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A NOTE ON COHEN-MACAULAY ASSOCIATED GRADED NEAR RINGS

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ABSTRACT

In thispaper we mainly discussed the characterization of Cohen-Macaulay Rees algebras about associated graded near rings.

Keywords: *Near -ring, Graded near ring, Rees algebra, Local Near-rings, superficial element.*

1. INTRODUCTION

In this paper we will show that normal ideals, while not always producing Cohen – Macaulay Rees algebras, do often yield depth information about associated graded near rings. A corollary will provide a 2 – dimensional version of Grauert – Riemenchneider vanishing. We will need to use concept of a superficial element. Recall that if (N, m) is a local Near ring and I is an ideal of N, then an element $x \in I \setminus I^2$ is said to be superficial (of order one) for I. .If there is a positive integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all n>c. If N/m is infinite then superficial elements exist for any ideal of N. An important property of a superficial element of I is the following; if x is superficial for I, and is also an n– regular element, then ($(I^n : x) = I^{n-1}$ for all n>>0. This follows by an application of the Artin – Rees lemma.

2. PRELIMINARIES

In this section we give the definitions and examples related to this topic to the next sections.

2.1 Definition: A nonempty set N is said to be a right near-ring with two binary operations '+'and '.' If
i) (N, +) is a group (not necessarily abelian)
ii) (N, .) is a semi group and
iii) (x + y)z = xz + yz for all x ,y ,z $\in$ N

2.2. Example: Let Z be the set of positive, negative integers with '0', then (z, +,) is a near ring with usual addition and multiplication.

2.3 Definition: A ring R is called graded (z-graded)if there exists a family of sub groups $\{R_n\}_{n\in Z}$ of R such that

1. $R = \oplus_n R_n$ (as abelian groups)
2. $R_n.R_m \subseteq R_{n+m}$ for all n, m $\in$ Z.

2.4. Example: Let R be a ring and $x_1, x_2, ...x_d$ indeterminates over R. For m= $(m_1, m_2, ...m_d) \in N^d$.

Let $x^m = x_1^{m_1} x_2^{m_2} ....x_d^{m_d}$. Then the polynomial ring S= R $[x_1, x_2, ...x_d]$is a graded ring where $S_n = \{ \sum_{m\in N^d} r_m x^m / r_m \in Randm_1 + ... + m_d = n\}$

2.5 Definition: A Near- ring Nis called graded near ring if there exists a family of sub groups $\{N_n\}_{n\in Z}$ of N such that

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1. $N = \oplus_n N_n (notnecessarilyabelian)$
2. $N_n . N_m \subseteq N_{n+m}$ for all n,m∈Z

2.6. Example: Every Polynomial near ring is a Graded Near ring with multiplicative identity 1.

2.7. Definition: Let (*R*, *M*, *k*) be a Noetherian local ring. (The notation means that the maximal ideal is *M* and the residue field is *k* = *R/M*.) If *d* is the dimension of *R*, then by the dimension theorem every generating set of *M* has at least *d* elements. If *M* does in fact have a generating set *S* of *d* elements, we say that *R* is *regular* and that *S* is a *regular system of parameters*

2.8 Example: If *R* has dimension 0, then *R* is regular iff {0}is a maximal ideal, in other words, If R has maximal ideal {0} iff *R* is a field.

2.9 Definition: A nearing N is local iff N has a unique maximal N-subgroup.

2.10 Example [Maxon [1]]: $M_{aff}(v)$

2.11Note: A local near ring has 0 and 1 as idempotants.

3. MAIN RESULTS

In this section we proved main results on local near rings.

Theorem 3.1: Let (N,m) be local Near ring and let I be a ideal of N. Assume I is normal, grade (I) ≥ 2, and I is integral over an ideal generated by an N – regular sequence. Then there exists n such that depth $(G(I^n)) \geq 2$.

Proof: By passing to N(x) = $N[x]_{m[x]}$ we may assume that N/m is infinite. In this case, if we choose a general element of I which has the same value as I on all the Rees valuations of I, we obtain that $I^n : x = I^{n-1}$ for all n, since $I^n : x$ will be contained in the integral closure of I^{n-1}.

Now choose $y \in I \setminus I^2$ such that the image of y in N/(x) is superficial for I/(x), y is an N/(x) – regular element, and {x,y} form part of a minimal generating set for a minimal reduction of I. Note that choosing y to be N/(x) – regular is where we require grade $(I) \geq 2$. There exists a positive integer b such that

$$((I^k : y) = (I^{k-1}, x) \; for \; all \, k \geq b \tag{1}$$

Now we have to show that $\{(x^b)', (y^b)'\}$ form a $N(I^b)$ - regular sequence.

Now $(x^b)'$ is a $G(I^b)$ - regular element.

Then to verify that

$$((I^{bn}, x^b) : y^b) = (I^{bn-b}, x^b) \; for \; all \; n \geq 2 \tag{2}$$

By our choice of x and y we may assume there is a minimal reduction J of I^b having a minimal generating set including $\{x^b, y^b\}$. In other words $J = (x^b, y^b, z_3, ... z_3)$ is a minimal reduction of I^b, and because I is integral over an ideal generated by a regular sequence $\{x^b, y^b, z_3, ... z_3\}$ must itself be a regular sequence. To prove (2) we will handle the cases n=2 and $n \geq 3$ separately. Suppose first that n=2. By the theorem of Itoh ([7]], $J \cap I^{2b} = JI^b$ (This uses that $I^b and I^{2b}$ are integrally closed)In fact ,we claim that

$$(x^b, y^b) \cap I^{2b} = (x^b, y^b) I^b \tag{3}$$

To prove this claim it suffices to show that if $\{z_1,...,z_s\}$ is an N – regular sequence contained in I^b such that

$$(z_1,...,z_i)\cap I^{2b} = (z_1,...,z_i)\cap I^b$$

For some i, $2 \le i \le s$ then

$$(z_1,...,z_{i-1})\cap I^{2b} = (z_1,...,z_{i-1})\cap I^b$$

Let $u \in (z_1,...,z_{i-1})\cap I^{2b}$ .Then

$$u \in (z_1,...,z_i)\cap I^{2b} = (z_1,...,z_i)\cap I^b$$

Hence we may write

$$u = \sum_{j=1}^{i-1} u_j z_j = \sum_{j=1}^{i} v_j z_j \text{ for some } u_j \in N, v_j \in I^b \tag{4}$$

Therefore $v_i z_i \in (z_1,...,z_{i-1})$, thus $v_i \in (z_1,...,z_{i-1})$.

Write $v_i = \sum_{j=1}^{i-1} w_j z_j$ for some $w_j \in N$ and substituting into (4) Then, we get

$$\sum_{j=1}^{i-1}(u_j - v_j - w_j z_i) z_j = 0$$

A consequence, because $\{z_1,...,z_s\}$ is a regular sequence, is that $u_j - v_j - w_j z_i \in (z_1...,z_{i-1})$ for each j, $i \le j \le i-1$. In particular this implies that $u \in (z_1,...,z_{i-1})I^b$, completing the proof of (3).

To complete the proof of (2) for the case n = 2, let $cy^b \in (I^{2b}, x^b)$.

Then $cy^b - dx^b \in I^{2b}$ for some d$\in$N, hence $cy^b - dx^b \in (x^b, y^b)I^b$ by (3).

Therefore $c \in I^b = (x^b, I^b)$ because $\{x^b, y^b\}$ is a regular sequence.

Now assume $n \ge 3$. We claim that

$$(I^{bn-i}, x^{b-i}) : y^b \subseteq I^{bn-b-i} + x((I^{bn-i-1}, x^{b-i-1}) : y^b)$$

For $0 \le i \le b-1$. Let $cy^b \in (I^{bn-i}, x^{b-i})$ and write $cy^b - dx^{b-i} \in I^{bn-i}$ for some $d \in N$.

Then $cy^b \in (I^{bn-i}, x)$ so $c \in (I^{bn-b-i}, x)$ by (1). Thus we write $c - c_0 x \in I^{bn-b-i}$ for some N, which leads to $cy^b - c_0 x y^b \in I^{bn-i}$.

Hence $(c_0 y^b - dx^{b-i-1})x \in I^{bn-i}$, therefore $c_0 y^b - dx^{b-i-1} \in I^{bn-i-1}$.

In other words, $c_0 y^b \in (I^{bn-i-1}, x^{b-i-1})$.

$$c \in I^{bn-b-i} + (c_0 x) \subseteq I^{bn-b-i} + x((I^{bn-b-i}, x^{b-i-1}) : y^b)$$

Proving (4). By applying (4) successively to the i = 0 and i = 1 cases we obtain

$$((I^{bn}, x^b) : y^b) \subseteq I^{bn-2} + x^2((I^{bn-2}, x^{b-2}) : y^b)$$

The pattern is now easily detected and by continuing we eventually otain.

$$((I^{bn}, x^b) : y^b) \subseteq I^{bn-b} + x^{b-1}((I^{bn-b+1}, x) : y^b)$$

But using (1) b successive times, $((I^{bn-b+1}, x) : y^b) = (I^{bn-2b+1}, x)$ (this is where we require that $n \geq 3$).

Therefore

$((I^{bn}, x^b) : y^b) \subseteq (I^{bn-b}, x^b)$, Hence the proof is complete.

Note 3.2: By looking at some special cases of Theorem 3.1 we are able to provide some interesting corollaries. The first is a 2 – dimensional version of Grauert – Riemenshneider vanishing

Corollary 3.3: Let (N,m) be a 2 – dimensional Cohen – Macaulay local near ring, and let I be a normal m – primary ideal of N. Assume I has a 2 – generated minimal reduction (automatic if N/m is infinite). Then there exists n such that $G(I^n)$ is Cohen – Macaulay.

The next corollary gives some information about the coefficient $e_3(I)$ of the Hilbert – Samuel polynomial of a normal m – primary ideal I of a Cohen – Macaulay local ring. Recall that for any d – dimensional local ring (N,m) and m – primary ideal I of N, the length of N / I^n is given by a polynomial in n, of degree d,for all large values of n. The Hilbert – Samuel polynomial is by definition that polynomial, and it can be expressed in the form

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + . \ . + (-1)^{d-1} e_{d-1}(I) n + (-1)^d e_d(I)$$

Where the coefficients $e_1(I)$ are integers. Certain bounds on the $e_1(I)$'s are known to hold if N is assumed to be Cohen – Macaulay. In particular it holds that $0 \leq i \leq 2$ (see [9] for i=1 and [6] for i=2). In [6] Narita showed that it is possible for $e_3(I)$ to be negative, but Itoh proved that $e_3(I) \geq 0$ if I is assumed to be normal.

In fact Itoh proved a stronger result by considering the filtration $\{\overline{I^n}\}$ of integral closures of the powers of I. If (N,m) is assumed to be analytically unramified then the legth of $N / \overline{I^n}$ is a polynomial of degree d for large n and takes the form.

$$\overline{P}_I(n) = \overline{e}_0(I)\binom{n+d-1}{d} - \overline{e}_1(I)\binom{n+d-2}{d-1} + . \ . + (-1)^{d-1}\overline{e}_{d-1}(I)_n + (-1)^d \overline{e}_d(I)$$

Where the coefficients $\overline{e}_1(I)$ are integers (the normalized Hilbert coefficients). In [7] Itoh proved $\overline{e}_3(I) \geq 0$. Corollaries 3.9 and 3.10 below give new proofs of Itoh's results.

Corollary 3.4: Let (N, m) be an analytically unramified d – dimensional Cohen – Macaulay local near ring. If I is an m – primary ideal of N then $\overline{e}_3(I) \geq 0$.

Proof: By using the usual machinery [2] we may assume d=3. Because N is analytically unramified the right

$$\overline{N^2} \oplus \overline{I}t \oplus \overline{I^2}t \oplus ....$$

is noetherian , thus there exists a positive integer k such that $\overline{I^{kn}} = (\overline{I^k})^n$ for all $n \geq 1$(see [5] for example). Furthermore we know that $\overline{e}_3(I) = \overline{e}_3(I^k)$ because $\overline{P}_I(kn) = \overline{P}_{I^k}(n)$. Hence we may replace I with I^k and therefore assume I is normal. By applying Theorem 3.1 we obtain that depth $((G(I)) \geq 2$.

Corollary 3.5: Let (N,m) be a d – dimensional Cohen – Macaulay local near ring. If I is a normal m – primary ideal of N then $e_3(I) \geq 0$

Proof: As in (3.4) we may assume that dim(N)=3. The result implies that $e_3(I) = e_3(I^k)$ for all k $k \geq 1$. Therefore we may assume that depth $(G(I)) \geq 2$ by using Theorem 3.1. The statement now follows from above corollary 3.5

We now give an example showing that Corollary 3.4 does not extend to higher dimensions. The existence of such an example, over C, was proved in cor 3.3. The purpose here is to provide an explicit example, in the sense of giving actual equations. The idea behind this construction is useful: to find an m – primary normal ideal having specified properties, one first finds a height two normal prime ideal p with the required properties (often a much easier task), then the takes the integral closure of $p + m^n$ for large n. This ideal 'should' have much the same properties as p. However, to make this philosophy work in practice, we depend on the following lemma.

Lemma 3.6: Let N be graded Near ring with homogeneous maximal ideal m and such that N_0 is a field. Let Abe a homogeneous ideal of N generated by forms of the same degree d. If A is a normal ideal $A + m^{d+1}$ is also a normal ideal.

Proof: Let $f \in \overline{(A+m^{d+1})^k}$ be homogeneous of degree n. Then there are elements $c_i \in (A+m^{d+1})^{ki}$ such that $f^m + c_1 f^{m-1} + ... + c_{m-1} f + c_m = 0$. Note that $n \geq kd$. If n = kd then by considering the homogeneous part of the equation having degree mkd we may assume that c_i is homogeneous of degree kdi for each i, $1 \leq i \leq m$. Write $c_i = a_i + b_i$ where $a_i \in A^{ki}$ and $b_i \in m^{d+1}(A+m^{d+1})^{ki-1}$. Then $\deg(b_i) \geq kdi + 1$, thus $b_i = 0$. It follows that $f \in \overline{A^k}$ so that $f \in A^k$ by normality. In particular $f \in (A+m^{d+1})^k$. If $n \geq k(d+1)$ then $f \in m^{k(d+1)} \subseteq (A+m^{d+1})^k$. Assume that $kd < n < k(d+1)$ and write n=kd+j for some j, 0<j<k. By decomposing $(A+m^{d+1})^{ki}$ into the sum

$$A^{(k-j)i}(A+m^{d+1})^{ji} + (m^{d+1})^{ji+1}(A+m^{d+1})^{ki-ji-1}$$

We may express $c_i = a_i + b_i$ for some

$$a_i \in A^{(k-j)i}(A+m^{d+1})^{ji} \quad and \quad b_i \in (m^{d+1})^{ji+1}(A+m^{d+1})^{ki-ji-1}$$

As above we may assume that c_i is homogeneous, this time having degree i(kd+j). But $\deg(b_i) \geq i(kd+j)+1$, thus $b_i = 0$ *and* $c_i = a_i$. In particular $c_i \in A^{(k-j)i}$, therefore $f \in \overline{A^{k-j}}$. By the normality of A again, $f \in A^{k-j}$. This means that $f \in A^{k-j} \cap m^{kd+j}$. But by using that Nis graded and m is its homogeneous maximal ideal it holds that $A^{k-j} \cap m^{kd+j} = A^{k-j} m^{(d+1)j}$. Therefore $f \in (A+m^{d+1})^k$ and the proof of Lemma 3.6 is complete.

Theorem 3.7: Let k be a field of characteristic not 3. Set N = k[x,y,z]. Let $A = (x^4, y(y^3+z^3), z(y^3+z^3)), z(y^3+z^3))$ And set $I = A + m^5$,, where m=(x,y,z)N. Then

1) I is a height 3 normal ideal of N.
2) $G(I^n)$ is not Cohen – Macaulay for any $n \geq 1$.
3) If X denotes the blow up of I, then X is normal but $H^2(X, O_x) \neq 0$

Proof: Let $L = (x^4, y^3 + z^3)$. We first show that L is normal. The powers of L are unmixed since L is generated by a regular sequence. Further, L is generically normal (i.e. locally normal at its minimal primes). This follows since $y^3 + z^3$ will be reduced in characteristic not equal to 3, and then the minimal primes above L are exactly generated by the minimal primes over $y^3 + z^3$ together with the element x, and locally at each such prime L is generated by x^4 together with a regular parameter. But all such ideals are normal. It follows that L is normal.

By induction on i we claim that $L^i \cap m^{4i} = A^i$. It follows immediately from this claim that the powers of J are also integrally closed and so J is normal.

For i = 1 the equation is clear. Assume i > 1. Clearly $A^i \subseteq L^i \cap m^{4i}$, so we prove the other containment. Let $u \in L^i \cap m^{4i}$. Write $u = rx^{4i} + (y^3 + z^3)v$ for some $r \in N$ and $v \in L^{i-1}$. Then $(y^3 + z^3)v \in m^{4i}$ and so $v \in m^{4i-3} \subseteq m^{4(i-1)}$. Hence $v \in L^{i-1} \cap m^{4(i-1)} = A^{i-1}$. Since $v \in m^{4i-3}$, we even obtain that $v \in mA^{i-1}$. Finally we need only to observe that $(y^3 + z^3)v \in (y^3 + z^3)mA^{i-1} \subseteq A^i$.

Therefore I is a normal ideal by Lemma 3.6. For each $s \geq 1$ we will prove that $G(I^s)$ is not Cohen – Macaulay by showing that $G(I_m{}^s)$ is not Cohen – Macaulay. For this it suffices to prove that the reduction number of $I_m{}^s$ is at least 3 Further , it will be enough to find a single minimal reduction L_s of $I^s{}_m$ such that $L_s I^{2s}{}_m \neq I^{3s}{}_m$. This is because if $LI^{2s}{}_m = I_m{}^{3s}$ for some minimal reduction L of $I_m{}^s$ then $L \cap I_m{}^{(j+1)s} = LI_m{}^{js}$ *for* $j \geq 1$, and by the result of Huneke and Itoh ([6] or [10, Theorem [1], therefore $G(I_m{}^s)$ is Cohen – Macaulay A consequence of $G(I_m{}^s)$ being Cohen – Macaulay is that every minimal reduction of $I_m{}^s$ has reduction number 2.

We proceed to construct the ideals J_s. First observe that $J = (x^4, y(y^3 + z^3), z(y^3 + z^3))$ is a reduction of A (in fact, $JA^3 = A^4$). For convenience set $a = x^4, b = y(y^3 + z^3), c = z(y^3 + z^3)$ and $d = z^5$. We claim that $J_1 = (a, c, b + d)$ is a minimal reduction of I. To see this it suffices to show that J is a reduction of the ideal K = (a, b, c, d), because K is already a reduction of $I(KI^3 = I^4)$. Clearly $0 = zc^4 - b^3d - c^3d$

In particular this implies that $b^3 d \in J_1 k^3$. But $b^4 = b^3(b + d) - b^3 d$, therefore $b^4 \in J_1 K^3$. In other words, $k^4 \in J_1 K^3$ which prove the claim.

Now define $J_s = (a^s, c^s, (b + d)^s)$. Then J_s is a minimal reduction of I^s. To show that $J_s I^{2s} \neq I^{3s}$,observe that $x^{3s}(y^3 + z^3)^{3s} \notin J^s I^{2s}$

Because the multiplies of x contained in $J^s I^{2s}$ must be of the form x^j or x^{4s+j} for some j, $0 \leq j \leq 2s$. To finish the proof, observe that $J^s I_m{}^{2s} \neq I^{3s}{}_m$ because I is m – primary.

To prove the last statement, the following lemma is useful.

Lemma 3.8: Let N be a normal local domain whose completion is reduced. Let $f_1 .... f_d \in N$ and for each integer t > 0 let I_t be the ideal $(f^t{}_1, ..., f_d{}^t)$. Let $Y \rightarrow$ Spec(N) be obtained by blowing up I and normalizing. Then, for all sufficiently large t we have (with I = I_1): $H^{d-1}(O_Y) = \overline{I_t{}^d} / I_t(\overline{I^{d-1}{}_t})$.

In the notation of this lemma, to prove $H^2(X, O_X) \neq 0$, it suffices to prove that $\overline{I_t{}^3} / I_t(\overline{I^2{}_t}) \neq 0$ for all large t. Since $\overline{I_t{}^3} = I^{3t}$ by above, we need only to prove that $I^{3t} \neq J_t I^{2t}$, which we have done above.

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