International Journal of Mathematical Archive-3(5), 2012, 1940-1956
Available online through www.ijma.info ISSN 2229 – 5046

NONDIFFERENTIABLE MULTIOBJECTIVE MIXED SYMMETRIC DUALITY FOR NONLINEAR PROGRAMMING INVOLVING (ϕ, ρ) -UNIVEX FUNCTION.

A. K. Tripathy[1*] & G. Devi[2]

[1]Department of Mathematics, Trident Academy of Technology, F2/A, Chandaka Industrial Estate, In front of Infocity, Patia, Bhubaneswar-751024, Odisha, India

[2]Department of Computer Science, Ajayabinayak Institute of Technology, Cuttack, Odisha, India

(Received on: 07-05-12; Accepted on: 31-05-12)

ABSTRACT

In this paper we have presented a pair of mixed symmetric dual for a class of nondifferentiable multiobjective programming involving square root term like $(x^TAx)^{1/2}$ .We established weak duality , strong duality and converse duality theorems with their proofs under (Φ, ρ)-univexity and (Φ, ρ)-pseudounivexity assumption. Also the self duality theorem with example is established. Discussion on some particular cases shows that our results generalize earlier results in related domain.

AMS 2002 Subject Classification: *90C29, 90C30, 90C46.*

Keywords- *(Φ, ρ)-univexity; (Φ, ρ)-pseudo-univexity; mixed symmetric dual, Schwartz inequality, quadratic term, efficient solution; self duality.*

---

1. INTRODUCTION

Duality theory has played an important role in the development of optimization theory. For nonlinear programming problems a number of duals have been suggested, among which the Wolfe dual proposed by Dorn [12] is well known. Subsequently Dantzing et al [13] and Bazarra et al [5] established symmetric duality results for convex/concave functions. Devi [11], Weir and Mond [34], Mond and Schechter [25] studied non differentiable symmetric duality for a class of optimization problem in which the objective function consist of support function. Husain et al [15] have formulated a pair of Mond Weir type second order symmetric dual and establish the duality results under pseudo convexity –pseudo concavity assumption.

In recent years, several extension and generalization have been considered for classical convexity. A significant generalization of convex function is that of in-vex function introduced by Hanson [14]. Bector et al [6] have introduced the concept of pre-univex function, univex functions and pseudo-univex function as a generalization of in-vex function. Further development on the application of univex function and generalized univex function can be found in Rueda et al [28], Mishra [22] and Mishra et al [23], [24]. Ojha[27] has established symmetric duality results for (Φ, ρ)-univex function and Thakur et al [32] have established second order symmetric duality results for second order (Φ, ρ)-univex function .Earlier Chandra et al [8] had formulated mixed symmetric duality for a class of nonlinear programming problems. Yang et al [33] have discussed a mixed symmetric duality for a class of nondifferentiable nonlinear programming problems. Later on Ahmad [4] has formulated mixed type symmetric dual in multiobjective programming problem ignoring nonnegative restriction of Bector et al [7].Very recently Mishra et al [20] and Mishra [21] have presented a mixed symmetric first and second order duality in nondifferentiable mathematical programming problem under F-convexity. Li et al [18] and Agarwal et al [1] introduced a model of mixed symmetric duality for a class of non differentiable multiobjective programming problem with multiple arguments.

In motivation of Bector et al.[7], Ahmed [4], Ahmed et al[3], Mishra[21], Ojha[27], Thakur et al [32], Agarwal et al [1] and Li et al [18], we formulated a pair of multiobjective mixed symmetric dual program using square root term and established weak and strong duality and the converse duality theorems under (Φ, ρ)-univexity and (Φ, ρ)-pseudo-univexity condition. These results generalized the known works of Thakur et al [32], Mishra et al [20], Mishra [21], Agarwal et al [1], Ojha [27], Suneja et al [27], Chandra et al [8], Ahmad et al [4] and Wier and Mond l [30].

Corresponding author: A. K. Tripathy[1*]
[1]Department of Mathematics, Trident Academy of Technology, F2/A, Chandaka Industrial Estate, In front of Infocity, Patia, Bhubaneswar-751024, Odisha, India

2.NOTATION AND PRELIMINARIES

Let R^n be n-dimensional Euclidean space and R_+^n its nonnegative orthant .The following conventions for vectors $x, y \in R^n$ will be followed throughout this paper: $x < y \Leftrightarrow x_i < y_i,\ i = 1,2,....n.$ $x \le y \Leftrightarrow x_i \le y_i$. For any vector we denote $x^T y = \sum_{i=1}^{n} x_i y_i$.

Let X and Y are open subset of R^n and R^m respectively. Let $f_i(x, y)$ be a real valued twice differentiable function defined on $X \times Y$. Let $\nabla_1 f_i(x, y)$ and $\nabla_2 f_i(x, y)$ denote the gradient vectors of $f_i(x, y)$ with respect to first variable x and second variable y respectively. Also let $\nabla_1^2 f_i(x, y)$ and $\nabla_2^2 f_i(x, y)$ denote the Hessian matrix of $f_i(x, y)$ with respect to the first variable x and second variable y respectively.

Let ϕ be a real valued function defined on $X \times X \times R^{n+1}$ such that $\phi(x,u,*)$ is convex on R^{n+1} and $\phi(x,u,(0,r)) \ge 0$ for every $(x,u) \in X \times X$ and $r \in R$. Also let $b : X \times X \to R_+$ and $\psi : R \to R$ satisfying $\psi(u) \le 0 \Rightarrow u \le 0$ and $\psi(-a) = -\psi(a)$.

Let $C \subset R^n$ be a compact convex set .The support function of C is defined by $s(x \mid C) = \max\{x^T y \mid y \in C\}$.
Consider the following multiobjective programming problem (MP):

P :(**Primal**) Minimize $f(x) = (f_1(x), f_2(x), ..., f_r(x))$
Subject to $h(x) \le 0, x \in X \subseteq R^n$, where $f : X \to R^r, g : X \to R^r$

Let X_0 be the set of all feasible solutions of problem (P); that i.e. $X_0 = \{x \in X \mid h(x) \le 0\}$.

Definition 2.1.A vector $\bar{x} \in X_0$ is said to be an efficient solution of problem (P) if there exists no $x \in X_0$ such that $f(x) \le f(\bar{x}), f(x) \ne f(\bar{x})$.

Definition 2.2.A vector $\bar{x} \in X_0$ is said to be a weakly efficient solution of problem (P) if there exists no $x \in X_0$ such that $f(x) < f(\bar{x})$.

Definition 2.3.A vector $\bar{x} \in X_0$ is said to be a properly efficient of problem (P) if it is efficient and there exists a positive constant M such that whenever $f_i(x) < f_i(\bar{x})$ for $x \in X_0$ and for $i \in \{1, 2, ...r\}$, there exist at least one $j \in \{1,2,...p\}$ such that $f_j(\bar{x}) < f_j(x)$ and $f_i(\bar{x}) - f_i(x) \le M(f_j(x) - f_j(\bar{x}))$.

Definition 2.4: A real-valued twice differentiable function $f_i(*, y) : X \times Y \to R$ is said to be (Φ, ρ)-univex at $u \in X$ with respect to $q \in R^n$ if for all $b : X \times X \to R_+$, $\phi : X \times X \times R^{n+1} \to R$, and $\psi : R \to R$ with ρ_i is a real number, we have $b(x,u)[\psi\{f_i(x, y) - f_i(u, y)\}] \ge \varphi(x,u;(\nabla_1 f_i(u, y)))$ (2.1)

Definition 2.5: A real-valued twice differentiable function $f_i(x,*) : X \times Y \to R$ is said to be (Φ, ρ)-univex at $y \in Y$ with respect to $p \in R^m$ if for all $b : Y \times Y \to R_+$, $\phi : Y \times Y \times R^{m+1} \to R$ and $\psi : R \to R$ with ρ_i is a real number, we have $b(v, y)[\psi\{f_i(x,v) - f_i(x, y)\}] \ge \varphi(v, y;(\nabla_2 f_i(x, y), \rho_i))$ (2.2)

Definition 2.6: A real-valued twice differentiable function $f_i(*, y) : X \times Y \to R$ is said to be (Φ- ρ)-pseudounivex at $u \in X$ with respect to $q \in R^n$ if for all $b : X \times X \to R_+$, $\phi : X \times X \times R^{n+1} \to R$ and $\psi : R \to R$ with ρ_i is a real number ,we have $\varphi(x,u;(\nabla_1 f_i(u, y), \rho_i)) \ge 0 \Rightarrow b(x,u)[\psi\{f_i(x, y) - f_i(u, y)\}] \ge 0$ (2.3)

Definition 2.7: A real-valued twice differentiable function $f_i(x,*) : X \times Y \to R$ is said to be (Φ, ρ)-pseudounivex at $y \in X$ with respect to $p \in R^m$ if for all $b : Y \times Y \to R_+$, $\phi : Y \times Y \times R^{m+1} \to R$, and $\psi : R \to R$ with ρ_i is a real number we have $\varphi(v, y; (\nabla_2 f_i(x, y), \rho_i)) \geq 0 \Rightarrow b(v, y)[\psi\{f_i(x, v) - f_i(x, y)\}] \geq 0$ (2.4)

Definition 2.8: A real valued twice differentiable function f is (ϕ, ρ) - unicave and (ϕ, ρ) -pseudounicave if $-f$ is (ϕ, ρ) -univex and (ϕ, ρ) pseudounivex respectively.

Example 2.1: We present here a function which is $(\phi, \rho) - univex$ function .We can proceed similarly for the other classes of function introduced.

Let $X = (0, \infty)$, $f : X \to R$ defined by $f(x) = x \log x$ .Obviously f is non convex.

We have $\nabla f(u) = 1 + \log x$.

Let $\phi : X \times X \times R \to R$ defined by $\phi(x, y; (a, \rho)) = a\rho$,

Let $b(x,u) = \begin{cases} \frac{1+\log u}{x\log x - u\log u} & if \quad (x\log x - u\log u) > 0 \\ 0 & if \quad x\log x - u\log u \leq 0 \end{cases}$, and $\psi(x) = 3x$

We have $\phi(x, u; (\nabla f(u), \rho)) = \rho(1 + \log u)$. So the definition of $(\phi, \rho) - univex$ function becomes

$$\frac{1+\log u}{x\log x - u\log u} \times 3(x\log x - u\log u) \geq \rho(1 + \log u) \Rightarrow \rho \leq 3$$

It now follows that for $\rho \leq 3$, the function $f(x) = x \log x$ becomes $(\phi, \rho) - univex$ function with respect to b, ψ .

Remark 1:

(i) If we consider the case $b \equiv 1$, $\varphi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ with F is sub linear in third argument, then the above definition reduce to F-convexity and F-pseudo convexity introduced by Mishra [21].

(ii) If $b \equiv 1$, $\varphi(x, u; (\nabla f(u), \rho)) = \eta(x, u)^T \nabla f(u))$, where $\eta : X \times X \to R^n$, the above definition reduces to η-bonvex function given by Devi [11].

Definition 2.9: (Schwartz Inequality)

Let $x, y \in R^n$ and $A \in R^n \times R^n$ be a positive semi definite matrix, then $x^T Ay \leq (x^T Ax)^{\frac{1}{2}} (y^T Ay)^{\frac{1}{2}}$, equality holds if for some $\lambda \geq 0, Ax = \lambda Ay.$

3. MIXED SYMMETRIC DUAL PROGRAM

For N={1,2,…n} and M={1,2,….m},let $J_1 \subseteq N$ and $J_2 = N \setminus J_1$, $K_1 \subseteq M$ and $K_2 = M \setminus K_1$ .Let $|J_1|$ denote the number of elements in J_1 . The numbers $|J_2|, |K_1|, |K_2|$ are defined similarly. Notice that, if $J_1 = \phi$, then $J_2 = N$ that is $|J_1| = 0$, and $|J_2| = n$. It is clear that any $x \in X \subseteq R^n$ can be written as, $x = (x^1, x^2)$, $x^1 \in R^{|J_1|}$ and $x^2 \in R^{|J_2|}$ .Similarly for any $y \in Y \subseteq R^m$, $y = (y^1, y^2)$, $y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}$. Let $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ are twice continuously differentiable functions.

Now we formulate the following pair of multiobjective mixed symmetric dual programs and prove duality theorems under some mild assumptions of (ϕ, ρ) -univexity and (ϕ, ρ) -pseudo univexity.

(SMSP): Mixed Symmetric Primal:

$$H(x,y,w) = Minimize\left\{H_1(x,y,w), H_2(x,y,w), \ldots\ldots, H_r(x,y,w)\right\}$$

Subject to $\sum_{i=1}^{r} \lambda_i [\nabla_2 f_i(x^1, y^1) - C_i^1 w_i^1] \leq 0$, (3.1)

$$\sum_{i=1}^{r} \lambda_i [\nabla_2 g_i(x^2, y^2) - C_i^2 w_i^2] \leq 0, \tag{3.2}$$

$$(y^1)^T \sum_{i=1}^{r} \lambda_i \left\{\nabla_2 f_i(x^1, y^1) - C_i^1 w_i^1\right\} \geq 0, \tag{3.3}$$

$$(y^2)^T \sum_{i=1}^{r} \lambda_i \left\{\nabla_2 g_i(x^2, y^2) - C_i^2 w_i^2\right\} \geq 0 \tag{3.4}$$

$$(x^1, x^2) \geq 0, \tag{3.5}$$

$$(w_i^1)^T C_i^1 w_i^1 \leq 1, i = 1, 2, \ldots r. \tag{3.6}$$

$$(w_i^2)^T C_i^2 w_i^2 \leq 1, i = 1, 2, \ldots r. \tag{3.7}$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1 \tag{3.8}$$

(SMSD): Mixed Symmetric Dual:

$$G(u,v,a,) = Maximize\left\{G_1(u,v,a), G_2(u,v,a), \ldots., G_r(u,v,a)\right\}$$

Subject to $\sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(u^1, v^1) + E_i^1 a_i^1] \geq 0,$ (3.9)

$$\sum_{i=1}^{r} \lambda_i [\nabla_1 g_i(u^2, v^2) + E_i^2 a_i^2] \geq 0, \tag{3.10}$$

$$(u^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(u^1, v^1) + E_i^1 a_i^2] \leq 0, \tag{3.11}$$

$$(u^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_1 g_i(u^2, v^2) + E_i^2 a_i^2] \leq 0, \tag{3.12}$$

$$(v^1, v^2) \geq 0, \tag{3.13}$$

$$(a_i^1)^T E_i^1 a_i^1 \leq 1, \quad i = 1,2,\ldots..r \tag{3.14}$$

$$(a_i^2)^T E_i^2 a_i^2 \leq 1, i = 1,2,\ldots..r \tag{3.15}$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1 . \tag{3.16}$$

Where

$$H_i(x,y,w) = \left\{ f_i(x^1, y^1) + g_i(x^2, y^2) + ((x^1)^T E_i^1 x^1)^{\frac{1}{2}} + ((x^2)^T E_i^2 x^2)^{\frac{1}{2}} - (y^1)^T C_i^1 w_i^1 - (y^2)^T C_i^2 w_i^2 \right\}$$

$$G_i(u,v,a) = \left\{ f_i(u^1, v^1) + g_i(u^2, v^2) - ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} - ((v^2)^T C_i^2 v^2)^{\frac{1}{2}} + (u^1)^T E_i^1 a_i^1 + (u^2)^T E_i^2 a_i^2 \right\}$$

$$\lambda_i \in R, w = (w^1, w^2), w_i^1 \in R^{|K_1|}, w_i^2 \in R^{|K_2|}, w^1 = (w_1^1, w_2^1, \ldots w_r^1), w^2 = (w_1^2, w_2^2, \ldots w_r^2),$$

$$a = (a^1, a^2), a^1 = (a_1^1, a_2^1, \ldots a_r^1), a^2 = (a_1^2, a_2^2, \ldots a_r^2), a_i^1 \in R^{|J_1|}, a_i^2 \in R^{|J_2|}$$

And $E_i^1, E_i^2, C_i^1, C_i^2, i = 1, 2, \ldots r.$ are positive semi definite matrices of order $|J_1|, |J_2|, |K_1|$ *and* $|K_2|$ respectively.

Remark 3.1

Since the objective function of (MSP) and (MSD) contains the square root terms like $(x^T Ax)^{\frac{1}{2}}$, these problems are nondifferentiable programming problems.

For the following theorems let assume ρ is a real number and ϕ_0^i, ϕ_1^i are a real valued function defined on $R^{|J_i|} \times R^{|J_i|} \times R^{|J_i|+1}$ and $R^{|K_i|} \times R^{|K_i|} \times R^{|K_i|+1}$ respectively such that $\phi_0^i(x^i, u^i, (0,r)) \ge 0$,for every $(x^i, u^i) \in R^{|J_i|} \times R^{|J_i|}$. $i = 1,2$ and $\phi_1^i(y^i, v^i, (0,r)) \ge 0$ for $(y^i, v^i) \in R^{|K_i|} \times R^{|K_i|}$ and $r \in R_+$. Let b_0^i, b_1^i be non negative real valued function defined on $R^{|J_i|} \times R^{|J_i|}$ and $R^{|K_i|} \times R^{|K_i|}$ respectable for $i = 1,2$

We assume that $\psi_0^i, \psi_1^i : R \to R$ satisfying $\psi_0^i(u) \le 0 \Rightarrow u \le 0$ and $\psi_1^i(u) \le 0 \Rightarrow u \le 0$ and assume that ψ_0^i, ψ_1^i are odd function, for $i = 1,2$.

Theorem 3.1**(Weak duality)**
Let (x, y, λ, w, p) be feasible solution of (SMSP) and (u, v, λ, a, q) be feasible solution (MSD) and

(I) $\sum_{i=1}^{r} \lambda_i [f_i(*, v^1) + (*)^T E_i^1 a_i^1]$ is (ϕ_0^1, ρ) -univex at u^1 for fixed v^1,

(II) $\sum_{i=1}^{r} \lambda_i [f_i(x^1, *) - (*)^T C_i^1 w_i^1]$ is (ϕ_1^1, ρ) -unicave at y^1 for fixed x^1,

(III) $\sum_{i=1}^{r} \lambda_i [g_i(*, v^2) + (*)^T E_i^2 a_i^2]$ is (ϕ_o^2, ρ) -pseudo univex at u^2 for fixed v^2,

(IV) $\sum_{i=1}^{r} \lambda_i [g_i(x^2, *) - (*)^T C_i^2 w_i^2]$ is (ϕ_1^2, ρ) -pseudo unicave at y^2 for fixed x^2,

(V) $\phi_0^1(x^1, u^1; (\xi^1, \rho)) + (u^1)^T \xi^1 \ge 0$, where $\xi^1 = \sum_{i=1}^{r} \lambda_i (\nabla_1 f_i(u^1, v^1) + E_i^1 a_i^1)$,

(VI) $\phi_0^2(x^2, u^2; (\xi^2, \rho)) + (u^2)^T \xi^2 \ge 0$, where $\xi^2 = \sum_{i=1}^{r} \lambda_i (\nabla_1 g_i(u^2, v^2) + E_i^2 a_i^2)$.

(VII) $\phi_1^1(v^1, y^1; (\varsigma^1, \rho)) + (y^1)^T \varsigma^1 \le 0$, where $\varsigma^1 = \sum_{i=1}^{r} \lambda_i (\nabla_2 f_i(x^1, y^1) - C_i^1 w_i^1)$,

(VIII) $\phi_1^2(v^2, y^2; (\varsigma^2, \rho)) + (y^2)^T \varsigma^2 \le 0$, where $\varsigma^2 = \sum_{i=1}^{r} \lambda_i (\nabla_2 g_i(x^2, y^2) - C_i^2 w_i^2)$.

Then $H(x, y, \lambda, a) \ge G(u, v, \lambda, w)$

Proof: Since $\sum_{i=1}^{r} \lambda_i [f_i(*, v^1) + (*)^T E_i^1 a_i^1]$ is (ϕ_0, ρ)-univex at u^1 for fixed v^1 and $\lambda > 0$,

Then for $b_0^1 : R^{|J_1|} \times R^{|J_1|} \to R_+, \phi_0^1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|+1} \to R$, $\psi_o^1 : R \to R$ and $\rho \in R$ we have

$$b_0^1(x^1, u^1) \psi_0^1 \{ \sum_{i=1}^{r} \lambda_i [f_i(x^1, v^1) + (x^1)^T E_i^1 a_i^1 - f_i(u^1, v^1) - (u^1)^T E_i^1 a_i^1] \}$$

$$\ge \phi_0^1(x^1, u^1; (\sum_{i=1}^{r} (\lambda_i [\nabla_1 f_i(u^1, v^1) + E_i^1 a_i^1], \rho)), \text{ for } i = 1,2, \ldots r . \tag{3.17}$$

Using (3.11) in hypothesis (V) we get

$$\phi_0^1(x^1, u^1; (\sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(u^1, v^1) + E_i^1 a_i^1], \rho) \ge 0 \tag{3.18}$$

So from (3.17) and (3.18) with the property of b_0^1 and ψ_0^1 we obtain

$$\sum_{i=1}^{r}\lambda_i[f_i(x^1,v^1)+(x^1)^T E_i^1 a_i^1] \geq \sum_{i=1}^{r}\lambda_i[f_i(u^1,v^1)+(u^1)^T E_i^1 a_i^1]\,. \tag{3.19}$$

Now it follows from the (ϕ_1^1,ρ) -unicavity of $\sum_{i=1}^{r}\lambda_i[f_i(x^1,*)-(*)^T C_i^1 w_i^1]$ at y^2 for fixed x^2, for $\lambda>0$, $b_1^1: R^{|K_1|}\times R^{|K_1|}\to R_+, \phi_1^1: R^{|K_1|}\times R^{|K_1|}\times R^{|K_1|+1}\to R$, $\psi_1^1: R\to R$ and $\rho\in R$ that

$$b_1^1(y^1,v^1)\psi_1^1\{\sum_{i=1}^{r}\lambda_i[f_i(x^1,v^1)-(v^1)^T C_i^1 w_i^1 - f_i(x^1,y^1)+(y^1)^T C_i^1 w_i^1]\} \leq \phi_1^1(v^1,y^1;\sum_{i=1}^{r}(\lambda_i[\nabla_2 f_i(x^1,y^1)-C_i^1 w_i^1],\rho)) \tag{3.20}$$

Hypothesis (VII) in light of (3.3) implies

$$\phi_1^1(v^1,y^1;(\sum_{i=1}^{r}\lambda_i[\nabla_2 f_i(x^1,y^1)-C_i^1 w_i^1],\rho)) \leq 0\,. \tag{3.21}$$

So from (3.20), (3.21) and with the property of b_1^1 and ψ_1^1 we get

$$\sum_{i=1}^{r}\lambda_i[f_i(x^1,v^1)-(v^1)^T C_i^1 w_i^1] \leq \sum_{i=1}^{r}[f_i(x^1,y^1)-(y^1)^T C_i^1 w_i^1]\,. \tag{3.22}$$

Subtracting (3.22) from (3.19) we get

$$\sum_{i=1}^{r}\lambda_i[(x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1] \geq \sum_{i=1}^{r}\lambda_i\left\{[f_i(u^1,v^1)+(u^1)^T E_i^1 a_i^1]-[f_i(x^1,y^1)-(y^1)^T C_i^1 w_i^1]\right\}\,. \tag{3.23}$$

Similarly hypothesis (VI) in light of (3.12) implies

$$\phi_0^2(x^2,u^2;(\sum_{i=1}^{r}\lambda_i[\nabla_1 g_i(u^2,v^2)+E_i^2 a_i^2],\rho)) \geq 0\,. \tag{3.24}$$

So the (ϕ_o^2,ρ) -pseudo univexity of $\sum_{i=1}^{r}\lambda_i[g_i(*,v^2)+(*)^T E_i^2 a_i^2]$ at u^2 for fixed v^2 and for $\lambda>0$, $b_0^2: R^{|J_2|}\times R^{|J_2|}\to R_+, \phi_0^2: R^{|J_2|}\times R^{|J_2|}\times R^{|J_2|+1}\to R$, $\psi_o^2: R\to R$ and $\rho\in R$, implies

$$b_0^2(x^2,u^2)\psi_0^2\left(\sum_{i=1}^{r}\lambda_i\left[g_i(x^2,v^2)+(x^2)^T E_i^2 a_i^2 - g_i(u^2,v^2)-(u^2)^T E_i^2 a_i^2\right]\right) \geq 0$$

and by the properties of b_0^2 and $\psi_0{}^2$, we get

$$\sum_{i=1}^{r}\lambda_i[g_i(x^2,v^2)+(x^2)^T E_i^1 a_i^1] \geq \sum_{i=1}^{r}\lambda_i[g_i(u^2,v^2)+(u^2)^T E_i^2 a_i^2]\,. \tag{3.25}$$

Similarly from (3.4) and from the hypothesis (iv), (viii) with the property of $b_1{}^2$ and $\psi_1{}^2$, we get

$$\sum_{i=1}^{r}\lambda_i[g_i(x^2,v^2)-(v^2)^T C_i^2 w_i^2] \leq \sum_{i=1}^{r}\lambda_i[g_i(x^2,y^2)-(y^2)^T C_i^2 w_i^2]\,. \tag{3.26}$$

Subtracting (3.26) from (3.25) we get

$$\sum_{i=1}^{r}\lambda_i[(x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2] \geq \sum_{i=1}^{r}\lambda_i\left\{[g_i(u^2,v^2)+(u^2)^T E_i^2 a_i^2]-[g_i(x^2,y^2)-(y^2)^T C_i^2 w_i^2]\right\} \tag{3.27}$$

Adding (3.23) and (3.27), we obtain

$$\begin{aligned}\sum_{i=1}^{r}\lambda_i[(x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1 + (x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2]\\ \geq \sum_{i=1}^{r}\lambda_i\{[f_i(u^1,v^1)+(u^1)^T E_i^1 a_i^1 - [f_i(x^1,y^1)-(y^1)^T C_i^1 w_i^1]+[g_i(u^2,v^2)\\ +(u^2)^T E_i^2 a_i^2]-[g_i(x^2,y^2)-(y^2)^T C_i^2 w_i^2]\}\,.\end{aligned} \tag{3.28}$$

From the Definition 2.9 (Schwartz inequality), we have

$$(x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1 + (x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2 \le ((x^1)^T E_i^1 x^1)^{\frac{1}{2}}((a_i^1)^T E_i^1 a_i^1)^{\frac{1}{2}}$$
$$+((v^1)^T C_i^1 v^1)^{\frac{1}{2}}((w_i^1)^T C_i^1 w_i^1)^{\frac{1}{2}} + ((x^2)^T E_i^2 x^2)^{\frac{1}{2}}((a_i^2)^T E_i^2 a_i^2)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}}((w_i^2)^T C_i^2 w_i^2)^{\frac{1}{2}} \quad (3.29)$$

Using (3.6), (3.7), (3.14) and (3.15) in (3.29) we obtain

$$(x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1 + (x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2$$
$$\le ((x^1)^T E_i^1 x^1)^{\frac{1}{2}} + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} + ((x^2)^T E_i^2 x^2)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}} \;. \quad (3.30)$$

Using (3.30) in (3.28), we get

$$((x^1)^T E_i^1 x^1)^{\frac{1}{2}} + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} + ((x^2)^T E_i^2 x^2)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}}$$
$$\ge \sum_{i=1}^r \lambda_i \{[f_i(u^1, v^1) + (u^1)^T E_i^1 a_i^1] - [f_i(x^1, y^1) - (y^1)^T C_i^1 w_i^1] + [g_i(u^2, v^2)$$
$$+ (u^2)^T E_i^2 a_i^2 - [g_i(x^2, y^2) - (y^2)^T C_i^2 w_i^2]$$

$$\Rightarrow \sum_{i=1}^r \lambda_i \{ f_i(x^1, y^1) + g_i(x^2, y^2) + ((x^1) E_i^1 x^1)^{\frac{1}{2}} + ((x^2) E_i^2 x^2)^{\frac{1}{2}} - (y^1)^T C_i^1 w_i^1 - (y^2)^T C_i^2 w_i^2$$
$$\ge \sum_{i=1}^r \lambda_i \{ f_i(u^1, v^1) + g_i(u^2, v^2) - ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} - ((v^2) C_i^2 v^2)^{\frac{1}{2}} + (u^1)^T E_i^1 a_i^1 - (u^2)^T E_i^2 a_i^2 \}$$

$$\Rightarrow \sum_{i=1}^r \lambda_i H_i(x, y, a) \ge \sum_{i=1}^r \lambda_i G_i(u, v, w)$$

That is $H(x, y, \lambda, a) \ge G(u, v, \lambda, w)$ □

Theorem 3.2 (Strong Duality): $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ be thrice differentiable function and let $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{w}_i^1, \hat{w}_i^2)$ be a weak efficient solution for (SMSP) .Let $\hat{\lambda} = \lambda$ be fixed in (MSD).Assume that (i) the matrix $\sum_{i=1}^r \lambda_i (\nabla_2^2 f_i)$ and $\sum_{i=1}^r \lambda_i (\nabla_2^2 g_i)$ are positive definite or negative definite and (ii) the set $\{\nabla_2 f_1 - C_1^1 \hat{w}_1^1, \ldots\ldots\ldots\ldots \nabla_2 f_r - C_r^1 \hat{w}_r^1\}$ and $\{\nabla_2 g_1 - C_1^2 \hat{w}_1^2, \ldots\ldots\ldots\ldots \nabla_2 g_r - C_r^2 \hat{w}_r^2 .\}$ are linearly independent. Then there exist $\hat{a}_i^1 \in R^{|J_1|}, \hat{a}_i^2 \in R^{|J_2|}$ such that $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{a}_i^1, \hat{a}_i^2)$ is feasible solution of (SMSD) and the two objective values are equal. Also if the hypothesis of Theorem 3.1 are satisfied for all feasible solution of (SMSP) and (SMSD), then $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{a}_i^1, \hat{a}_i^2)$ is properly efficient solution of (SMSD).

Proof: Since $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{w}_i^1, \hat{w}_i^2)$ is a weak efficient solution of (SMSP) by Fritz-John constraint condition (Mangasarian, (1969)) there exist

$\alpha \in R^r, \tau \in R^r, \delta \in R^r, \gamma \in R, \beta^1 \in R^{|K_1|}, \beta^2 \in R^{|K_2|}, \xi^1 \in R^{|J_1|}, \xi^2 \in R^{|J_2|}$

Such that
$$\sum_{i=1}^r \alpha_i [\nabla_1 f_i + E_i^1 \hat{a}_i^1] + \sum_{i=1}^r \lambda_i [\nabla_1 (\nabla_2 f_i)(\beta^1 - \gamma \hat{y}^1)] = \xi^1 \quad (3.31)$$

$$\sum_{i=1}^r \alpha_i [\nabla_1 g_i + E_i^2 \hat{a}_i^2] + \sum_{i=1}^r \lambda_i [\nabla_1 (\nabla_2 g_i)(\beta^2 - \gamma \hat{y}^2)] , \quad (3.32)$$

$$\sum_{i=1}^r (\alpha_i - \gamma \hat{\lambda}_i)[\nabla_2 f_i - C_i^1 \hat{w}_i^1]) + \sum_{i=1}^r [(\nabla_2^2 f_i)][(\beta^1 - \gamma \hat{y}^1) \hat{\lambda}_i] = 0, \quad (3.33)$$

$$\sum_{i=1}^r (\alpha_i - \gamma \hat{\lambda}_i)[\nabla_2 g_i - C_i^2 \hat{w}_i^2] + \sum_{i=1}^r [(\nabla_2^2 g_i)][(\beta^2 - \gamma \hat{y}^2) \hat{\lambda}_i] = 0, \quad (3.34)$$

$(\beta^1 - \gamma\hat{y}^1)^T [\nabla_2 f_i - C_i^1 \hat{w}_i^1] - \delta_i^1 = 0$, (3.35)

$(\beta^2 - \gamma\hat{y}^2)^T [\nabla_2 g_i - C_i^2 \hat{w}_i^2] - \delta_i^2 = 0$, (3.36)

$[(\beta^1 - \gamma\hat{y}^1)\hat{\lambda}_i]^T \nabla_2^2 f_i = 0, i = 1,2,...r$, (3.37)

$[(\beta^2 - \gamma\hat{y}^2)\hat{\lambda}_i]^T \nabla_2^2 g_i = 0, i = 1,2,...r$ (3.38)

$\alpha_i C_i^1 \hat{y}^1 + (\beta^1 - \gamma\hat{y}^1)\lambda_i C_i^1 = 2\tau_i C_i^1 \hat{w}_i^1$, (3.39)

$\alpha_i C_i^2 \hat{y}^2 + (\beta^2 - \gamma y^2)^T \lambda_i C_i^2 = 2\tau_i C_i^2 \hat{w}_i^2$, (3.40)

$(\hat{x}^1)^T E_i^1 \hat{w}_i^1 = ((\hat{x}^1)^T E_i^1 \hat{x}^1)^{\frac{1}{2}}$, (3.41)

$(\hat{x}^2)^T E_i^2 \hat{w}_i^2 = ((\hat{x}^2)^T E_i^2 \hat{x}^2)^{\frac{1}{2}}$, (3.42)

$(\beta^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_2 f_i - C_i^1 \hat{w}_i^1] = 0$, (3.43)

$(\beta^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_2 g_i - C_i^2 \hat{w}_i^2] = 0$, (3.44)

$(\gamma\hat{y}^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_2 f_i - C_i^1 \hat{w}_i^1] = 0$, (3.45)

$((\gamma\hat{y}^2)^T) \sum_{i=1}^{r} \lambda_i [\nabla_2 g_i - C_i^2 \hat{w}_i^2] = 0$,
(3.46)

$\tau_i [(\hat{w}_i^1)^T C_i^1 \hat{w}_i^1 - 1] = 0, i = 1,2,...r$,
(3.47)

$\tau_i [(\hat{w}_i^2)^T C_i^2 \hat{w}_i^2 - 1] = 0, i = 1,2,...r$, (3.48)

$\delta^{1^T} \overline{\lambda} = 0,$ (3.49)

$\delta^{2^T} \hat{\lambda} = 0$, (3.50)

$\hat{x}^1 \xi^1 = 0$, (3.51)

$\hat{x}^2 \xi^2 = 0$, (3.52)

$\hat{w}_i^{1^T} E_i^1 \hat{w}_i^1 \le 1$, (3.53)

$\hat{w}_i^{2^T} E_i^2 \hat{w}_i^2 \le 1,$ (3.54)

$(\alpha, \beta, \gamma, \tau, \delta, \xi) \ge 0$, (3.55)

and $(\alpha, \beta, \gamma, \tau, \delta, \xi) \ne 0$. (3.56)

Since $\hat{\lambda} > 0$, from (3.49) and (3.50), we get $\delta^1 = 0, \delta^2 = 0$. Consequently (3.35) and (3.36) implies

$(\beta^1 - \gamma\hat{y}^1)^T [\nabla_2 f_i - C_i^1 \hat{w}_i^1] = 0$ (3.57)

and $(\beta^2 - \gamma\hat{y}^2)^T [\nabla_2 g_i - C_i^2 \hat{w}_i^2] = 0$. (3.58)

Pre-multiplying (3.33) by $(\beta^1 - \gamma\hat{y}^1)^T$ and then using (3.57) we get

$(\beta^1 - \gamma\hat{y}^1)^T \sum_{i=1}^{r} \hat{\lambda}_i (\nabla_2^2 f_i)(\beta^1 - \gamma\hat{y}^1) = 0.$ (3.59)

Again pre- multiplying (3.34) by $(\beta^2 - \gamma\hat{y}^2)^T$ and then using (3.58) we get

$(\beta^2 - \gamma\hat{y}^2)^T \sum_{i=1}^{r} \hat{\lambda}_i (\nabla_2^2 g_i)(\beta^2 - \gamma\hat{y}^2) = 0.$ (3.60)

Using the hypothesis (i) in (3.59) and (3.60) we get

$$\beta^1 = \gamma\hat{y}^1 \tag{3.61}$$

and $\beta^2 = \gamma\hat{y}^2$ (3.62)

Therefore using (3.61) and (3.62) in (3.33) and (3.34), we get

$$\sum_{i=1}^{r}(\alpha_i - \gamma\hat{\lambda}_i)[\nabla_2 f_i - C_i^1\hat{w}_i^1] = 0 \tag{3.63}$$

and $\sum_{i=1}^{r}(\alpha_i - \gamma\hat{\lambda}_i)[\nabla_2 g_i - C_i^2\hat{w}_i^2] = 0$ (3.64)

Using hypothesis (ii) in (3. 63) and (3.64) we get $\alpha_i = \gamma\hat{\lambda}_i$, $i = 1,2,.....r.$ (3.65)

If $\gamma = 0,$ then $\alpha_i = 0$, $i = 1,2,.....r.$ and (3.61) and (3.62) implies $\beta^1 = \beta^2 = 0$

Therefore (3.31) and (3.32) implies $\xi^1 = \xi^2 = 0$ and (3.39), (3.40) implies $\tau_i = 0$, $i = 1,2,.....r.$ Thus $(\alpha,\beta,\gamma,\tau,\delta,\xi) = 0$ contradicting (3.56).

Hence $\gamma > 0$ (3.66)

Since $\hat{\lambda}_i > 0$, $i = 1,2,....r$ (3.65) implies $\alpha_i > 0$, $i = 1,2,....r$. (3.67)

Again using (3.61), (3.62) in (3.31) and (3.32) it gives $\sum_{i=1}^{r}\alpha_i[\nabla_1 f_i + E_i^1\hat{a}_i^1] = \xi^1$ and $\sum_{i=1}^{r}\alpha_i[\nabla_1 g_i + E_i^2\hat{a}_i^2] = \xi^2$ which by (3.63) and (3. 55) gives

$$\sum_{i=1}^{r}\lambda_i[\nabla_1 f_i + E_i^1\hat{a}_i^1] = \frac{\xi^1}{\gamma} \geq 0 \quad , \tag{3.68}$$

and $\sum_{i=1}^{r}\lambda_i[\nabla_1 g_i + E_i^2\hat{a}_i^2] = \dfrac{\xi^2}{\gamma} \geq 0$. (3.69)

Now $(\hat{x}^1)^T\sum_{i=1}^{r}\lambda_i[\nabla_1 f_i + E_i^1\hat{a}_i^1] = \dfrac{\hat{x}^{1^T}\xi^1}{\gamma} = 0$, (using (3.51)) (3.70)

and $(\hat{x}^2)^T\sum_{i=1}^{r}\lambda_i[\nabla_1 g_i + E_i^2\hat{a}_i^2] = \dfrac{\hat{x}^{2^T}\xi^2}{\gamma} = 0$. (Using (3.52)) (3.71)

Also from (3.61), (3.62) and (3.55) we have $\hat{y}^1 = \dfrac{\beta^1}{\gamma} \geq 0$ and $y^2 = \dfrac{\beta^2}{\gamma} \geq 0$. (3.72)

Hence from (3.53), (3.54) and from (3.68) to (3.72) we conclude that $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{\lambda},\hat{a}_i^1,\hat{a}_i^2)$ satisfies the dual constraint from (3.9) to (3.16).So it is a feasible solution of (SMSD).

Let $\dfrac{2\tau_i}{\alpha_i} = t$, then $t \geq 0$ .From (3.39), (3.40), (3.61) and (3.62) we get

$$C_i^1\hat{y}^1 = tC_i^1\hat{w}_i^1, C_i^2\hat{y}^2 = tC_i^2\hat{w}_i^2 \quad . \tag{3.73}$$

This is a condition of Schwartz Inequality (Definition 2.9). Therefore

$(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}} (\hat{w}_i^{1^T} C_i^1 \hat{w}_i^1)^{\frac{1}{2}}$ and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}} (\hat{w}_i^{2^T} C_i^2 \hat{w}_i^2)^{\frac{1}{2}}$. (3.74)

In case $\tau_i > 0$, from (3.47) and (3.48) we get $\hat{w}_i^{1^T} C_i^1 \hat{w}_i^1 = 1, \hat{w}_i^{2^T} C_i^2 \hat{w}_i^2 = 1$ and so we get

$(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}}$ and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}}$.

In case $\tau_i = 0$ we get $\frac{2\tau_i}{\alpha_i} = t = 0$, so (3.77) implies $C_i^1 \hat{y}^1 = C_i^2 \hat{y}^2 = 0$.

Hence $(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}} = 0$ and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}} = 0$.

Thus in either case $(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}}$ (3.75)

and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}}$. (3.76)

Therefore using (3.41), (3.42), (3.75) and (3.76) we get

$$f_i(\hat{x}^1, \hat{y}^1) + g_i(\hat{x}^2, \hat{y}^2) + (\hat{x}^{1^T} E_i^1 \hat{x}^1)^{\frac{1}{2}} + (\hat{x}^{2^T} E_i^2 \hat{x}^2)^{\frac{1}{2}} - \hat{y}^{1^T} C_i^1 \hat{w}_i^1 - \hat{y}^{2^T} C_i^2 \hat{w}_i^2$$

$$= f_i(\hat{x}^1, \hat{y}^1) + g_i(\hat{x}^2, \hat{y}^2) + (\hat{x}^{1^T} E_i^1 \hat{a}_i^1) + (\hat{x}^{2^T} E_i^2 \hat{a}_i^2) - (\hat{y}^{1^T} C_i^1 \hat{y}^1)^{\frac{1}{2}} - (\hat{y}^{2^T} C_i^2 \hat{y}^2)^{\frac{1}{2}}, \quad (3.77)$$

for each $i = 1,2,3,......r$.

Or $H_i(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{w}^1, \hat{w}^2) = G_i(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2)$

for each $i = 1,2,3,......r$,

Or $H(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{w}^1, \hat{w}^2) = G(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2)$. (3.78)

So the objective values of both problems are equal.

Now we claim that $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2, \lambda)$ is properly efficient solution for (SMSD).

First we have to show it is an efficient solution of (SMSD).

If this would not be the case, then there would exist a feasible solution $(\hat{u}^1, \hat{u}^2, \hat{v}^1, \hat{v}^2, \hat{a}^1, \hat{a}^2, \hat{\lambda})$ of (SMSD) such that $G(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2) \leq G(\hat{u}^1, \hat{u}^2, \hat{v}^1, \hat{v}^2, \hat{a}^1, \hat{a}^2)$

This by (3.78) gives

$$H(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{w}^1, \hat{w}^2) \leq G(\hat{u}^1, \hat{u}^2, \hat{v}^1, \hat{v}^2, \hat{a}^1, \hat{a}^2).$$

This is a contradiction to Theorem 3.1.Hence $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2, \lambda)$ is an efficient solution of (SMSD).

Now we have to claim $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2, \lambda)$ is properly efficient for (SMSD).

For that rewriting the objective function of (SMSD) into minimization form we get

$$\min G_i^*(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2) = -\max G_i(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2).$$

If $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\lambda)$ were not properly efficient for (SMSD), then for every scalar $M>0$, there exist a feasible solution $(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,\hat{\lambda})$ in (SMSD) and an index i such that

$$\{G_i^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)-G_i^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)\}<M\{G_j^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)-G_j^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)\} \quad (3.79)$$

and for all j satisfying

$$G_j^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)<G_j^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)\}, \quad (3.80)$$

whenever

$$G_i^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)<G_i^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2). \quad (3.81)$$

This implies

$$\begin{aligned}&\{G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)-G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)\}\\ &>M\{G_j(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)-G_j(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)\},\end{aligned} \quad (3.82)$$

for all j satisfying

$$G_j(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)>G_j(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)\}, \quad (3.83)$$

whenever

$$G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,\hat{q}^1=0,\hat{q}^2=0)>G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2,\hat{p}^1=0,\hat{p}^2=0). \quad (3.84)$$

Since $M>0$ and using (3.83) in (3.82) we get

$$\{G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)-G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)\}>0. \quad (3.85)$$

Using (3.83) in (3.90) we obtain

$$\{G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)-H_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2)>0$$

$$\Rightarrow\{G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)>H_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2).$$

That can be made arbitrary large and for each $\hat{\lambda}_i>0$, we have

$$\sum_{i=1}^{r}\hat{\lambda}_i\{G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2)\}>\sum_{i=1}^{r}\hat{\lambda}_iH_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2).$$

This again contradicts Theorem 3.1.Hence $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\lambda)$ is properly efficient solution of (SMSD).

□

Theorem 3. 3. (Converse Duality).

Let $f_i:R^{|J_1|}\times R^{|K_1|}\to R$ and $g_i:R^{|J_2|}\times R^{|K_2|}\to R$ be thrice differentiable function and let $(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{\lambda},\hat{a}_i^1,\hat{a}_i^2)$ be a weak efficient solution for (SMSD) .Let $\hat{\lambda}=\lambda$ be fixed in (SMSD).Assume that

(i) The matrix $\sum_{i=1}^{r}\lambda_i(\nabla_1^2 f_i)$ and $\sum_{i=1}^{r}\lambda_i(\nabla_1^2 g_i)$ are positive definite or negative definite and

(ii) The set $\left\{\nabla_1 f_1+E_1^1\hat{a}_1^1,\ldots\ldots\ldots\nabla_1 f_r+E_r^1\hat{a}_r^1\right\}$ and $\left\{\nabla_1 g_1+E_1^2\hat{a}_1^2,\ldots\ldots\ldots\nabla_1 g_r+E_r^2\hat{a}_r^2.\right\}$ are linearly independent.

Then there exist $w_i^1\in R^{|K_1|}, w_i^2\in R^{|K_2|}$ such that $(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{\lambda},\hat{w}_i^1,\hat{w}_i^2)$ is feasible solution of (SMSD) and the two objective values are equal. Also if the hypothesis of Theorem 3.1 are satisfied for all feasible solution of (SMSP) and (SMSD), then $(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{\lambda},\hat{w}_i^1,\hat{w}_i^2)$ is properly efficient solution of (SMSP).

Proof: It follows on the lines of Theorem 3.2

4. SELF DUALITY

A primal (dual) problem having equivalence dual(primal) formulation is said to be self dual i.e ,if the dual can be recast in the form of the primal .In general (SMSP) and (SMSD) are not self dual without some added restriction on f and g.

Theorem 4.1: Let $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ be skew symmetric and $B_i^1 = C_i^1, B_i^2 = C_i^2$. Then (SMSP) is a self dual. Furthermore if (SMSP) and (SMSD) are dual programs and $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{w}, \hat{\lambda})$ is an efficient solution of (SMSP), then $(\hat{y}^1, \hat{y}^2, \hat{x}^1, \hat{x}^2, \hat{w}, \hat{\lambda})$ is efficient solution of (SMSD) and the values of two objective function are equal to zero.

Proof: By recasting the dual problem (SMSD) as minimization problem, we have

(SMSD$_0$): $$\min imize G_i^*(u,v,a) = \begin{Bmatrix} -f_i(u^1,v^1) - g_i(u^2,v^2) + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}} \\ -(u^1)^T E_i^1 a_i^1 - (u^2)^T E_i^2 a_i^2, i = 1,2,...r \end{Bmatrix}$$

Subject to $$-\sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(u^1,v^1) + E_i^1 a_i^1] \leq 0, \tag{4.1}$$

$$-\sum_{i=1}^{r} \lambda_i [\nabla_1 g_i(u^2,v^2) + E_i^2 a_i^2] \leq 0 \ , \tag{4.2}$$

$$-(u^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(u^1,v^1) + E_i^1 a_i^2] \geq 0 \ , \tag{4.3}$$

$$-(u^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_1 g_i(u^2,v^2) + E_i^2 a_i^2] \geq 0, \tag{4.4}$$

$$(v^1, v^2) \geq 0 \tag{4.5}$$

$$(a_i^1)^T E_i^1 a_i^1 \leq 1 \ , \quad i = 1,2,.....r \tag{4.6}$$

$$(a_i^2)^T E_i^2 a_i^2 \leq 1 \, , i = 1,2,.....r \tag{4.7}$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1 \tag{4.8}$$

Since $f_i(u^1,v^1)$ and $g_i(u^2,v^2)$ are skew symmetric, the above problem becomes (SMSD$_0$):

$$\min imize G_i^*(u,v,a) = \begin{Bmatrix} f_i(v^1,u^1) + g_i(v^2,u^2) + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}} \\ -(u^1)^T E_i^1 a_i^1 - (u^2)^T E_i^2 a_i^2; i = 1,2,...r \end{Bmatrix}$$

Subject to $$\sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(v^1,u^1) - E_i^1 a_i^1] \leq 0 \tag{4.9}$$

$$\sum_{i=1}^{r} \lambda_i [\nabla_1 g_i(v^2,u^2) - E_i^2 a_i^2] \leq 0 \tag{4.10}$$

$$(u^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_1 f_i(v^1,u^1) - E_i^1 a_i^2] \geq 0 \tag{4.11}$$

$$(u^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_1 g_i(v^2,u^2) - E_i^2 a_i^2] \geq 0 \tag{4.12}$$

$$(v^1, v^2) \geq 0 \tag{4.13}$$

$$(a_i^1)^T E_i^1 a_i^1 \leq 1 \ , \quad i = 1,2,.....r \tag{4.14}$$

$$(a_i^2)^T E_i^2 a_i^2 \le 1, i = 1,2,\ldots..r \tag{4.15}$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1 \tag{4.16}$$

This shows that ($SMSD_0$) is formally identical to (SMSP) i.e. the objective function and the constraints are identical Thus, the problem (SMSD) becomes self dual. Hence if $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$ is efficient solution for (SMSP) then $(\bar{y}, \bar{x}, \bar{w}, \bar{\lambda})$ is efficient solution for (SMSD).By similar argument $(\bar{x}, \bar{y}, \bar{a}, \bar{\lambda})$ is efficient solution for (SMSP) implies $(\bar{y}, \bar{x}, \bar{a}, \bar{\lambda})$ is efficient solution for (SMSD).

If (SMSP) and (SMSD) are dual programs and $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$ is efficient solution, then by theorem 3.2 we get

$$H(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda}) = G(\bar{x}, \bar{y}, \bar{a}, \bar{\lambda}) \tag{4.17}$$

Now we claim that $H(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda}) = 0$

From the definition of Schwarz inequality and (3.6), (3.7), (4.14) and (4.15) we have

$$(y^1)^T C_i^1 w_i^1 \le ((y^1)^T C_i^1 y^1)^{\frac{1}{2}} ((w_i^1)^T C_i^1 w_i^1)^{\frac{1}{2}} \le ((y^1)^T C_i^1 y^1)^{\frac{1}{2}}, \tag{4.18}$$

$$(y^2)^T C_i^2 w_i^2 \le ((y^2)^T C_i^2 y^2)^{\frac{1}{2}} ((w_i^2)^T C_i^1 w_i^2)^{\frac{1}{2}} \le ((y^2)^T C_i^1 y^2)^{\frac{1}{2}}, \tag{4.19}$$

$$(x^1)^T E_i^1 a_i^1 \le ((x^1)^T E_i^1 y^1)^{\frac{1}{2}} ((a_i^1)^T E_i^1 a_i^1)^{\frac{1}{2}} \le ((x^1)^T E_i^1 x^1)^{\frac{1}{2}}, \tag{4.20}$$

$$(x^2)^T E_i^2 a_i^2 \le ((x^2)^T E_i^2 x^2)^{\frac{1}{2}} ((a_i^2)^T E_i^1 a_i^2)^{\frac{1}{2}} \le ((x^2)^T E_i^1 x^2)^{\frac{1}{2}}, \tag{4.21}$$

So

$$H_i(\bar{x}, \bar{y}, \bar{w}) = f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) + ((\bar{x}^1)^T E_i^1 \bar{x}^1)^{\frac{1}{2}} + ((\bar{x}^2)^T E_i^2 \bar{x}^2)^{\frac{1}{2}} - (\bar{y}^1)^T C_i^1 \bar{w}_i^1 - (\bar{y}^2)^T C_i^2 \bar{w}_i^2$$

That implies in light of (4.18) and (4.19)

$$H_i(\bar{x}, \bar{y}, \bar{w}) \ge f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) + ((\bar{x}^1)^T E_i^1 \bar{x}^1)^{\frac{1}{2}} + ((\bar{x}^1)^T E_i^1 \bar{x}^1)^{\frac{1}{2}} - ((\bar{y}^1)^T C_i^1 \bar{y}^1)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i^2 \bar{y}^2)^{\frac{1}{2}} \tag{4.22}$$

Similarly

$$G_i(\bar{x}, \bar{y}, \bar{a}) = f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) + (\bar{x}^1)^T E_i^1 \bar{a}_i^1) + (\bar{x}^2)^T E_i^2 \bar{a}_i^2 - ((\bar{y}^1)^T C_i^1 \bar{y}^1)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i^2 \bar{y}^2)^{\frac{1}{2}}$$

Which implies $G_i(\bar{x}, \bar{y}, \bar{a}) \le f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) + ((\bar{x}^1)^T E_i^1 \bar{x}^1)^{\frac{1}{2}} + ((\bar{x}^2)^T E_i^2 \bar{x}^2)^{\frac{1}{2}}$

$$- ((\bar{y}^1)^T C_i^1 \bar{y}^1)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i^2 \bar{y}^2)^{\frac{1}{2}} \tag{4.23}$$

Hence from (4.17) (4.22) and (4.23) we get

$$\begin{aligned} H_i(\bar{x}, \bar{y}, \bar{w}) = G_i(\bar{x}, \bar{y}, \bar{a}) = {} & f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) + ((\bar{x}^1)^T E_i^1 \bar{x}^1)^{\frac{1}{2}} + ((\bar{x}^2)^T E_i^2 \bar{x}^2)^{\frac{1}{2}} \\ & - ((\bar{y}^1)^T C_i^1 \bar{y}^1)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i^2 \bar{y}^2)^{\frac{1}{2}} \end{aligned} \tag{4.24}$$

Similarly $(\bar{y}, \bar{x}, \bar{a}, \bar{\lambda})$ is also efficient solution for (SMSP) and (SMSD) implies

$$\begin{aligned} H_i(\bar{x}, \bar{y}, \bar{w}) = G_i(\bar{x}, \bar{y}, \bar{a}) = {} & f_i(\bar{y}^1, \bar{x}^1) + g_i(\bar{y}^2, \bar{x}^2) + ((\bar{y}^1)^T E_i^1 \bar{y}^1)^{\frac{1}{2}} + ((\bar{y}^2)^T E_i^2 \bar{y}^2)^{\frac{1}{2}} \\ & - ((\bar{x}^1)^T C_i^1 \bar{x}^1)^{\frac{1}{2}} - ((\bar{x}^2)^T C_i^2 \bar{x}^2)^{\frac{1}{2}} \end{aligned}$$

$$= -f_i(\bar{x}^1, \bar{y}^1) - g_i(\bar{x}^2, \bar{y}^2) + ((\bar{y}^1)^T C_i^1 \bar{y}^1)^{\frac{1}{2}} + ((\bar{y}^2)^T C_i^2 \bar{y}^2)^{\frac{1}{2}} - ((\bar{x}^1)^T E_i^1 \bar{x}^1)^{\frac{1}{2}} - ((\bar{x}^2)^T E_i^2 \bar{x}^2)^{\frac{1}{2}}$$

(4.25)

Now adding (4.24) and (4.25) we get $H_i(\bar{x},\bar{y},\bar{w}) = G_i(\bar{x},\bar{y},\bar{a}) = 0$ □

Example 4.1: Let $n = m = 3, r = 1, x^1 = (x_1, x_2), x^2 = x_3, y^1 = (y_1, y_2), y^2 = y_3$

$f(x,y) = e^{x_1} + e^{x_2} - e^{y_1} - e^{y_2}$, $g(x,y) = \sin x_3 - \sin y_3$, $E_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = C_1, E_2 = C_2 = (1).$

$w^1 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, z^1 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, w^2 = w_3, z^2 = z_3$

Then the problem (MSP) and (MSD) become

Primal Problem (MSP)

$$\min imize \begin{Bmatrix} e^{x_1} + e^{x_2} - e^{y_1} - e^{y_2} + \sin x_3 - \sin y_3 + \sqrt{2x_1^{\,2} + 2x_1x_2 + x_2^{\,2}} \\ + x_3 + (2y_1 + y_2)w_1 + (y_1 + y_2)w_2 + y_3w_3 \end{Bmatrix}$$

Subject to $-e^{y_1} - 2w_1 - w_2 \le 0$, (4.26)

$-e^{y_2} - w_1 - w_2 \le 0$, (4.27)

$-\cos y_3 - w_3 \le 0,$ (4.28)

$-y_1e^{y1} - 2y_1w_1 - y_1w_2 \ge 0,$ (4.29)

$-y_2e^{y_2} - 2y_2w_1 - y_2w_2 \ge 0$, (4.30)

$-y_3 \cos y_3 - y_3w_3 \ge 0$, (4.31)

$2w_1^{\,2} + 2w_1w_2 + w_2^{\,2} \le 1$, $w_3 \le 1$, (4.32)

$(x_2, x_3) \ge 0$, (4.33)

Dual Problem (MSD)

$$\max imize \begin{Bmatrix} e^{u_1} + e^{u_2} - e^{v_1} - e^{v_2} + \sin u_3 - \sin v_3 - \sqrt{2v_1^{\,2} + 2v_1v_2 + v_2^{\,2}} \\ - v_3 + (2u_1 + u_2)a_1 + (u_1 + u_2)a_2 + u_3a_3 \end{Bmatrix}$$

Subject to $-e^{u_1} - 2a_1 - a_2 \ge 0$, (4.34)

$-e^{u_2} - a_1 - a_2 \ge 0$, (4.35)

$-\cos u_3 - a_3 \ge 0$, (4.36)

$-u_1e^{u_1} - 2u_1a_1 - u_1a_2 \le 0$, (4.37)

$-u_2e^{u_2} - 2u_2a_1 - u_2a_2 \le 0$, (4.38)

$-u_3 \cos u_3 - u_3a_3 \le 0$, (4.39)

$2a_1^{\,2} + 2a_1a_2 + a_2^{\,2} \le 1$, $a_3 \le 1$, (4.40)

$(v_2, v_3) \ge 0$, (4.41)

Clearly $f(x,y) = -f(y,x)$, Therefore the problem (S P) is a self dual and hence theorem 4.1 is applicable for this pair.

5. SPECIAL CASE

(i) If $|J_2|=0, |K_2|=0$,then our problem reduces to a pair of (MP) and (MD) given by Thakur et al(2011).

(ii) If we take $C_i^j=\{E_i^j y^j; y^{jT} E_i^j y^j \le 1\}$ where E_i^j, are positive semi definite matrices, and $(x^{jT} E_i^j x^j)^{\frac{1}{2}} = s(x^j \mid C_i^j)$ and $(y^{jT} C_i^j y^j)^{\frac{1}{2}} = s(y^j \mid D_i^j)$, $i=1,2,.....r$, $j=1,2$.
$b=1, \psi = I$, $C_i^j w_i^j = z_i^j, E_i^j a_i^j = w_i^j$ with $\phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ for $\rho = 0$ and $i=1,2,.....r$, $j=1,2$, then our problem (SMSP) and (SMSD) reduces to the pair of dual and dual results given by Li (2011) , .Mishra et al.(2006) and Mishra(2007).

(iii) If we take $C_i^j=\{E_i^j y^j; y^{jT} E_i^j y^j \le 1\}$ where E_i^j, are positive semi definite matrices and $(x^{jT} E_i^j x^j)^{\frac{1}{2}} = s(x^j \mid C_i^j)$ and $(y^{jT} C_i^j y^j)^{\frac{1}{2}} = s(y^j \mid D_i^j)$, $i=1,2,.....r$, $j=1,2$.
$b=1, \psi = I$, $C_i^2 w_i^j = z_i^2, E_i^2 a_i^2 = w_i^2$,
$C_i^1 w_i^1 = \nabla_2 f(x^1,y^1) + \nabla_2^2 f(x^1,y^1)p, E_i^1 a_i^1 = \nabla_1 f(u^1,v^1) + \nabla_1^2 f(u^1,v^1)q$
with $\phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ for $\rho = 0$,

then our problem (MMSP) and (MMSD) reduces to the problem (MP) and (MD) given by Agarwal et al (2011).

(iv) If $|J_2|=0, |K_2|=0$, $(x^T E_i x)^{\frac{1}{2}} = s(x \mid C_i)$, $(y^T C_i y)^{\frac{1}{2}} = s(y \mid D_i)$, $C_i w_i = z_i, E_i a_i = w_i$ $i=1,2,.....r$, then the problem (MMSP) and (MMSD) reduces to a pair of problems (MP) and (MD) and its results given by Ojha (2010).

(v) If $\phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ for $\rho = 0$, $|J_2|=0, |K_2|=0$, $b=1, \psi = I$ and, $C_i = E_i = 0, i=1,2,...r$ then the problem (SMSP) and (SMSD) reduces to a pair of problems (MP) and (MD) and the results studied by Suneja et al (2003) and dual given by Chandra et al(1998).

(vi) If $|J_2|=0, |K_2|=0$, $b=1, \psi = I$ and $\phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ for $\rho = 0$ in (SMSP) and (SMSD) ,then we obtain a pair of nondifferentiable symmetric dual in multiobjective program considered by Ahmad et al (2005).

(vii) If we set $|J_2|=0, |K_2|=0$ in (SMSP) and (MSD, then we obtain a pair of first order symmetric dual nondifferentiable multiobjective programs considered by Mond et al (1991).

5. CONCLUSION

In this article, a new pair of nondifferentiable multiobjective mixed symmetric dual programs is presented and duality relations between primal and dual problems are established. The results developed in this paper improve and generalize a number of existing results in the literature. The results discussed in this paper can be extended to second order and higher order as well as to other generalized convexity assumptions .These results can be extended to the case of continuous-time problems as well.

6. REFERENCES

[1] Agarwal, R.P., Ahmad, I., Gupta, S. and Kali, N., Generalized Second order Mixed symmetric duality in Nondifferentiable Mathematical Programming, *Abstract and Applied Analysis,* Hindawi Pub. Cor., doi: 10, (2011), 1155/2011/103587

[2] Ahmad, I., Multiobjective mixed duality for with invexity, *New Zealand J. Math*. 34(2005), 1-9.

[3] Ahmad. I., and Husain, Z., Multiobjective mixed symmetric duality involving cones. *Computer and Mathematics with Applications*, 59(2010), 319-*326.*

[4] Ahmad, I. and Husain, Z., Nondifferentiable second order duality in multiobjective programming, *Applied Mathematical Letter*, 18 (2005), 721-728.

[5] Bazaars, M.S., Goode, J.J., On symmetric duality in nonlinear Programming, *Oper.Res.*21 (1973), 1-9.

[6] Bector, C.R., Chandra, S., Gupta, S. and Suneja, S.K., Univex sets, functions, and univex non linear Programming, *Lecture Notes in Economics and Mathematical system*, Springer Verlag, Berlin, 405(1994)1-8.

[7] Bector, C.R., Chandra, S., .Goyal, A. On mixed symmetric duality in multiobjective programming, *Opsearch*, 36 (1999), 399-407.

[8] Chandra, S., Husain, I. and Abha, On mixed symmetric duality in mathematical programming, *Opsearch,* Vol-36, no 2,(1999),pp 165-171.

[9] Chandra, S., Goyal, A. and Husain, I. On symmetric duality in mathematical programming with F-convexity , *Optimization, 43* (1998), 1.

[10] Chandra, S., Kumar, V. , A note on pseudo invexity and Symmetric duality, *European Journal of Operation Research*, 105(1998),626-629.

[11] Devi, G., Symmetric duality for nonlinear programming problem involving η -bonvex function, *European Journal Operation Research*, 104(1998), 615-621.

[12] Dorn, W.S. A symmetric dual theorem for quadratic Programs, J. Oper. Res. Soc. Japan 2, (1960), 93-97.

[13] Dantzig, G.B., Eisenberg, E., Cottle, R.W., Symmetric dual nonlinear programs, *Pacific Journal of Mathematics*, *vol*-15(1965), 809-812.

[14] Hanson, M.A., On sufficiency of the Kuhn –Tucker condition, Journal *of Mathematical Analysis and Application*, 30(1981), 545-550.

[15] Husain, I.Goyel, A, and Masoodi, M., On Nondifferentiable Multiobjective Second order symmetric duality, *International Journal of Operation Research,* Vol-5, No-2(2008), 91-98.

[16] Khorana, S., Symmetric duality in multiobjective Programming involving generalized cone-in vex function, *European Journal of Operation Research*, 165(2005), 592-597.

[17] Kim, M.H., Kim, D.S., Non –differentiable symmetric duality. For multiobjective programming with cone constraint, *European Journal of Operation Research,*188,(2008)652-661.

[18] Li, J. and.Gao, Y. Non-differentiable multiobjective mixed symmetric duality under generalized Convexity, *Journal of Inequality and Applications,*doi:10.1186/1029-242x-2011-23.

[19] Mangasarian, O.L, *Nonlinear Programming*, McGraw Hill, New York, 1969.

[20] Mishra, S.K., Wang, S.K., and Lai, K.K., Mond-Weir type mixed Symmetric first and second order duality in Nondifferentiable mathematical programming, Journal *of nonlinear and convex Analysis,* 7,no-3,(2006)189-198.

[21] Mishra, S.K., Second order mixed symmetric duality in non-differentiable multiobjective Mathematical Programming, Journal *of Applied Analysis,*Vol-13,No-1(2007),pp.117-132.

[22] Mishra, S.K., On multiple objective optimization with generalized univexity, *Journal of Mathematical Analysis and Application,* 224; (1998), 131-148.

[23] Mishra, S.K. and Giorgi,G.,Optimality and duality with generalized semi –univexity, *Opsearch 37(2000),340-350*

[24] Mishra, S.K, Wang, S.Y. and Lai K.K., Nondifferentiable multiobjective programming under generalized d-Univexity, *European Journal of Operation Research*, 160, (2005), 218-226.

[25] Mond, B.and Schecther, M., Non-differentiable symmetric duality, *Bulletin of the Australian Mathematical Society, 53*(1996), 177-188.

[26] Mond, B., Hussion, I. and Durga.M.V.P, Duality for a class of nondifferentiable multiobjective Programming, *Utilitas Math, 39,(1996) 3.*

[27] Ojha, D.B.Duality for nondifferentiable multiobjective programming involving (ϕ_0, ρ)-Univexity, *J Sci. Math. Res,* vol-4(2010), pp1-9.

[28] Rueda, N.G.; Hanson, M.A. and Sing., Optimality and duality with generalized convexity, *Journal of Optimization theory and Application, 86, (1995)491-500.*

[29] Suneja, S.K., Agarwal, S.K., Davar, S., Multiobjective Symmetric duality involving cones, *European Journal of Operation Research*, 141(2002), 471-479
.
[30] Schter, M., More on sub gradient duality, *Journal of Mathematical Analysis and Application*, vol.71, no- 1(1979), pp.251-262

[31] Suneja, S.K., Lalita, C.S. and Khorana, S., Second order symmetric dual in multiobjective programming, *European Journal of Operation Research*, 144, (2003)492

[32] Thakur, G.K, Priya, B, B., Second order duality for nondifferentiable multiobjective programming involving (Φ, ρ)-univexity, *Kathmandu University Jr. of Sc., Engg And Technology,* vol.7, No-1(2011), 92-104

[33] Yang, X.M., Teo, K.L. and Yang, X.Q.Mixed symmetric duality in nondifferentiable mathematical Programming, *Indian Journal of Pure and Applied Mathematics,* vol-34, no-5(2003), pp.805-815

[34] Weir, T. and Mond, B.; Symmetric and self duality in Multiobjective programming, *Asia Pacific Journal of Operation Research,*vol-5(1991),75-87

Source of support: Nil, Conflict of interest: None Declared