] proved that if $P\in P_n$ and P (z) does not vanish in $|z|<1$, then for arbitrary real or complex numbers α and β with $|\alpha|\le 1$, $|\beta|\le 1$, $R>r\ge 1$ and $p>0$,

$$\|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)\|_p \le \frac{C_p\|P(z)\|_p}{\|1+z\|_p} \tag{11}$$

where

$$C_p = \|\left(R^n+\varphi(R,r,\alpha,\beta)r^n\right)z+\left(1+\varphi(R,r,\alpha,\beta)\right)\|_p \tag{12}$$

and

$$\phi(R,r,\alpha,\beta)=\beta\left\{\left(\frac{R+1}{r+1}\right)^n-|\alpha|\right\}-\alpha \tag{13}$$

In this paper, we prove the following interesting result which includes not only a generalization of the inequality (11) as a special case but also leads to some refinements and generalizations of certain known polynomial inequalities.

Theorem 1: If $P\in P_n$, does not vanish in $|z|<1$ and $m=\min_{|z|=1}|P(z)|$, then for arbitrary complex numbers α, β, δ with $|\alpha|\le 1$, $|\beta|\le 1$, $|\delta|\le 1$, $R>r\ge 1$ and $p>0$,

$$\|P(Rz)+\varphi(R,r,\alpha,\beta)P(rz)+\delta m\frac{|R^n+\varphi(R,r,\alpha,\beta)r^n|-|1+\varphi(R,r,\alpha,\beta)|}{2}\|_p \le \frac{C_p}{\|1+z\|_p}\|P(z)\|_p, \tag{14}$$

where C_p and $\phi(R,r,\alpha,\beta)$ are defined by (12) and (13) respectively. The result is best possible and the equality in (14) holds for the polynomial $P(z)=az^n+b$, where $|a|=|b|=1$.

Theorem 1 has various interesting consequences. Here we mention few of these.

For δ = 0, the inequality (14) reduces to inequality (11). Next, we mention the following compact generalization of inequalities (5), (6), (7), (8), (9) and (10), which follows from Theorem 1 by setting $\beta = 0$.

Corollary 1: If $P \in P_n$ does not vanish in $|z| < 1$ and $m = \min_{|z|=1} |P(z)|$, then for arbitrary complex numbers α, δ with $|\alpha| \le 1$, $|\delta| \le 1$, $R > r \ge 1$ and $p > 0$,

$$\| P(Rz) - \alpha P(rz) + \frac{\delta}{2}\left(|R^n - \alpha r^n| - |1-\alpha|\right) m \|_p \le \frac{\left\|\left(R^n - \alpha r^n\right) z + (1-\alpha)\right\|_p}{\|1+z\|_p} \|P\|_p \ . \tag{15}$$

The result is best possible and equality in (15) holds for $P(z) = z^n + 1$.

Remark 1: Corollary 3 includes as a special case a result due to Rather [17, Theorem 1], which is obtained by taking α = 0 in (15).

Next if we set α = 1 and divide two sides of (15) by R – r and let R ⟶ r, we obtain,

Corollary 2: If $P \in P_n$ does not vanish in $|z| < 1$ and $m = \min_{|z|=1} |P(z)|$, then for each $p > 0$ and $r \ge 1$,

$$\| z P'(rz) + \frac{\delta}{2} n m r^{n-1} \|_p \le \frac{n r^{n-1}}{\|1+z\|_p} \|P(z)\|_p \ .$$

The result is sharp.

Corollary 2 is an interesting generalization of inequality (7) due to De Bruijn [9]. Inequality (8) can also be obtained from inequality (15) by setting α = δ = 0.

Making p ⟶ ∞ in (14) and choosing the argument of δ with |δ| = 1 suitably, we obtain:

Corollary 3: If $P \in P_n$ does not vanish in $|z| < 1$, then for $\alpha, \beta \in C$ with $|\alpha| \le 1$, $|\beta| \le 1$ and $R > r \ge 1$,

$$\begin{aligned} \left\|P(Rz) + \varphi(R,r,\alpha,\beta) P(rz)\right\|_\infty \le & \left\{\frac{|R^n + \varphi(R,r,\alpha,\beta) r^n| + |1+\varphi(R,r,\alpha,\beta)|}{2}\right\} \|P(z)\|_\infty \\ & - \left\{\frac{|R^n + \varphi(R,r,\alpha,\beta) r^n| - |1+\varphi(R,r,\alpha,\beta)|}{2}\right\} \min_{|z|=1} |P(z)|, \end{aligned} \tag{16}$$

where $\phi(R,r,\alpha,\beta)$ is the same as defined in Theorem 1. The result is sharp and the equality holds for $P(z) = a z^n + b$, where |a| = |b| =1.

Corollary 3 is a refinement as well as a generalization of a result due to Aziz and Rather [4, Theorem 3]. For $\alpha = \beta = 0$, it reduces to (10). If we divide the two sides of inequality (16) by R – r with α = 1 =r and let R⟶r, we get inequality (9).

Finally we mention the result which is a refinement as well as a generalization of a result due to Jain [11, Theorem 2], which follows from corollary 3 as a special case.

Corollary 4: If $P \in P_n$, does not vanish in $|z| < 1$ and $m = \min_{|z|=1} |P(z)|$, then for every $\beta \in C$ with $|\beta| \le 1$, $R > r \ge 1$ and for $|z| = 1$,

$$\left| z P'(rz) + \frac{n\beta}{1+r} P(rz) \right| \le \frac{n}{2}\left(\left| r^{n-1} + \beta \frac{r^n}{1+r}\right| + \left|\frac{\beta}{1+r}\right|\right) \|P\|_\infty - \frac{n}{2}\left(\left| r^{n-1} + \beta \frac{r^n}{1+r}\right| - \left|\frac{\beta}{1+r}\right|\right) m \tag{17}$$

and

$$\left|P(Rz)+\beta\left(\frac{R+1}{r+1}\right)^n P(rz)\right|\le\frac{1}{2}\left[\left\{\left|R^n+\beta\left(\frac{R+1}{r+1}\right)^n r^n\right|+\left|1+\beta\left(\frac{R+1}{r+1}\right)^n\right|\right\}\|P\|_\infty - \left\{\left|R^n+\beta\left(\frac{R+1}{r+1}\right)^n r^n\right|-\left|1+\beta\left(\frac{R+1}{r+1}\right)^n\right|\right\}m\right] \tag{18}$$

The result is sharp and the extremal polynomial is $P(z)=a\,z^n+b$ where |a| = |b| =1.

2. LEMMAS

For the proofs of these theorems, we need the following lemmas.

Lemma 1: If F (z) is a polynomial of degree n having all its zeros in |z| ≤ 1 and f (z) is a plynomial of degree at most n such that

$$|f(z)|\le|F(z)| \quad \text{for } |z|=1,$$

then for every $R>r\ge 1$, α, $\beta\in C$ with |α| ≤ 1, |β| ≤ 1 and |z|≥1,

$$|f(Rz)+\phi(R,r,\alpha,\beta)f(rz)|\le|F(Rz)+\phi(R,r,\alpha,\beta)F(rz)| \tag{19}$$

where $\phi(R,r,\alpha,\beta)$ is defined by (13).

Lemma 1 is due to A.Aziz and N.A. Rather [7].

Lemma 2: If P (z) is a polynomial of degree n having all its zeros in |z| ≤1 and $m=\min_{|z|=1}|P(z)|$, then for every $R>r\ge 1$, α, $\beta\in C$ with |α| ≤ 1, |β| ≤ 1 and |z|≥1,

$$|P(Rz)+\phi(R,r,\alpha,\beta)p(rz)|\ge m\,\left|R^n+\phi(R,r,\alpha,\beta)\,r^n\right||z|^n \tag{20}$$

where $\phi(R,r,\alpha,\beta)$ is defined by (13).

Proof of Lemma 2: For m=0, there is nothing to prove. Assume m > 0, so that all the zeros of P (z) lie in |z| <1 and we have

$$m\,|z|^n\le|P(z)| \quad \text{for} \quad |z|=1.$$

Applying Lemma 2 with F (z) replaced by P(z) and f (z) by $m\,z^n$, we obtain for |z|≥1,

$$m\left|R^n z^n+\phi(R,r,\alpha,\beta)\,r^n z^n\right|\le|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)|.$$

That is,

$$|P(Rz)+\varphi(R,r,\alpha,\beta)P(rz)|\ge m|z|^n\left|R^n+\varphi(R,r,\alpha,\beta)\,r^n\right|$$

for α, $\beta\in C$ with |α| ≤ 1, |β| ≤ 1, R > r ≥ 1 and |z|≥1. That proves Lemma 2.

Lemma 3: If P∈P_n does not vanish in |z| < 1 and $m=\min_{|z|=1}|P(z)|$, then for every $R>r\ge 1$, α, $\beta\in C$ with |α| ≤ 1, |β| ≤ 1 and |z| = 1,

$$\left|P(Rz)+\varphi(R,r,\alpha,\beta)P(rz)\right|\le\left|Q(Rz)+\varphi(R,r,\alpha,\beta)Q(rz)\right|-\{\left|R^n+\varphi(R,r,\alpha,\beta)\,r^n\right|-\left|1+\varphi(R,r,\alpha,\beta)\right|\}m \tag{21}$$

where $Q(z)=z^n\,\overline{P(1/\overline{z})}$.

Proof of Lemma 3: Since $m=\min\limits_{|z|=1}|P(z)|$, we have

$m\le|P(z)|$ for $|z|=1$.

Therefore, for every complex number λ with $|\lambda|<1$, the polynomial $H(z)=P(z)-\lambda m$ of degree n does not vanish in $|z|<1$. If

$$G(z)=z^n\,\overline{H(1/\overline{z})}=Q(z)-\overline{\lambda}\,m\,z^n,$$

then all the zeros of polynomial G (z) of degree n lie $|z|\le1$ and

$|H(z)|=|G(z)|$ for $|z|=1$.

Applying Lemma 1 with f (z) replaced by H(z) and F (z) by G (z), we get for every $R>r\ge1$, $\alpha,\beta\in C$ with $|\alpha|\le1$, $|\beta|\le1$ and $|z|\ge1$,

$$\left|H(Rz)+\phi(R,r,\alpha,\beta)H(rz)\right|\le\left|G(Rz)+\phi(R,r,\alpha,\beta)G(rz)\right|.$$

That is,

$$\begin{aligned}&|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)-\lambda m\{1+\phi(R,r,\alpha,\beta)\}|\\&\qquad\le|Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)-\overline{\lambda}\,m\,z^n\left\{R^n+\phi(R,r,\alpha,\beta)r^n\right\}|\end{aligned} \tag{22}$$

for every $R>r\ge1$, $\alpha,\beta\in C$ with $|\alpha|\le1$, $|\beta|\le1$ and $|z|\ge1$. Since all the zeros of Q(z) lie in $|z|\le1$, we choose argument of λ with $|\lambda|<1$, such that

$$\begin{aligned}&|Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)-\overline{\lambda}\,m\,z^n\left\{R^n+\phi(R,r,\alpha,\beta)r^n\right\}|\\&\qquad=|Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)|-|\lambda|m\,\left|R^n+\phi(R,r,\alpha,\beta)r^n\right||z|^n\end{aligned}$$

for $|z|\ge1$, which is possible by Lemma 3, we have from (22),

$$\begin{aligned}&\left|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)\right|-|\lambda|m\left|1+\phi(R,r,\alpha,\beta)\right|\\&\qquad=|Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)|-|\lambda|m\,\left|R^n+\phi(R,r,\alpha,\beta)r^n\right||z|^n\end{aligned} \tag{23}$$

for $|z|\ge1$. Equivalently for every $R>r\ge1$, $\alpha,\beta\in C$ with $|\alpha|\le1$, $|\beta|\le1$ and $|z|\ge1$,

$$\begin{aligned}\left|P(Rz)+\varphi(R,r,\alpha,\beta)P(rz)\right|&\le\left|Q(Rz)+\varphi(R,r,\alpha,\beta)Q(rz)\right|\\&\quad-|\lambda|\{\left|R^n+\varphi(R,r,\alpha,\beta)\,r^n\right||z|^n-\left|1+\varphi(R,r,\alpha,\beta)\right|\}m.\end{aligned}$$

Letting $|\lambda|\longrightarrow1$, we get the conclusion of lemma 4.

We also need the following two Lemmas due to A.Aziz and N.A.Rather [8, 9].

Lemma 4 [7]: If $P\in P_n$, then for arbitrary real or complex numbers α, β, with $|\alpha|\le1$, $|\beta|\le1$, $R>r\ge1$, $p>0$ and γ real,

$$\int_0^{2\pi} | P(Re^{i\theta}) + \phi(R,r,\alpha,\beta) P(re^{i\theta}) + e^{i\gamma}\{R^n P(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta}) r^n P(e^{i\theta}/r)\}|^p \, d\theta$$

$$\leq | R^n + \phi(R,r,\alpha,\beta) r^n + e^{i\gamma}\left(1+\phi(R,r,\overline{\alpha},\overline{\beta}) r^n\right)| \int_0^{2\pi} \left| P(e^{i\theta})\right| d\theta. \qquad (24)$$

The result is sharp and the extremal polynomial is $P(z) = \lambda z^n$, $\lambda \neq 0$.

Lemma 5 [6]: If A, B, C are non-negative real numbers with B+C $\leq$ A, then for every realnumber α,

$$\left|(A-C)\, e^{i\alpha} + (B+C)\right| \leq \left| Ae^{i\alpha} + B \right| .$$

3. PROOF OF THE THEOREM

Proof of Theorem 1: By hypothesis $P \in P_n$ and P (z) $\neq$ 0 in $|z| < 1$, therefore by Lemma 3, we have for arbitrary real or complex numbers α, β, with $|\alpha|\leq 1$, $|\beta|\leq 1$, $R>r\geq 1$ and $0 \leq \theta < 2\pi$,

$$\left| P\left(Re^{i\theta}\right) + \varphi(R,r,\alpha,\beta) P\left(re^{i\theta}\right)\right| \leq \left| Q\left(Re^{i\theta}\right) + \varphi(R,r,\alpha,\beta) Q\left(re^{i\theta}\right)\right|$$
$$- \left\{\left|R^n + \varphi(R,r,\alpha,\beta)\, r^n\right| - \left|1+\varphi(R,r,\alpha,\beta)\right| \right\} m$$

where $m = \min_{|z|=1} |P(z)|$, $Q(z) = z^n \overline{P\left(1/\overline{z}\right)}$ and $\phi(R,r,\alpha,\beta)$ is defined by (13). This implies,

$$\left| P\left(Re^{i\theta}\right) + \varphi(R,r,\alpha,\beta) P\left(re^{i\theta}\right)\right| \leq \left| R^n P\left(\frac{e^{i\theta}}{R}\right) + \varphi\left(R,r,\overline{\alpha},\overline{\beta}\right) r^n P\left(\frac{e^{i\theta}}{r}\right)\right|$$
$$- \left\{\left|R^n + \varphi(R,r,\alpha,\beta)\, r^n\right| - \left|1+\varphi(R,r,\alpha,\beta)\right| \right\} m \qquad (25)$$

which gives,

$$\left| P\left(Re^{i\theta}\right) + \varphi(R,r,\alpha,\beta) P\left(re^{i\theta}\right)\right| + \frac{m}{2}\left\{\left|R^n + \varphi(R,r,\alpha,\beta) r^n\right| - \left|1+\varphi(R,r,\alpha,\beta)\right|\right\}$$
$$\leq \left|R^n P(e^{i\theta}/R) + \phi\left(R,r,\overline{\alpha},\overline{\beta}\right) r^n P(e^{i\theta}/r)\right| - \frac{m}{2}\left\{ |R^n + \phi(R,r,\alpha,\beta)\, r^n| - |1+\phi(R,r,\alpha,\beta)|\right\} \qquad (26)$$

Taking

$$A = \left| R^n P\left(\frac{e^{i\theta}}{R}\right) + \phi\left(R,r,\overline{\alpha},\overline{\beta}\right) r^n P\left(\frac{e^{i\theta}}{r}\right)\right|,$$

$$B = \left| P\left(Re^{i\theta}\right) + \varphi(R,r,\alpha,\beta) P\left(re^{i\theta}\right)\right| ,$$

and $C = \dfrac{\left|R^n + \phi(R,r,\alpha,\beta)\, r^n\right| - \left|1+\phi(R,r,\alpha,\beta)\right|}{2} m$

in Lemma 5 and noting by (26), that B+C $\leq$ A, we obtain for every real γ,

$$\left|\left\{\left| R^n P\left(\frac{e^{i\theta}}{R}\right) + \phi\left(R,r,\overline{\alpha},\overline{\beta}\right) r^n P\left(\frac{e^{i\theta}}{r}\right)\right| - \left(\frac{\left|R^n + \phi(R,r,\alpha,\beta)\, r^n\right| - \left|1+\phi(R,r,\alpha,\beta)\right|}{2}\right) m\right\} e^{i\gamma}\right.$$
$$\left. + \left\{ \left| P\left(Re^{i\theta}\right) + \phi(R,r,\alpha,\beta) P\left(re^{i\theta}\right)\right| + \left(\frac{\left|R^n + \phi(R,r,\alpha,\beta)\, r^n\right| - \left|1+\phi(R,r,\alpha,\beta)\right|}{2}\right) m\right\}\right|$$

$$\leq \left| \left| R^n P(e^{i\theta}/R) + \phi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P(e^{i\theta}/r) \right| e^{i\gamma} + \left| P\left(Re^{i\theta}\right) + \phi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) \right| \right|$$

This implies for each p>0,

$$\int_0^{2\pi} | F(\theta) + e^{i\gamma} G(\theta) |^p \, d\theta \leq \int_0^{2\pi} \Big| | R^n P(e^{i\theta}/R) + \varphi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P(e^{i\theta}/r) | e^{i\gamma} + | P\left(Re^{i\theta}\right) + \varphi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) | \Big|^p d\theta \tag{27}$$

$$F(\theta) = \left| P\left(Re^{i\theta}\right) + \varphi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) \right| + \frac{\left| R^n + \varphi(R, r, \alpha, \beta)\, r^n \right| - \left| 1 + \varphi(R, r, \alpha, \beta) \right|}{2} m$$

where

and $G(\theta) = \left| R^n P\left(\frac{e^{i\theta}}{R}\right) + \phi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P\left(\frac{e^{i\theta}}{r}\right) \right| - \frac{\left| R^n + \phi(R, r, \alpha, \beta)\, r^n \right| - \left| 1 + \phi(R, r, \alpha, \beta) \right|}{2} m$

Integrating both sides of (27) with respect to γ from 0 to 2π, we get with the help of lemma 4, for each $p > 0$, $R > r \geq 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and γ real,

$$\int_0^{2\pi}\int_0^{2\pi} \left| F(\theta) + e^{i\gamma} G(\theta) \right|^p d\theta\, d\gamma \leq \int_0^{2\pi}\int_0^{2\pi} \Big| | R^n P(e^{i\theta}/R) + \varphi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P(e^{i\theta}/r) | e^{i\gamma} + | P\left(Re^{i\theta}\right) + \varphi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) | \Big|^p d\theta\, d\gamma$$

$$\leq \int_0^{2\pi} \Big\{ \int_0^{2\pi} \Big| | R^n P(e^{i\theta}/R) + \varphi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P(e^{i\theta}/r) | e^{i\gamma} + | P\left(Re^{i\theta}\right) + \varphi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) | \Big|^p d\gamma \Big\} d\theta$$

$$\leq \int_0^{2\pi} \Big\{ \int_0^{2\pi} \Big| (R^n P(e^{i\theta}/R) + \varphi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P(e^{i\theta}/r)) e^{i\gamma} + P\left(Re^{i\theta}\right) + \varphi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) \Big|^p d\gamma \Big\} d\theta$$

$$\leq \int_0^{2\pi} \Big\{ \int_0^{2\pi} \Big| (R^n P(e^{i\theta}/R) + \varphi\left(R, r, \overline{\alpha}, \overline{\beta}\right) r^n P(e^{i\theta}/r)) e^{i\gamma} + P\left(Re^{i\theta}\right) + \varphi(R, r, \alpha, \beta) P\left(re^{i\theta}\right) \Big|^p d\theta \Big\} d\gamma$$

$$\leq \int_0^{2\pi} \left| R^n + \phi(R, r, \alpha, \beta)\, r^n + e^{i\gamma}\left(1 + \phi\left(R, r, \overline{\alpha}, \overline{\beta}\right)\right) \right|^p d\gamma \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta \tag{28}$$

Now for every real γ, $t \geq 1$ and $p > 0$, we have

$$\int_0^{2\pi} | t + e^{i\gamma} |^p \, d\gamma \geq \int_0^{2\pi} | 1 + e^{i\gamma} |^p \, d\gamma .$$

If $F(\theta) \neq 0$, we take $t = \frac{|G(\theta)|}{|F(\theta)|}$, then by (26), $t \geq 1$ and we get

$$\int_0^{2\pi} | F(\theta)+e^{i\gamma} G(\theta) |^p \, d\gamma = | F(e^{i\theta}) |^p \int_0^{2\pi} \left| 1+\frac{G(e^{i\theta})}{F(e^{i\theta})} e^{i\gamma} \right|^p d\gamma$$
$$= | F(e^{i\theta}) |^p \int_0^{2\pi} \left| \left| \frac{G(e^{i\theta})}{F(e^{i\theta})} \right| + e^{i\gamma} \right|^p d\gamma$$
$$\geq | F(e^{i\theta}) |^p \int_0^{2\pi} \left| 1+e^{i\gamma} \right|^p d\gamma.$$

For F (θ) = 0, this inequality is trivially true. Using this in (28), we conclude that for arbitrary real or complex numbers α, β, with |α|≤1, |β|≤1, R>r≥1 and 0 ≤ θ < 2π,

$$\int_0^{2\pi} |1+e^{i\gamma} |^p \, d\gamma \int_0^{2\pi} \{| P(Re^{i\theta})+\varphi(R,r,\alpha,\beta)P(re^{i\theta})+\frac{| R^n+\varphi(R,r,\alpha,\beta)r^n |-|1+\varphi(R,r,\alpha,\beta)}{2} m\}^p \, d\theta$$
$$\leq \left\{ \int_0^{2\pi} \left| R^n+\phi(R,\, r,\, \alpha,\, \beta)\, r^n + e^{i\gamma}\left(1+\phi\left(R,\, r,\, \overline{\alpha},\, \overline{\beta}\right)\right) \right|^p d\gamma \right\} \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p \, d\theta \right\}. \tag{29}$$

Since

$$\int_0^{2\pi} \left| R^n+\varphi(R,\, r,\, \alpha,\, \beta)\, r^n + e^{i\gamma}\left(1+\varphi\left(R,\, r,\, \overline{\alpha},\, \overline{\beta}\right)\right) \right|^p d\gamma = \int_0^{2\pi} \left| R^n+\varphi(R,\, r,\, \alpha,\, \beta)\, r^n + e^{i\gamma}\left(1+\varphi(R,\, r, \alpha,\, \beta\,)\right) \right|^p d\gamma,$$

From (29), we obtain

$$\int_0^{2\pi} \left\{ \left| P\left(Re^{i\theta}\right)+\phi(R,r,\alpha,\beta)P\left(r e^{i\theta}\right) \right| + \frac{\left| R^n+\phi(R,r,\alpha,\beta)\, r^n \right| - \left| 1+\phi(R,r,\alpha,\beta) \right|}{2} m \right\}^p d\theta$$
$$\leq \frac{\int_0^{2\pi} \left| \left(R^n+\varphi(R,\, r,\, \alpha,\, \beta)\, r^n\right) e^{i\gamma} + \left(1+\varphi(R,\, r,\, \alpha,\, \beta\,)\right) \right|^p d\gamma}{\int_0^{2\pi} \left| 1+e^{i\gamma} \right|^p d\gamma} \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p \, d\theta$$

.
This gives for every real or complex number δ with |δ| ≤1,

$$\left\| \left| P(Rz)+\varphi(R,r,\alpha,\beta)\, P(rz) \right| + \delta \left\{ \frac{\left| R^n+\varphi(R,r,\alpha,\beta\,)\, r^n \right| - \left| 1+\varphi(R,r,\alpha,\beta\,) \right|}{2} \right\} m \right\|_p \leq \frac{C_p}{\left\| 1+z \right\|_p} \left\| P(z) \right\|_p .$$

That proves the Theorem completely.

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