International Journal of Mathematical Archive-3(5), 2012, 1833-1837
IJMA Available online through www.ijma.info ISSN 2229 – 5046

GENERALIZED ENTIRE METHOD OF SUMMATION

J. K. Srivastava* & Yogendra Yadav

Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur, India

(Received on: 20-04-12; Accepted on: 14-05-12)

ABSTRACT

In this paper, our aim is to extend the classical entire method of summation involving complex entire sequences to entire method of summation consisting of bounded linear operators involving Banach space valued entire sequences.

Keywords and Phrases: *Sequence space, Matrix transformation, Entire method.*

2010 AMS Subject Classification: *46A45.*

---

1. INTRODUCTION AND PRELIMENARIES

The concept of entire method of summation has been introduced by Brown [1] where he obtained a necessary and sufficient condition that an infinite matrix $A = (a_{n,k})$ be an entire method. Later on Fricke and Powell [2] also proved the necessity and sufficiency of a different condition which ensures that the matrix A is an entire method.

Let $\overline{x} = (x_k)_{k=0}^{\infty}$ be a sequence of complex numbers. The sequence $\overline{x}$ is entire $(\overline{x} \in \xi)$ provided $\sum_{k=0}^{\infty} |x_k| q^k < \infty$ for every positive integer *q*. The infinite matrix $A = (a_{n,k})$ is an entire method provided the $A-$transform of $\overline{x} \in \xi$ (written $A(\overline{x})$) is an entire sequence, i.e., the sequence $\overline{y} = (y_k)_{k=0}^{\infty} \in \xi$ where $y_n = \sum_{k=0}^{\infty} a_{n,k} x_k$.

Let *X* and *Y* be Banach spaces over the field *C* of complex numbers and $B(X,Y)$ be the set of all bounded linear operators from *X* into *Y* with usual operator norm $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| = 1\}$.

Let $A = (A_{nk}),\ (n,k = 0,1,2,3,...)$ be an infinite matrix of linear operators A_{nk} on a Banach space *X* into the Banach space *Y*. Let $\overline{x} = (x_k)_{k=0}^{\infty}$, $x_k \in X$, $k \geq 0$ be a sequence. Then for the classes of *X*–valued sequences, *E(X)* and *Y*–valued sequences *F(Y),* we define the matrix class *(E(X), F(Y))* by saying that $A = (A_{nk}) \in (E(X), F(Y))$ if for every $\overline{x} = (x_k) \in E(X)$, $y_n = A_n(\overline{x}) = \sum_{k=0}^{\infty} A_{nk} x_k$ converges in the norm of *Y* for each *n*, and the sequence $\overline{y} = (y_n) = (A_n(\overline{x}))$ belongs to *F(Y).* In such a case $\overline{y} = A\overline{x}$ is called the *A* transform of $\overline{x}$. We shall need the following form of Banach Steinhaus theorem:

Theorem 1.1: Let *X* be a Banach space. *Y* a normed space and let $(A_i)_{i \in I}$ be a family of bounded linear operators from *X* into *Y*. Suppose that for each $x \in X$ the family $(A_i(x))$ is bounded in *Y*. Then there exists a constant *M > 0* such that $\| A_i(x) \| \leq M \| x \|$ for all $x \in X$ and $i \in I$, (i.e., the family of real numbers $(\| A_i \|)_{i \in I}$ is bounded) (see [4]).

2. OPERATOR VERSION OF ENTIRE METHOD

Corresponding to entire sequence of complex number, we introduce here the Banach space valued entire sequence.

Corresponding author: J. K. Srivastava*
Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur, India

Definition: A sequence $\overline{x}=(x_k)_{k=0}^{\infty}$ in X is called an entire sequence if for every positive integer q, $\sum_{k=0}^{\infty}\|x_k\|q^k<\infty$.

Following the complex case we shall call the Infinite matrix $A=(A_{nk})$, $A_{nk}\in B(X,Y)$, $n,k=0,1,2,3,...,\infty$, is an entire method provided the $A-$transform of $\overline{x}\in\xi(X)$ (written $\overline{y}=A(\overline{x})$) is an entire sequence in Y. Such type of matrices of operators between various Banach space valued sequence spaces have been investigated by several workers for instance see Maddox [3], Robinson [5], Srivastava and Srivastava [6].

In order to characterize the class $(\xi(X),\xi(Y))$ (i.e. Theorem 2.3), we prove the following two lemmas:

Lemma 2.1: If $A=(A_{nk})$ is an entire method then

(a) for every integer $q>0$ and each fixed $k=0,1,2,...$ $\lim_{n\to\infty}q^nA_{nk}x=0$ for every $x\in X$, and

(b) for each $n=0,1,...$ there exists an integer $p_n>0$ such that $\|A_{nk}\|\le p_n^{k+1}$ for each $k=0,1,...$

Proof: (a) Let x be a fixed vector in X. For each $k\ge 0$ consider $\delta_k(x)=(0,0,...,0,x,0,...)$ where x is at k[th] place. Clearly $\delta_k(x)\in\xi(X)$ and so $A(\delta_k(x))\in\xi(Y)$ as A is an entire method. Therefore for an integer $q>0$

$$\sum_{n=0}^{\infty}\|y_n\|q^n<\infty$$

and so

$$\|A_{nk}x\|q^n=\|y_n\|q^n\to 0,\text{as } n\to\infty.$$

Hence

$$q^nA_{nk}x\to 0\,\text{as}\,n\to\infty,\text{fore ach}\,k\ge 0.$$

(b) Suppose (b) does not hold. Then there exists a non-negative integer N such that for each integer $p>0$ there exists an integer $k_p\ge 0$ satisfying

$$\|A_{N,k_p}\|>p^{k_p+1}$$

So, for $p_1=1$ there exists k_1 such that $\|A_{N,k_1}\|>1^{k_1+1}=1$. In general, choose $p_m>p_{m-1},(m\ge 2)$ such that $\max\{\|A_{N,k}\|:k\le k_{m-1}\}<p_m$.

There exists $k_m>k_{m-1}$ such that $\|A_{N,k_m}\|>p_m^{k_m+1}$ and for each k_m we can find z_{k_m} such that $\|z_{k_m}\|=1$ and

$$\|A_{N,k_m}\|\ge\|A_{N,k_m}z_{k_m}\|>p_m^{k_m+1}\text{ for all } m\ge 1.$$

Now define the sequence $\overline{x}=(x_n)_{n=0}^{\infty}$ by

$$x_n=\begin{cases}p_m^{-(k_m+1)}z_{k_m} & \text{for}\,n=k_m,m=1,2,...,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\overline{x}\in\xi(X)$ since for each integer $q>0$, we have

$$\sum_{n=0}^{\infty}\|x_n\|q^n=\sum_{m=1}^{\infty}\|p_m^{-(k_m+1)}z_{k_m}\|q^{k_m}=\sum_{m=1}^{\infty}p_m^{-(k_m+1)}q^{k_m}<\infty.$$

Since A is an entire method, we have that $A(\overline{x})\in\xi(Y)$, however

$$\sum_{k=0}^{\infty}\|A_{N,k}\,x_k\|=\sum_{m=1}^{\infty}\|A_{N,k_m}\,z_{k_m}\|\,p_m^{-(k_m+1)}\ge\sum_{m=1}^{\infty}\|A_{N,k_m}\|\,p_m^{-(k_m+1)}=\sum_{m=1}^{\infty}1=\infty.$$

Lemma 2.2: If $A=(A_{nk})$ has properties (a) and (b) and in addition (c) there exists an integer *q > 0* such that for each integer *p > 0* and each constant *M > 0* there exists integers *n, k*, where $\|A_{n,k}\| q^n > p^k M$ then for a given integer *p > 0*, constant *M > 0*, and integers n_0, k_0 there exist integers $N > n_0$, $K > k_0$ such that

$$\|A_{N,K}\| q^N > p^K M.$$

Proof: By (a) $\lim_{n\to\infty} q^n A_{nk} x = 0$ for each *k = 0, 1, 2, ...*, then there exist $B_k(x)$ for $k \ge 0$ such that $q^n \|A_{nk} x\| \le B_k(x)$ for all $n \ge 0$ and for each $x \in X$.

For each $k \ge 0$, $(q^n A_{nk})_{n=0}^{\infty}$ is pointwise bounded on *X* and therefore by Banach-Steinhaus Theorem $(q^n A_{nk})_{n=0}^{\infty}$ is uniformly bounded i.e., there exists B_k such that $q^n \|A_{nk}\| \le B_k$ for all $n \ge 0$ and for each $k \ge 0$.

Let $B = \max\{1, B_0, B_1, ..., B_{k_0}\}$. By (b) for every $n \ge 0$ there exist p_n such that $\|A_{nk}\| \le p_n^{k+1}$ for every $k \ge 0$.

Take $P = \max\{1, p_0, p_1, ..., p_{n0}\}$. Then $\|A_{n,k}\| \le P^{k+1}$ for each $k \ge 0$ but $n = 0,1,...n_0$. Also $\|A_{n,k}\| q^n < B$ for all $n \ge 0$ and $k = 0,1,\cdots k_0$.

Therefore $\|A_{nk}\| q^n \le BP^{k+1} q^{n_0}$ for every $k = 0,1,....,k_0$ or $n = 0,1,...n_0$.

By (c) taking $M = BPq^{n_0}$, and *P = p*, we get *N* and *K* such that $\|A_{N,K}\| q^N > p^K M$. Clearly $N > n_0$ and $K > k_0$ because for $k = 0,1,...k_0$ or $n = 0,1,...,n_0$, we have reverse inequality.

Theorem 2.3: A matrix $A=(A_{nk})$ is an entire method if and only if for each integer *q > 0* there exists an integer *p = p(q) > 0* and a constant *M = M(q) > 0* such that

$$\|A_{n,k}\| q^n \le p^k M, \qquad \text{for all } n,k = 0,1,...$$

Proof: Let $A=(A_{nk})$ be entire method, $\overline{x} \in \xi(X)$, $\overline{y} = A(\overline{x})$ and *q > 0* be an arbitrary fixed integer. We have

$$\sum_{n=0}^{\infty} \| y_n \| q^n \le \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \| A_{nk} \| \| x_k \| \right) q^n$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \sum_{k=0}^{\infty} \| A_{nk} \| (2q)^n \| x_k \|$$

there exists $p = p(2q) > 0$ and $M = M(2q) > 0$ such that

$$\|A_{nk}\| (2q)^n \le p^k M, \qquad \text{for all } n,k = 0,1,....$$

since $\overline{x} \in \xi(X)$, i.e. $\overline{y} = A(\overline{x}) \in \xi(Y)$. Therefore $\sum_{n=0}^{\infty} \| y_n \| q^n \le 2M \sum_{k=0}^{\infty} \| x_k \| p^k < \infty$.

For converse suppose there exists an integer *q > 0* such that for each integer *p > 0* and each constant *M > 0* there exist integers *n, k* where $\|A_{nk}\| q^n > p^K M$. By Lemma 2.1(b) choose a sequence of positive integers $(p_n)_{n=0}^{\infty}$ such that

(2.1) $$\|A_{nk}\| \le p_n^{k+1} \text{ for all } k = 0,1,...$$

Let $\overline{p}_n = \max\{p_i : i = 0,1,...,n\}$. We now construct the sequence $\{A_{n_j k_j}\}_{j=1}^{\infty}$ as follows:

Choose n_1, k_1 such that $A_{n_1 k_1} \neq 0$. Suppose that $n_1,...,n_j; k_1,...,k_j,\ j \geq 1$ have been choosen. By Lemma 2.1 (a) given

$$\varepsilon = (8(j+1))^{-1} \min\{\| A_{n_t k_t} \| : 1 \leq t \leq j\} > 0$$

there exists $\overline{n}_j = \overline{n}_j(\varepsilon, k_j)$ such that $\| A_{nk} \| q^n < \varepsilon$ for all $n \geq \overline{n}_j, k \leq k_j$.

By Lemma 2.2, there exists $n_{j+1} > \max\{\overline{n}_j, n_j\}$ and $k_{j+1} > k_j$ such that

$$\| A_{n_{j+1}, k_{j+1}} \| q^{n_{j+1}} > q^{n_j} [16(k_j + 1)\, \overline{p}_{n_j}]^{k_{j+1}+1}$$

So, we have a sequence $\{A_{n_j, k_j}\}_{j=1}^{\infty}$ such that

(2.2) $$\| A_{n,k_t} \| q^n \leq (8j)^{-1} \| A_{n_t, k_t} \|, \qquad \text{for all } n > n_j; t < j$$

and

(2.3) $$\| A_{n_j, k_j} \| q^{n_j} > q^{n_{j-1}} (16(k_{j-1} + 1)\, \overline{p}_{n_{j-1}})^{k_j + 1} \quad \text{for } j \geq 2.$$

Define the sequence $x = (x_m)_{m=1}^{\infty}$ by

$$x_m = \begin{cases} \| q^{n_j} A_{n_j k_j} \|^{-1} z_j & \text{for } m = k_j \ (j = 1,2,3,...) \\ 0, & \text{otherwise.} \end{cases}$$

By (2.3), it follows that

$$\| x_{k_{j+1}} \| < [q^{n_j} (16(k_j + 1)\, \overline{p}_{n_j})^{k_{j+1}+1}]^{-1} \leq (k_j + 1)^{-(k_j+1)},\ j \geq 1.$$

Therefore $\| x_n \| \leq \left(\frac{1}{n}\right)^n$ $(n \geq 1)$ and hence $\overline{x} \in \xi(X)$. So, $\overline{y} = A(\overline{x}) \in \xi(Y)$. Now

(2.4) $$\| y_{n_j} \| = \| \sum_{k=0}^{\infty} A_{n_j k} x_k \|$$

$$\geq \| A_{n_j, k_j} \| \| x_{k_j} \| - \sum_{t=1}^{J-1} \| A_{n_j k_t} \| \| x_{k_t} \| - \sum_{t=j+1}^{\infty} \| A_{n_j k_t} \| \| x_{k_t} \|$$

From (2.2), we have

(2.5) $$\sum_{t=1}^{j-1} \| A_{n_j, k_t} \| \| x_{k_t} \| \leq (8jq^{n_j})^{-1} \sum_{t=1}^{j-1} \| A_{n_t k_t} \| \| x_{k_t} \| \leq (8q^{n_j})^{-1}$$

and from (2.1) and (2.3), we have that

(2.6) $$\sum_{t=J+1}^{\infty} \| A_{n_j k_t} \| \| x_{k_t} \| \leq \sum_{t=j+1}^{\infty} (\overline{p}_{n_j})^{k_t+1} \| x_{k_t} \|$$

$$\leq \sum_{t=j+1}^{\infty} (\overline{p}_{n_j})^{k_t+1} [q^{n_t - 1} (16\, \overline{p}_{n_t - 1} (k_{t-1} + 1)^{k_t+1}]^{-1}$$

$$\leq \sum_{t=j+1}^{\infty} [q^{n_t - 1} (16(k_{t-1} + 1))^{k_t + 1}]^{-1}$$

$$\leq q^{-n_j} \sum_{t=j+1}^{\infty} 16^{-(k_t+1)} \leq (8q^{n_j})^{-1}$$

also

(2.7)
$$\begin{aligned}\| A_{n_jk_j} x_{k_j} \| &= \| A_{n_jk_j} (\| q^{n_j} A_{n_jk_j} \|)^{-1} z_j \| \\ &= q^{-n_j} \| A_{n_jk_j} \|^{-1} \| A_{n_jk_j} z_j \| \\ &> q^{-n_j} \| A_{n_jk_j} \|^{-1} \frac{1}{2} \| A_{n_jk_j} \| \\ &= \frac{1}{2} q^{-n_j}\end{aligned}$$

So, from (2.4), (2.5), (2.6) and (2.7), it follows that

$$\begin{aligned}\| y_{n_j} \| &\geq \frac{1}{2} q^{-n_j} - \frac{1}{8} q^{-n_j} - \frac{1}{8} q^{-n_j} \\ &= \frac{1}{4} q^{-n_j}\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} \| y_n \| q^n \geq \sum_{n=0}^{\infty} \frac{1}{4}$, which diverges, i.e. $\bar{y} = A(\bar{x}) \notin \xi(Y)$. This contradicts the fact that *A* is an entire method. This completes the proof.

REFERENCES

[1] Brown, H. I.: *A note on type P methods of summation*, Compositio Math., **22** (1970), 23-28.

[2] Fricke, G. and Powell, R. E.: *A theorem on entire method of summation*, Compositio Math., 22(3) (1970), 253-259.

[3] Maddox, I. J.: *Infinite Matrices of Operators*, Lecture notes in Mathematics, **786**, Springer-Verlag (1980).

[4] Maddox, I. J.: *Elements of Functional Analysis*, Camb. Univ. Press. (1970).

[5] Robinson, A.: *On functional transformations and summability*, Proc. London Math. Soc., 52 (1950), 132-160.

[6] Srivastava, J. K. and Srivastava, B. K.: *Generalized sequence spaces* $c_0(X, \lambda, p)$, Indian J. Pure and Appl. Math., 27(1) (1996), 73-84.

Source of support: Nil, Conflict of interest: None Declared