

A NOTE ON NORMAL IDEALS IN REGULAR NEAR RINGS

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ABSTRACT

In this paper we study and characterize the cohen-macaulay property of the Rees algebras of a normal ideal of regular local near ring in the 3-dimensional case by assuming the ideal is 4-generated, has height 2, is u mixed and is generally a complete intersection.

Keywords: Ring, Near-ring, Regular near ring, Regular -local -ring, Regular local near ring, Rees algebra.

1. INTRODUCTION

This paper studies the depth of Rees algebras associated to normal ideals. If I is an ideal in a commutative Near ring N , the integral closure of I , denoted \bar{I} , is the set of elements $x \in N$ such that x satisfies an equation of the form $x^n + a_1 x^{n-1} + \dots + a_n = 0$ where $a_j \in I^j$ for $1 \leq j \leq n$. If $x \in \bar{I}$ we say x is integral over I . It is not difficult to prove that \bar{I} is an ideal. An ideal I of a commutative Noetherian near ring N is said to be normal if all of its powers are integrally closed. If N is an integrally closed domain then the normality of I is equivalent to the normality of the Rees algebra of N with respect to I , $N[It] = \bigoplus_{n \geq 0} I^n t^n$. This paper was originally motivated by trying to prove that normality of the Rees algebra implied the Cohen-Macaulay property of the Rees algebra for a particular class of ideals in regular local Near rings.

2. PRELIMINARIES

In this section we shall give the definitions and required examples related to the next section topics.

2.1 Definition: A non empty set R with two binary operations '+' and '.' Is said to be a ring if i) $(R, +)$ is a commutative ring ; ii) (R, \cdot) is a semi group
 iii) Distributive laws hold good.

2.2 Example: The set of all integers modulo m under addition and multiplication modulo m is a ring

2.3 Definition: A near ring N is said to be Regular ring if for each element $x \in N$ then there exists an element $y \in N$ such that $x = xyx$.

2.4 Example:

- (i) $M(\Gamma)$ and $M_0(\Gamma)$ are regular rings (Beidleman (10) NR Text)
- (ii) Constant rings
- (iii) Direct sum and product of fields.

2.5. Definition: A nonempty set N is said to be a Right near-ring with two binary operations '+' and '.' If

- i) $(N, +)$ is a group (not necessarily abelian)
- ii) (N, \cdot) is a semi group and
- iii) $(x + y)z = xz + yz$ for all $x, y, z \in N$

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2.6 Example: Let Z be the set of positive, negative integers with '0', then $(z, +, \cdot)$ is a near ring with usual addition and multiplication.

2.7 Definition: A near ring N is said to be Regular near ring if for each element $x \in N$ then there exists an element $y \in N$ such that $x = xyx$.

2.8 Example: (i) $M(\Gamma)$ and $M_0(\Gamma)$ are regular near rings (Beidleman (10) NR Text)

2.9 Definition: Let (R, M, k) be a Noetherian local ring. (The notation means that the maximal ideal is M and the residue field is $k = R/M$.) If d is the dimension of R , then by the dimension theorem every generating set of M has at least d elements. If M does in fact have a generating set S of d elements, we say that R is *regular* and that S is a *regular system of parameters*

2.10 Example: If R has dimension 0, then R is regular iff $\{0\}$ is a maximal ideal, in other words, iff R is a field.

2.11 Definition: A Local near ring which satisfy regular property then it is called a regular local near ring.

2.12 Definition: The Rees algebra of an ideal $I \subset R$ by definition a graded algebra satisfying

$$R(I) = R \oplus I \oplus I^2 \oplus \dots$$

3. NON COHEN – MACAULAY REES ALGEBRAS

Given a three dimensional regular local Near ring (N, m) and normal four – generated height two unmixed ideal I of N , it is natural to ask about the Cohen – Macaulayness of the Rees algebra $N[I]$. Assume further that I is generically a complete intersection (that is, I_p is a complete intersection for every prime ideal P minimal over I). Our main result of this section will characterize, in terms of a presentation matrix of I , when $N[I]$ is Cohen – Macaulay for such an ideal. It builds on work of Vasconcelos and Aberbach – Huneke where techniques for studying Rees algebras via presentation matrices were staged. The assumption that I is normal is not needed for the proof of our theorem however. Instead we need only the weaker condition that $(mI^2 : m) = I^2$. If I^2 is integrally closed then it satisfies this condition. For if $w \in (mI^2 : m)$ then $\omega m \subset mI^2$, thus $\omega \in \overline{I^2} = I^2$ by the determinant trick.

Theorem 3.1: Let (N, m) be a d -dimensional regular local near ring containing a field and I a height $d - 1$ unmixed ideal of N . Assume that I is generically a complete intersection, I has a $d - 1$ generated reduction, $\mu(I) = d + 1$, and $I^2 m : m = I^2$, but I^2 is not unmixed. Then there is a generating set $\{x_1, \dots, x_d\}$ for m , positive integers $m \geq n$, and a presentation matrix ϕ for I , such that $I_1 \phi = (x_2, \dots, x_d, x_1^n)$ and

$$\phi = \begin{pmatrix} x_2 & \dots & x_d & x_1^m & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \end{pmatrix}$$

Proof: By our assumption, $m \in \text{Ass}(N/I^2)$. In particular, $(I^2 : m) \neq I^2$. Choose an element $e \in (I^2 : m) \setminus I^2$. Because $(mI^2 : m) = I^2$, $em \not\subset mI^2$. Choose $x \in m \setminus m^2$ such that $ex \notin mI^2$. Expand $x = x_1$ to a minimal generating set $\{x_1, \dots, x_d\}$ for m . Using that $ex \in I^2 \setminus mI^2$ we may assume that $I = (p_1, \dots, p_{d+1})$ is a minimal generating set for I , and that $ex = ap_1^2 + p$ for some $p \in (p_2, \dots, p_{d+1})I$ and unit a . Set $ex_i = g_i \in I^2$. Consider the homomorphism

$$\psi : R[T_1, \dots, T_{d+1}] \rightarrow R[It]$$

Given by $T_i \rightarrow p_i t$. Let $F, G_i \in R[T_1, \dots, T_{d+1}]$ be homogeneous of degree 2 (in the T_i 's) such that $\psi(F) = pt^2$, $\psi(G_i) = g_i t^2$. By our choice of p we can assume that $F \in (T_2, \dots, T_{d+1})R[T_1, \dots, T_{d+1}]$. Let Q be the kernel of ψ ,

and let Q_j denote the ideal generated by the homogeneous elements of Q having degree at most j . The trivial relation $(ex)x_i = (ex_i)x$ forces $x_i(\alpha T_1^2 + P) - xG_i \in Q_2$. But $Q_2 = Q_1$ because I is syzygetic [8], thus there are linear homogeneous polynomials $A_i \in R[T_1, \dots, T_{d+1}]$ such that $x_i(\alpha T_1^2 + P) - xG_i = A_1 L_1 + \dots + A_s L_s$ where $Q_1 = (L_1 \dots L_s)$. The coefficient of T_1^2 must be $x_i - xb_{1i}$ for some $b_{1i} \in R$ and the coefficients of L_i all lie in m , therefore Q_1 contains polynomials of the form $(x_i - xb_{1i})T_1 + b_{2i}T_2 + \dots b_{si}T_s$. The existence of these linear polynomials implies that $(x_i - xb_{1i})x \in ((p_2, \dots, p_{d+1}) : p_1)$. Replacing x_i with $x_i - xb_{1i}x$ yields that $x_i \in ((p_2, \dots, p_{d+1}) : p_1)$. Therefore invertible row and column operations yield that ϕ may be reduced to the form described in (1). In particular, $I_1(\phi) = (x_2, \dots, x_d, x^n)$ for some $n \leq m$.

We are particularly interested in applying (3.1) to curves in 3 – space, where several of our assumptions automatically are valid.

Corollary 3.2. Let (N, m) be a 3 – dimensional regular local near ring containing a field and I a height 2 unmixed ideal of N . Assume that I is generically a complete intersection, I has a 3 – generated reduction, $\mu(I) = 4$, and $I^2 m : m = I^2$. There is a generating set $\{x, y, z\}$ for m , positive integers $m \geq n$, and a presentation matrix ϕ for I ,

$$\text{such that } I_1(\phi) = (y, z, x^n) \text{ and } \phi = \begin{pmatrix} y & z & x^m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Proof: This follows at once from Proposition 3.1 as soon as we observe that the assumption that I^2 is not unmixed is automatic in this case we first analyze to have precisely three generators by the Hilbert – Burch theorem.

We recall Vasconcelos' construction. That is, let ϕ be a 4 X 3 presentation matrix of $I = (p_1, p_2, p_3, p_4)$ and assume $I_1(\phi)$ (the ideal generated by the entries of ϕ is a complete intersection, generated by $\{a, b, c\}$. Consider the homomorphism

$$\psi : N[T_1, T_2, T_3, T_4] \rightarrow N[It]$$

given by $T_i \rightarrow p_i t$. The symmetric algebra of I is a complete intersection whose defining ideal is generated by three elements L_1, L_2 and L_3 . These elements satisfy the matrix equation

$$(L_1 \ L_2 \ L_3) = (T_1 \ T_2 \ T_3 \ T_4) \cdot \phi$$

Build an associated matrix $B(\phi)$ via the matrix equation

$$3.3 (T_1 \ T_2 \ T_3 \ T_4) \cdot \phi = (L_1 \ L_2 \ L_3) = (a \ b \ c) \cdot B(\phi)$$

We also define $C(\phi)$ to be the image of $B(\phi)$ in $k(T_1 \ T_2 \ T_3 \ T_4)$, i.e the matrix $B(\phi)$ reduced modulo the maximal ideal of N

Remark 3.4: We will need to perform various changes on the generators of I , the choice of presentation ϕ , and the matrix $B(\phi)$. It is convenient to record exactly what changes we will use and how they affect the rest of the data.

First, consider column operations upon $B(\phi)$. Let θ be a 3 by 3 invertible matrix with coefficients in R . Column operations upon $B(\phi)$ coming from the base ring R are obtained by replacing $B(\phi)$ by $B(\phi)\theta$. To preserve equation (3.3) we must also multiply ϕ by θ and then the corresponding equation (2.3) is valid. This only changes the generators for the syzygies of I , and not the chosen generating set. Otherwise stated, $B(\phi\theta) = B(\phi)\theta$.

Row operations on $B(\phi)$ coming from \sim Correspond to multiplying $B(\phi)$ by an invertible 3 by 3 matrix θ with coefficients in R on the left side of $B(\phi)$. To insure that (3.3) continues to hold, we must then multiply the matrix (a, b, c) by θ^{-1} on the right, basically changing the choice of generators for the ideal (a, b, c) .

Finally, we are free to change the chosen generators of I . In this case we replace ϕ by $\theta\phi$, with θ in this case a 4 by 4 invertible matrix with coefficients in N . If we change the corresponding T_i by multiplying on the right by θ^{-1} , then (3.3) is still valid without change to the complete intersection a, b, c or the matrix $B(\phi)$.

Vasconcelos proved that if I is a prime ideal such that $\det(C(\phi)) \neq 0$ then $N[It]$ is Cohen – Macaulay. Implicit in the work of Aberbach and Huneke is an improvement of this statement.

Proposition 3.5: Let (N, m) be three – dimensional regular local Near ring containing a field of characteristic not 2 and I a four – generated height two unmixed ideal of N which is generically a complete intersection. Assume further that I has a three – generated minimal reduction (automatic if N/m is infinite) and $I_1(\phi)$ is a complete intersection. Then $N[It]$ is Cohen – Macaulay if and only if $\det(C(\phi)) \neq 0$.

Proof: The proof of the forward direction is contained in the proof of the (1) implies (3) part of the argument given in [2]. Conversely, if $\det(B(\phi)) \notin mN[T_1, T_2, T_3, T_4]$ then we may assume (after changing the generators of I if necessary) that $\det(B(\phi)) = T_1^3 + A$ for some homogeneous (in the T_i 's) degree 3 polynomial A contained in $(T_2, T_3, T_4)N[T_1, T_2, T_3, T_4]$. Because the elements $\psi(f), \psi(g)$ and $\psi(h)$ vanish, $\psi(\det(B(\phi))) = 0$. Hence $p_1^3 \in (p_2, p_3, p_4)I^2$ which means I has reduction number two, therefore $N[It]$ is Cohen – Macaulay by [3].

The following theorem is the key theorem of this paper upon which the examples are based.

Theorem 3.6: Let (N, m) be a 3 – dimensional regular local ring containing a field of characteristic not 2 and I a height 2 unmixed ideal of N . Assume that I is generically a complete intersection, I has a 3 – generated reduction, $\mu(I) = 4$, and $I^2m : m = I^2$. Further assume that a generating set $\{x, y, z\}$ for m and a presentation matrix ϕ for I have been chosen as in (3.2). Then $R[It]$ is not C – M if and only if $I_1\phi = (y, z, x^n) = (u, v, w)$ and there is a presentation matrix θ for I of the form

$$\theta = \begin{pmatrix} v & w & 0 \\ u & 0 & w \\ 0 & u & -v \\ 0 & 0 & 0 \end{pmatrix} \mod mI_1(\theta)$$

Proof: If θ has the form described in (3.6) then it is easy to see that $\det(C(\theta)) = 0$. Therefore $R[It]$ is not Cohen – Macaulay by Proposition 3.5.

Assume $R[It]$ is not Cohen – Macaulay and that ϕ is a presentation matrix for I having the form prescribed in (3.2), that is

$$\phi = \begin{pmatrix} y & z & x^m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

Where $m \geq n$ and $I_1(\phi) = (y, z, x^n)$. By using invertible row operations we may also assume that m is the least power of x appearing in the last column of ϕ . We will analyze the matrix $B(\phi) = \begin{pmatrix} T_1 + A & B & C \\ D & T_1 + E & F \\ G & H & x^{m-n}(T_1 + K) \end{pmatrix},$

Where $A, B, \dots, K \in N[T_2, T_3, T_4]$. If $m = n$ the ϕ satisfies the row condition, leading to the contradiction that $N[It]$ is Cohen – Macaulay. Therefore we assume $m > n$. Let $C = C(\phi)$ denote the image of B modulo $mN[T_1, T_2, T_3, T_4]$, and use lower-case letters to denote images modulo $mN[T_1, T_2, T_3, T_4]$. Then

$$(3.7) \quad C(\phi) = \begin{pmatrix} t_1 + a & b & c \\ d & t_1 + e & f \\ g & h & 0 \end{pmatrix}.$$

Here $a, b, c, \dots, h \in k[t_1, t_2, t_3, t_4]$ are liner forms. In addition, the fact that $I_1(\phi) = (y, z, x^n)$ implies that either $g \neq 0$ or $h \neq 0$ (else the least pure power of x in $I_1(\phi)$ would be greater than n).

Since $R[It]$ is not Cohen – Macaulay the determinant of $C(\phi)$ is zero by Proposition 3.5, therefore

$$(3.8) \quad g(bf - ct_1 - ce) - h(ft_1 + af - cd) = 0$$

We consider two cases; either g and h are relatively prime or not. First suppose that g and h are relatively prime. Because $a, b, \dots, h \in k[t_2, t_3, t_4]$, (2.8) implies that $gc + hf = 0$. Therefore there exists an $\alpha \in k$ such that $c = -\alpha h$ and $f = \alpha g$. Substituting into (3.7) yields

$$C(\phi) = \begin{pmatrix} t_1 + a & b & -\alpha h \\ d & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}$$

Thus,

$$0 = \det(C) = \alpha(bg^2 + ghe - gha - dh^2)$$

Using that g and h are relatively prime we obtain that h divides b , hence $b = \beta h$ for some $\beta \in k$.

Substituting above and factoring out h implies that

$$\beta g^2 + ge - ga - dh = 0$$

Therefore g divides d , hence $d = \gamma g$ for some $\gamma \in k$.

Substituting and factoring our g implies that $\beta g + e - a - \gamma h$, therefore

$$e - \gamma h = a - \beta g$$

Further, after making the substitutions $b = \beta h$ and $d = \gamma g$, $C(\phi)$ takes the form

$$C(\phi) = \begin{pmatrix} t_1 + a & \beta h & -\alpha h \\ \gamma g & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}$$

By using row operations $C(\phi)$ may be reduced to

$$C(\phi) = \begin{pmatrix} t_1 + a - \beta g & 0 & -h \\ 0 & t_1 + e - \gamma h & g \\ g & h & 0 \end{pmatrix}$$

As in Remark 2.4, these row operations will change the choice of generators of the ideal (y, z, x^n) . Let us call the new generators u, v, w . Set $q = a - \beta g = e - \gamma h$. By the above calculation

$$C(\phi) = \begin{pmatrix} t_1 + q & 0 & -h \\ 0 & t_1 + q & g \\ g & h & 0 \end{pmatrix}.$$

Note that $\{t_1 + q, g, h\}$ are independent linear forms in $k[t_1, t_2, t_3, t_4]$ because g and h are relatively prime linear forms and do not involve t_1 . Therefore by changing variables we may replace $t_1 + q, g$ and h with t_1, t_2 and t_3 (respectively replace $T_1 + Q, G$ and H with T_1, T_2 and T_3). As in Remark 2.4 this change also will change our original choice of generators for I . Substituting this variable – change yields that

$$C(\phi) = \begin{pmatrix} t_1 & 0 & -t_3 \\ 0 & t_1 & t_2 \\ t_2 & t_3 & 0 \end{pmatrix}$$

And lifting back to $NN[T_1, T_2, T_3, T_4]$

$$B(\phi) = \begin{pmatrix} T_1 & 0 & -T_3 \\ 0 & T_1 & T_2 \\ T_2 & T_3 & 0 \end{pmatrix} \pmod{mN[T_1, \dots, T_4]}.$$

Finally, to see that the matrix ϕ may be chosen to have the form described in the statement of the theorem, use the matrix equation.

$$B(\phi) = (T_1 \ T_2 \ T_3 \ T_4)\phi$$

to rebuild ϕ .

This completes the proof of theorem 3.6 in the case that g and h are relatively prime.

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