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ON GEODETIC SETS AND POLYNOMIALS OF CENTIPEDES

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ABSTRACT

Let G = (V,E) be a simple graph. A set of vertices S of a graph G is geodetic, if every vertex of G lies on a shortest path between two vertices in S. The geodetic number of G is the minimum cardinality of all geodetic sets of G, and is denoted by g (G). In (8), the concept of geodetic polynomial is defined as $g(G,x)=\sum_{i=g(G)}^{n} g_e(G,i)x^i$ *where* $g_e(G,i)$ *is the number of geodetic sets of cardinality i, and* G^* *be the graph obtained by appending a single pendant edge to each vertex of graph G. We call* P_n^* *a centipede, where* P_n *is a path with n vertices. In this paper, we obtain the geodetic sets and polynomials of the centipedes. Also, we study some properties of geodetic sets and the coefficients of the polynomials. It is also derived that the geodetic polynomial of the centipede* P_n^* *is* $x^n(1+x)^n$.

Keywords*: Geodetic sets, geodetic number, centipede, Recursive formula.*

1. INTRODUCTION

Let G=(V, E) be a simple graph of order |V| = n. The distance d(u, v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. A u-v path of length d(u, v) is called u-v geodesic. The closed interval I[u, v] consists of all vertices lying on some u- v geodesic of G, while for S $\subseteq$ V, I [S] = $\bigcup_{u,v}$ I [u, v]. A set S of vertices is a geodetic set if I [S] = V, and the minimum cardinality of a geodetic set is the geodetic number g(G). The geodetic number of a graph was introduced in [4,5]. In [I], the domination polynomial was introduced and some properties have been derived. In [8], the concept of geodetic polynomial was introduced. It is defined as $g(G,x)=\sum_{i=g(G)}^{n} g_e(G,i)x^i$ where G is a graph of order n and $g_e(G,i)$ is the number of geodetic sets of G of cardinality i. As $g(P_n^*)=n$ and P_n^* has 2n vertices, the geodetic polynomial of P_n^* is of the form $g(P_n^*,x)=\sum_{i=n}^{2n} g_e(P_n^*,i)x^i$ where $g_e(P_n^*,i)$ is the geoditic sets of P_n^* with cardinality i. The geodetic number g(G) is the minimum cardinality of a geodetic set in G. A geodetic set with cardibality g(G) is called a g-set.

Let G^* denote the graph obtained by appending a single pendent edge to each vertex of a graph G. In this paper we consider the labeled centipede as shown in figure 1. We denote the graph obtained from P_n^* by deleting the vertex labeled 2n as $P_n^*-\{2n\}$.

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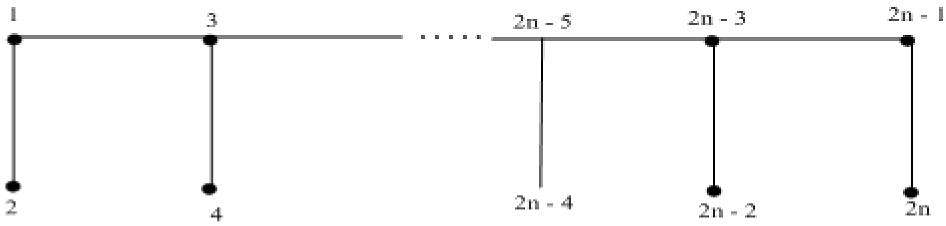

Figure: 1
Labeled centipede, P_n^*

In the next section we study geodetic sets of $P_n^* - \{2n\}$, which is needed for the study of geodetic sets of centipedes.

In section 3, by using the results in section2, we investigate the geodetic sets of centipedes. In the last section we study the geodetic polynomial of centipedes.

As usual, we use $\lceil x \rceil$ for the smallest integer greater than or equal to *x*. In this paper, we denote the set {1,2,…n} simply by $[n]$.

2. GEODETIC SETS OF $P_n^* - \{2n\}$

For the construction of the geodetic sets of centipede P_n^*, we need to investigate the geodetic sets of $P_n^* - \{2n\}$. In this section, we investigate geodetic sets of $P_n^* - \{2n\}$. Let $g_e(P_n^* - \{2n\}, i)$ be the family of geodetic sets of $P_n^* - \{2n\}$ with cardinality i. We shall find recursive formula for $|g_e(P_n^* - \{2n\}, i)|$. We need the following lemmas to obtain the main results of this section.

Lemma 2.1: [1] $g(P_n) = 2$

Lemma 2.2: For every $n \in N$:

(i) $g(P_n^*) = n$

(ii) $g(P_n^* - \{2n\}) = n$

(iii) $g_e(P_n^*) = \phi$ if and only if $i < n \text{ or } i > 2n$

(iv) $g_e(P_n^* - \{2n\}, i) = \phi$ if and only if $i < n \text{ or } i > 2n-1$

Proof:

(i) Every g-set of P_n^* must contain the vertices labeled 2i, for every $1 \le i \le n$. Therefore $g(P_n^*) = n$.

(ii) Every g-set of $P_n^* - \{2n\}$ must contain the vertex labeled 2n-1 and the vertices labeled 2i, for every $1 \le i \le n-1$.

(iii) It follows from part (i) and the definition of geodetic set.

(iv). It follows from part (ii) and the definition of geodetic set. □

Lemma 2.3: If $g_e(P_{n-2}^*, i-2) = \phi$ and $g_e(P_{n-1}^* - \{2n-2\}, i-2) = \phi$ then $g_e(P_n^* - \{2n\}, i) = \phi$

Proof: Since $g_e(P_{n-2}^*, i-2) = \phi$ and $g_e(P_{n-1}^* - \{2n-2\}, i-2) = \phi$ by lemma 2.2 (iii),(iv), we have $i-2 < n-2$ or $i-2 > 2n-3$. If $i-2 < n-2$ then $i < n$ and by lemma 2.2 (iv), $g_e(P_n^* - \{2n\}, i) = \phi$. Also, if $i-2 > 2n-3$, then $i > 2n-1$ and by lemma 2.2 (iv), we have $g_e(P_n^* - \{2n\}, i) = \phi$. □

Lemma 2.4: Suppose $g_e(P_n^* - \{2n\}, i) \ne \phi$ then

(i) $g_e(P_{n-2}^*, i-2) \ne \phi$ and $g_e(P_{n-1}^* - \{2n-2\}, i-2) = \phi$ iff $i = n$

(ii) $g_e(P^*_{n-2}, i-2) = \phi$ $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$ iff $i = 2n-1$

(iii) $g_e(P^*_{n-2}, i-2) \neq \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$ iff $n+1 \leq i \leq 2n-2$

Proof: Suppose $g_e(P^*_n - \{2n\}, i) \neq \phi$

(i) $(\Rightarrow)$ Since $g_e(P^*_{n-1} - \{2n-2\}, i-2) = \phi$, by lemma 2.2 (iv), we have $i-2 < n-1$ or $i-2 > 2n-3$. If $i-2 > 2n-3$ then $i > 2n-1$ and by lemma 2.2 (iv), $g_e(P^*_n - \{2n\}, i) = \phi$, which is a contradiction. So we must have $i-2 < n-1$. Also, since $g_e(P^*_{n-2}, i-2) \neq \phi$, We have $n-2 \leq i-2 \leq 2n-4$. Together, we have $i = n$.

$(\Leftarrow)$ If $i = n$ then by lemma 2.2 (iii),(iv), we have $g_e(P^*_{n-2}, i-2) \neq \phi$, and $g_e(P^*_{n-1} - \{2n-2\}, i-2) = \phi$

(ii) $(\Rightarrow)$ Since $g_e(P^*_{n-2}, i-2) = \phi$, by lemma 2.2 (iii), we have $i-2 < n-2$ or $i-2 > 2n-4$. If $i-2 < n-2$, then $i < n$ and by lemma 2.2 (iv), $g_e(P^*_n - \{2n\}, i) = \phi$, which is a contradiction. So, we must have $i-2 > 2n-4$. Also, since $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$, we have $n-1 \leq i-2 \leq 2n-3$. Together, we have $i = 2n-1$.

$(\Leftarrow)$If $i = 2n-1$ then by lemma 2.2 (iii),(iv), we have $g_e(P^*_{n-2}, i-2) = \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$.

(iii) $(\Rightarrow)$ Since $g_e(P^*_{n-2}, i-2) \neq \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$, by lemma 2.2 (iii), (iv), we have $n-1 \leq i-2 \leq 2n-4$. So $n+1 \leq i \leq 2n-2$.

$(\Leftarrow)$ If $n+1 \leq i \leq 2n-2$, then by lemma 2.2(iii),(iv) we have $g_e(P^*_{n-2}, i-2) \neq \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$. □

Theorem 2.5: Suppose $g_e(P^*_n - \{2n\}, i) \neq \phi$,

(i) If $g_e(P^*_{n-2}, i-2) \neq \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) = \phi$, then

$$g_e(P^*_n - \{2n\}, i) = \left\{\{2n-1, 2n-2\} \cup X \,/\, X \in g_e(P^*_{n-2}, i-2)\right\}$$

(ii) If $g_e(P^*_{n-2}, i-2) = \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$, then

$$g_e(P^*_n - \{2n\}, i) = \left\{\{2n-1, 2n-2\} \cup X \,/\, X \in g_e(P^*_{n-1} - \{2n-2\}, i-2)\right\}$$

(iii) If $g_e(P^*_{n-2}, i-2) \neq \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$, then

$$g_e(P^*_n - \{2n\}, i) = \left\{\{2n-1, 2n-2\} \cup X \,/\, X \in g_e(P^*_{n-2}, i-2)\right\} \cup \left\{\{2n-1, 2n-2\} \cup X \,/\, X \in g_e(P^*_{n-1} - \{2n-2\}, i-2)\right\}$$

Proof:

(i) Since $g_e(P^*_{n-2}, i-2) \neq \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) = \phi$, by lemma 2.4 (i), $i = n$. Therefore, $g_e(P^*_n - \{2n\}, i) = g_e(P^*_n - \{2n\}, n)$. Every geodetic set of $P^*_n - \{2n\}$ with cardinality n must contain the vertices labeled 2n-1, 2n-2. Therefore,

$$g_e(P^*_n - \{2n\}, i) = \left\{\{2n-1, 2n-2\} \cup X \,/\, X \in g_e(P^*_{n-2}, i-2)\right\}$$

(ii) Since $g_e(P^*_{n-2}, i-2) = \phi$ and $g_e(P^*_{n-1} - \{2n-2\}, i-2) \neq \phi$, by lemma 2.4 (ii), i = 2n-1.

Therefore, $g_e(P^*_n - \{2n\}, i) = g_e(P^*_n - \{2n\}, 2n-1)$ Every geodetic set of $P^*_n - \{2n\}$ with cardinality 2n-1 must contain the vertices 2n-1, 2n-2.

Therefore, $g_e(P_n^* - \{2n\}, i) = \{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-2}^*, i-2)\}$.

(iii) Denote the families $\{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-2}^*, i-2)\}$ and $\{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-1}^* - \{2n-2\}, i-2)\}$ Simply by Y_1 and Y_2 respectively. It is obvious that $Y_1 \cup Y_2 \subseteq g_e(P_n^* - \{2n\}, i)$.

Now let $Y \in g_e(P_n^* - \{2n\}, i)$, Suppose $2n-3 \notin Y_1$ then $Y - \{2n-1, 2n-2\} \in g_e(P_{n-2}^*, i-2)$.

Therefore $Y \in Y_1$. If $2n\text{-}3 \in Y$ then $Y - \{\{2n-1, 2n-2\} \in g_e(P_{n-1}^* - \{2n-2\}, i-2)\}$ that is $Y \in Y_2$.

Therefore, $g_e(P_n^* - \{2n\}, i) \subseteq Y_1 \cup Y_2$. Hence, we have $g_e(P_n^* - \{2n\}, i) = Y_1 \cup Y_2$.

Theorem 2.6: For every $n \geq 4$, $|g_e(P_n^* - \{2n\}, i)| = |g_e(P_{n-2}^*, i-2)| + |g_e(P_{n-1}^* - \{2n-2\}, i-2)|$

Proof:

Case (i): If $g_e(P_{n-2}^*, i-2) = \phi$ and $g_e(P_{n-1}^* - \{2n-2\}, i-2) = \phi$ then by theorem 2.5 (i) we have $g_e(P_n^* - \{2n\}, i) = \{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-2}^*, i-2)\}$. In this case the number of geodetic sets of $P_n^* - \{2n\}$ with cardinality i is equal to the number of geodetic sets of P_{n-2}^* with cardinality $i-2$.

Case (ii): If $g_e(P_{n-2}^*, i-2) = \phi$ and $g_e(P_{n-1}^* - \{2n-2\}, i-2) \neq \phi$ then by theorem 2.5 (i) we have $g_e(P_n^* - \{2n\}, i) = \{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-1}^* - \{2n-2\}, i-2)\}$. In this case the number of geodetic sets of $P_n^* - \{2n\}$ with cardinality i is equal to the number of geodetic sets of $P_{n-2}^* - \{2n-2\}$ with cardinality i - 2.

Case (iii): If $g_e(P_{n-2}^*, i-2) \neq \phi$ and $g_e(P_{n-1}^* - \{2n-2\}, i-2) \neq \phi$ then by theorem 2.5 (iii), we have $g_e(P_n^* - \{2n\}, i) = Y_1 \cup Y_2$,

where
$Y_1 = \{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-2}^*, i-2)\}$ and $Y_2 = \{\{2n-1, 2n-2\} \cup X / X \in g_e(P_{n-1}^* - \{2n-2\}, i-2)\}$.

Therefore, $|Y_1| = |g_e(P_{n-2}^*, i-2)|$ and y_2 = $|Y_2| = |g_e(P_{n-1}^* - \{2n-2\}, i-2)|$ since for every $X_1 \in Y_1$ and $X_2 \in Y_2$, we have $2n-3 \in X_2$ but $2n-3 \notin X_1$, then $Y_1 \cap Y_2 = \phi$.

Therefore $|g_e(P_n^* - \{2n\}, i)| = |g_e(P_{n-2}^*, i-2)| + |g_e(P_{n-1}^* - \{2n-2\}, i-2)|$.

3. GEODETIC SETS OF CENTIPEDE

In this section, we investigate geodetic sets of centipedes. We construct $g_e(P_n^*, i)$ from $g_e(P_{n-1}^*, i-1)$ and $g_e(P_n^* - \{2n\}, i-1)$. The families of these geodetic sets can be empty or otherwise. Thus we have four combinations, whether these two families are empty or not.

Lemma 3.1: If $g_e(P_{n-1}^*, i-1) = \phi$ and $g_e(P_n^* - \{2n\}, i-1) = \phi$ then $g_e(P_n^*, i) = \phi$.

Proof: Since $g_e(P^*_{n-1}, i-1) = \phi$ and $g_e(P^*_n - \{2n\}, i-1) = \phi$, by lemma 2.2 (iii) (iv), we have $i-1 < n-1$ or $i-1 > 2n-1$. In either case, we have $g_e(P^*_n, i) = \phi$. □

Lemma 3.2: Suppose $g_e(P^*_n, i) \neq \phi$ then

(i) $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) = \phi$ iff $i = n$

(ii) $g_e(P^*_{n-1}, i-1) = \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ iff $i = 2n$

(iii) $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ iff $n+1 \leq i \leq 2n-1$

Proof: Suppose $g_e(P^*_n, i) \neq \phi$ then

(i) $(\Rightarrow)$ Since $g_e(P^*_n - \{2n\}, i-1) = \phi$, by lemma 2.2(iv), we have $i-1 < n$ or $i-1 > 2n-1$. If $i-1 > 2n-1$ then $i > 2n$ and by lemma 2.2 (iii), $g_e(P^*_n, i) = \phi$, which is a contradiction. So, we have $i-1 < n$. On the other hand, since $g_e(P^*_{n-1}, i-1) \neq \phi$ we have $n-1 \leq i-1 \leq 2n-2$. Together we have $i-1 < n \leq i \leq 2n-1$, therefore $i = n$.

$(\Leftarrow)$ If $i = n$ then by lemma 2.2(iii),(iv), we have $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) = \phi$.

(ii) $(\Rightarrow)$ Since $g_e(P^*_{n-1}, i-1) = \phi$, by lemma 2.2(iii), we have $i-1 < n-1$ or $i-1 > 2n-2$. If $i-1 < n-1$ then $i < n$ and by lemma 2.2 (iii), $g_e(P^*_n, i) = \phi$, which is a contradiction. So, we have $i-1 > 2n-2$. On the other hand, since $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ we have $n \leq i-1 \leq 2n-1$. Together we have have $i = 2n$.

$(\Leftarrow)$ If $i = 2n$ then by lemma 2.2(iii),(iv), we have $g_e(P^*_{n-1}, i-1) = \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$.

(iii) $(\Rightarrow)$ Since $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$, by lemma 2.2(iii)(iv), we have $n+1 \leq i \leq 2n$.

$(\Leftarrow)$ If $n+1 \leq i \leq 2n$ then by lemma 2.2(iii),(iv), we have $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$. □

Theorem 3.3:

(i) If $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) = \phi$ then $g_e(P^*_n, i) = \left\{X \cup \{2n\} / X \in g_e(P^*_{n-1}, i-1)\right\}$

(ii) If $g_e(P^*_{n-1}, i-1) = \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ then $g_e(P^*_n, i) = \left\{X \cup \{2n\} / X \in g_e(P^*_n - \{2n\}, i-1)\right\}$

(iii) If $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ then

$$g_e(P^*_n, i) = \left\{X \cup \{2n\} / X \in g_e(P^*_{n-1}, i-1)\right\} \cup \left\{X \cup \{2n\} / X \in g_e(P^*_n - \{2n\}, i-1)\right\}$$

Proof:

(i) Since $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) = \phi$ by theorem 3.2 (i), $i = n$. Therefore $g_e(P^*_n, i) = g_e(P^*_n, n)$. Every geodetic sets of P^*_n with cardinality n must contain the vertex labeled 2n. Therefore $g_e(P^*_n, i) = \left\{X \cup \{2n\} / X \in g_e(P^*_{n-1}, i-1)\right\}$.

(ii) Since $g_e(P^*_{n-1}, i-1) = \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ by theorem 3.2(ii), $i = 2n$

Therefore $g_e(P^*_n, i) = g_e(P^*_n, 2n)$. Every geodetic sets of P^*_n with cardinality 2n must contain the vertex labeled 2n.

Therefore $g_e(P^*_n, 2n) = \left\{X \cup \{2n\} / X \in g_e(P^*_n - \{2n\}, i-1)\right\}$.

(iii) Denote the families $\left\{X \cup \{2n\} / X \in g_e(P^*_{n-1}, i-1)\right\}$ and $\left\{X \cup \{2n\} / X \in g_e(P^*_n - \{2n\}, i-1)\right\}$ simply by Y_1 and Y_2 respectively. It is obvious that $Y_1 \cup Y_2 \subseteq g_e(P^*_n, i)$. Now let $Y \in g_e(P^*_n, i)$. So at least two of the vertices labeled 2n, 2n-1 or 2n, 2n-2 are in Y. If $2n\text{-}1 \in Y$, then $Y\text{-}\{2n\} \in g_e(P^*_{n-1}, i-1)$, $Y \in Y_1$.

If $2n\text{-}1 \in Y$, then $Y\text{-}\{2n\} \in g_e(P^*_n - \{2n\}, i-1)$ that is $Y \in Y_2$. Therefore $g_e(P^*_n, i) \subseteq Y_1 \cup Y_2$. Hence we have $g_e(P^*_n, i) = Y_1 \cup Y_2$. □

Example 3.4: Consider the centipede P^*_4 we use theorem 3.3 to construct P^*_4, for $4 \le i \le 8$.

Since $g_e(P^*_4 - \{8\}, 3) = \phi$ and $g_e(P^*_3, 3) = \{\{2,4,6\}\}$, by theorem 3.3(i),

$$g_e(P^*_4, 4) = \left\{X \cup \{8\} / X \in g_e(P^*_3, 3)\right\} = \{\{2,4,6,8\}\}$$

Since $g_e(P^*_3, 7) = \phi$ and $g_e(P^*_4 - \{8\}, 7) = \{\{1,2,3,4,5,6,7\}\}$ by theorem 3.3 (ii)

$$g_e(P^*_4, 8) = \left\{X \cup \{8\} / X \in g_e(P^*_4 - \{8\}, 7)\right\} = \{\{1,2,3,4,5,6,7,8\}\}$$

Since $g_e(P^*_3, 4) = \{\{1,2,4,6\},\{2,3,4,6\},\{2,4,5,6\}\}$, and $g_e(P^*_4 - \{8\}, 4) = \{\{2,4,6,7\}\}$, by theorem 3.3 (iii) $g_e(P^*_4, 5) = \left\{X \cup \{8\} / X \in g_e(P^*_3, 4)\right\} \cup \left\{X \cup \{8\} / X \in g_e(P^*_4 - \{8\}, 4)\right\}$

$$= \{\{1,2,4,6,8\},\{2,3,4,6,8\},\{2,4,5,6,8\},\{2,4,6,7,8\}\}$$

Since $g_e(P^*_3, 5) = \{\{1,2,3,4,6\},\{1,2,4,5,6\},\{2,3,4,5,6\}\}$ } and

$g_e(P^*_4 - \{8\}, 5) = \{\{1,2,4,6,7\},\{2,3,4,6,7\},\{2,4,5,6,7\}\}$ by theorem 3.3(iii),

$$g_e(P^*_4, 6) = \left\{X \cup \{8\} / X \in g_e(P^*_3, 5)\right\} \cup \left\{X \cup \{8\} / X \in g_e(P^*_4 - \{8\}, 5)\right\}$$

$$= \{\{1,2,3,4,6,8\},\{1,2,4,5,6,8\},\{2,3,4,5,6,8\},\{1,2,4,6,7,8\},\{2,3,4,6,7,8\},\{2,4,5,6,7,8\}\}$$

Since $g_e(P^*_3, 6) = \{\{1,2,3,4,5,6\}\}$ and

$g_e(P^*_4 - \{8\}, 6) = \{\{1,2,3,4,6,7\},\{1,2,4,5,6,7\},\{2,3,4,5,6,7\}\}$ by theorem 3.3 (iii)

$$g_e(P^*_4, 7) = \left\{X \cup \{8\} / X \in g_e(P^*_3, 6)\right\} \cup \left\{X \cup \{8\} / X \in g_e(P^*_4 - \{8\}, 6)\right\}$$

$$= \{\{1,2,3,4,5,6,8\},\{1,2,3,4,6,7,8\},\{1,2,4,5,6,7,8\},\{2,3,4,5,6,7,8\}\}.$$ □

Theorem 3.5: For every $n \ge 3$, $\left|g_e(P^*_n, i)\right| = \left|g_e(P^*_{n-1}, i-1)\right| + \left|g_e(P^*_n - \{2n\}, i-1)\right|$.

Proof:

Case (i): If $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) = \phi$ then by theorem 3.3 (i), we have $g_e(P^*_n, i) = \left\{X \cup \{2n\} / X \in g_e(P^*_{n-1}, i-1)\right\}$. In this case the number of geodetic sets of P^*_{n-1} with cardinality $i\text{-}1$ is equal to the number of geodetic sets of P^*_n with cardinality i.

Case (ii): If $g_e(P^*_{n-1}, i-1) = \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ then by theorem 3.3 (ii), we have $g_e(P^*_n, i) = \left\{X \cup \{2n\} / X \in g_e(P^*_n - \{2n\}, i-1)\right\}$. In this case the number of geodetic sets of P^*_n with cardinality i is equal to the number of geodetic sets of $P^*_n - \{2n\}$ with cardinality $i\text{-}1$.

Case (iii): If $g_e(P^*_{n-1}, i-1) \neq \phi$ and $g_e(P^*_n - \{2n\}, i-1) \neq \phi$ then by theorem 3.3 (iii), we have $g_e(P_n^*, i) = A_1 \cup A_2$ where $A_1 = \{X \cup \{2n\} / X \in g_e(P_{n-1}^*, i-1)\}$ and $A_2 = \{X \cup \{2n\} / X \in g_e(P_n^* - \{2n\}, i-1)\}$. Therefore $|A_1| = |g_e(P_{n-1}^*, i-1)|$ and $|A_2| = |g_e(P_n^* - \{2n\}, i-1)|$. Since for every $X_1 \in A_1$ and $X_2 \in A_2$, we have $2n-1 \in X_2$ but $2n-1 \notin X_1$. Therefore $A_1 \cap A_2 = \phi$. Hence $|g_e(P_n^*, i)| = |g_e(P_{n-1}^*, i-1)| + |g_e(P_n^* - \{2n\}, i-1)|$.

Using theorem 2.6 and 3.5 we obtain the $|g_e(P_n^*, i)|$ and $|g_e(P_n^* - \{2n\}, i)|$ in Table 1 for $2 \le n \le 9$. There are interesting relationships between these numbers in Table 1.

Table 1

$\left|g_e\left(P_n^*, i\right)\right|$ *and* $\left|g_e\left(P_n^* - \{2n\}, i\right)\right|$

j	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*
P_2^*	1	2	1														
$P_3^* - \{6\}$	0	1	2	1													
P_3^*	0	1	3	3	1												
$P_4^* - \{8\}$	0	0	1	3	3	1											
P_4^*	0	0	1	4	6	4	1										
$P_5^* - \{10\}$	0	0	0	1	4	6	4	1									
P_5^*	0	0	0	1	5	10	10	5	1								
$P_6^* - \{12\}$	0	0	0	0	1	5	10	10	5	1							
P_6^*	0	0	0	0	1	6	15	20	15	6	1						
$P_7^* - \{14\}$	0	0	0	0	0	1	6	15	20	15	6	1					
P_7^*	0	0	0	0	0	1	7	21	35	35	21	7	1				
$P_8^* - \{16\}$	0	0	0	0	0	0	1	7	21	35	35	21	7	1			
P_8^*	0	0	0	0	0	0	1	8	28	56	70	56	28	8	1		
$P_9^* - \{18\}$	0	0	0	0	0	0	0	1	8	28	56	70	56	28	8	1	
P_9^*	0	0	0	0	0	0	0	1	9	36	84	126	126	84	36	9	1

4. GEODETIC POLYNOMIAL OF CENTIPEDES AND $P_n^* - \{2n\}$

In this section, we investigate the geodetic polynomial of centipedes and $P_n^* - \{2n\}$. Let $g_e(P_{n-1}^*, i)$ and $g_e(P_n^* - \{2n\}, i)$ be the family of geodetic sets of centipedes and $P_n^* - \{2n\}$ with cardinality i. Then the geodetic polynomials are defined as follows.

$$g(P_n^*, x) = \sum_{i=n}^{2n} |g_e(P_n^*, i)| x^i \quad and \quad g(P_n^* - \{2n\}, x) = \sum_{i=n}^{2n-1} |g_e(P_n^* - \{2n\}, i)| x^i$$

Where $\left|g_e\left(P_n^*, i\right)\right|$ be the number of geodetic sets of centipedes with cardinality i and $\left|g_e\left(P_n^* - \{2n\}, i\right)\right|$ be the number of geodetic sets of $P_n^* - \{2n\}$ with cardinality i.

Theorem 4.1:

(i) For every $n \geq 3$, $g(P_n^*, x) = x[g(P_{n-1}^*, x) + g(P_n^* - \{2n\}, x)]$ with the initial values $g(P_2^*, x) = x^2 + 2x^3 + x^4$ and $g(P_3^* - \{6\}, x) = x^3 + 2x^4 + x^5$.

(ii) For every $n \geq 4$, $g(P_n^* - \{2n\}, x) = x^2[g(P_{n-2}^*, x) + g(P_{n-1}^* - \{2n-2\}, x)]$ with the initial values $g(P_2^*, x) = x^2 + 2x^3 + x^4$ and $g(P_3^* - \{6\}, x) = x^3 + 2x^4 + x^5$.

(iii) For every $n \geq 2$, $g(P_n^*, x) = x^n(1+x)^n$

(iv) For every $n \geq 3$, $(P_n^* - \{2n\}, x) = x^n(1+x)^{n-1}$

Proof:

(i) By theorem 3.5, $|g_e(P_n^*, i)| = |g_e(P_{n-1}^*, i-1)| + |g_e(P_n^* - \{2n\}, i-1)|$

$$\sum_{i=n}^{2n} |g_e(P_n^*, i)| x^i = \sum_{i=n}^{2n} |g_e(P_{n-1}^*, i-1)| x^i + \sum_{i=n}^{2n} |g_e(P_n^* - \{2n\}, i-1)| x^i$$

$$g(P_n^*, x) = x\left[\sum_{i=n}^{2n} |g_e(P_{n-1}^*, i-1)| x^{i-1} + \sum_{i=n}^{2n} |g_e(P_n^* - \{2n\}, i-1)| x^{i-1}\right]$$

$$= x[g(P_{n-1}^*, x) + g(P_n^* - \{2n\}, x)]$$

(ii) By theorem 2.6,

$$|g_e(P_n^* - \{2n\}, i)| = |g_e(P_{n-2}^*, i-2)| + |g_e(P_{n-1}^* - \{2n-2\}, i-2)|$$

$$\sum_{i=n}^{2n-1} |g_e(P_n^* - \{2n\}, i)| x^i = \sum_{i=n}^{2n-1} |g_e(P_{n-2}^*, i-2)| x^i + \sum_{i=n}^{2n-1} |g_e(P_{n-1}^* - \{2n-2\}, i-2)| x^i$$

$$g(P_n^* - \{2n\}, x) = x^2\left[\sum_{i=n}^{2n-1} |g_e(P_{n-2}^*, i-2)| x^{i-2} + \sum_{i=n}^{2n-1} |g_e(P_{n-1}^* - \{2n-2\}, i-2)| x^{i-2}\right]$$

$$= x^2[g(P_{n-2}^*, x) + g(P_{n-1}^* - \{2n-2\}, x)].$$

(iii) By induction on n. The result is true for n=2, because $g(P_2^*, x) = x^2(1+x)^2 = x^2 + 2x^3 + x^4$. Assume that the result is true for all natural numbers less than n. we prove the result for n. We have $g(P_{n-1}^*, x) = x^{n-1}(1+x)^{n-1}$.

Now
$$\begin{aligned} g(P_n^*, x) &= x[g(P_{n-1}^*, x) + g(P_n^* - \{2n\}, x)] \\ &= x[x^{n-1}(1+x)^{n-1} + x^n(1+x)^{n-1}] \\ &= x^n(1+x)^{n-1} + x^{n+1}(1+x)^{n-1} \\ &= x^n(1+x)^{n-1}[1+x] \\ &= x^n[1+x]^n. \end{aligned}$$

(iv) By induction on n. The result is true for n=3, because $g(P_3^* - \{6\}, x) = x^3(1+x)^2 = x^3 + 2x^4 + x^5$. Assume that the result is true for all natural numbers less than n. We prove the result for n, we have $g(P_{n-1}^* - \{2n-2\}, x) = x^{n-1}(1+x)^{n-2}$

Now
$$\begin{aligned} g(P_n^* - \{2n\}, x) &= x^2[g(P_{n-2}^*, x) + g(P_{n-1}^* - \{2n-2\}, x)] \\ &= x^2[x^{n-2}(1+x)^{n-2} + x^{n-1}(1+x)^{n-2}] \end{aligned}$$

$$= x^n(1+x)^{n-2} + x^{n+1}(1+x)^{n-2}]$$
$$= x^n(1+x)^{n-2}[1+x]$$
$$= x^n[1+x]^{n-1}. \quad \square$$

Theorem 4.2: Suppose that $n \geq 2$. Then for every $n \leq i \leq 2n$, $|g_e(P_n^*, i)| = \binom{n}{i-n}$ and for every $n \leq i \leq 2n-1$, $|g_e(P_n^* - \{2n\}, i)| = \binom{n-1}{i-n}$

Proof: We shall prove both equalities together by induction on n. We have the result for $n = 2$ from Table 1. Now suppose the results are true for all natural numbers less than n, and prove them for n. By theorem 2.6 and induction hypothesis we have

$$|g_e(P_n^* - \{2n\}, i)| = |g_e(P_{n-2}^*, i-2)| + |g_e(P_{n-1}^* - \{2n-2\}, i-2)|$$
$$= \binom{n-2}{i-n} + \binom{n-2}{i-n-1}$$
$$= \binom{n-1}{i-n}$$

Now by theorem 3.5, we have $|g_e(P_n^*, i)| = |g_e(P_{n-1}^*, i-1)| + |g_e(P_n^* - \{2n\}, i-1)|$

$$= \binom{n-1}{i-n} + \binom{n-1}{i-n-1}$$
$$= \binom{n}{i-n}$$

Therefore, we have the result.

By theorem 4.2 we can obtain many properties of $|g_e(P_n^*, i)|$ and $|g_e(P_n^* - \{2n\}, i)|$. We state some of these properties in the following corollary.

Corollary 4.3: The following properties hold for coefficients of $g(P_n^*, x)$ and $g(P_n^* - \{2n\}, x)$.

(i) $|g_e(P_n^*, 2n)| = 1$, $|g_e(P_n^*, n)| = 1$ for every $n \geq 2$

(ii) $|g_e(P_n^*, 2n-1)| = n$, $|g_e(P_n^*, n+1)| = n$ for every $n \geq 2$

(iii) $|g_e(P_n^*, 2n-2)| = \frac{n(n-1)}{2}$, $|g_e(P_n^*, n+2)| = \frac{n(n-1)}{2}$ for every $n \geq 2$

(iv) $|g_e(P_n^*, 2n-3)| = \frac{n(n-1)(n-2)}{6}$, $|g_e(P_n^*, n+3)| = \frac{n(n-1)(n-2)}{6}$ for every $n \geq 2$

(v) $|g_e(P_n^* - \{2n\}, 2n-1)| = 1$, $|g_e(P_n^* - \{2n\}, n)| = 1$,

(vi) $|g_e(P_n^* - \{2n\}, 2n-2)| = n-1$, $|g_e(P_n^* - \{2n\}, n+1)| = n-1$,

(vii) $|g_e(P_n^* - \{2n\}, 2n-3)| = \frac{(n-1)(n-2)}{2}$,

$|g_e(P_n^* - \{2n\}, n+2)| = \frac{(n-1)(n-2)}{2}$

(viii) $|g_e(P_n^* - \{2n\}, 2n-4)| = \frac{(n-1)(n-2)(n-3)}{6}$,

$$\left|g_e(P_n^* - \{2n\}, n+3)\right| = \frac{(n-1)(n-2)(n-3)}{6}$$

(ix) If $S_n = \sum_{j=n}^{2n} \left|g_e(P_n^*, j)\right|$ then for every $n \geq 3$, $S_n = 2(S_{n-1})$ with the initial value $S_2 = 4$.

(x) If $S_n = \sum_{i=n}^{2n-1} \left|g_e(P_n^* - \{2n\}, i)\right|$ then for every $n \geq 4$, $S_n = 2(S_{n-1})$

Proof: The properties (i) to (viii) hold, by theorem 4.2

(ix) $S_n = \sum_{j=n}^{2n} \binom{n}{i-n}$

$= \sum_{i=0}^{n} nc_i = 2^n = 2 \times 2^{n-1} = 2S_{n-1}$.

(x) If $S_n = \sum_{i=n}^{2n-1} \left|g_e(P_n^* - \{2n\}, i)\right|$ then for every $n \geq 4$, $S_n = 2\,(S_{n-1})$

$S_n = \sum_{i=n}^{2n} \binom{n}{i-n}$

$= \binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{n-1}$

$= (1+1)^{n-1} = 2^{n-1} = 2 \times 2^{n-2} = 2S_{n-1}$.

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