International Journal of Mathematical Archive-3(5), 2012, 1789-1794
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A NOTE ON NORMAL IDEALS IN REGULAR NEAR RINGS

P. Jyothi*
Asst. Prof. in Mathematics, Laqshya Institute of Technology and Sciences, Khammam (AP)

T. V. Pradeep Kumar
Asst. Prof. In Mathematics, ANU college of engineering and Technology
Acharya Nagarjuna University (Guntur AP)

(Received on: 18-04-12; Accepted on: 09-05-12)

ABSTRACT

In this paper we study and characterize the cohen-macaulay property of the Rees algebras of a normal ideal of regular local near ring in the 3-dimensional case by assuming the ideal is 4-generated, has height2, is u mixed and is generally a complete intersection.

Keywords: *Ring, Near-ring, Regular near ring, Regular -local –ring, Regular local near ring, Rees algebra.*

---

1. INTRODUCTION

This paper studies the depth of Rees algebras associated to normal ideals .If I is an ideal in a commutative Near ring N, the integral closure of I, denoted $\overline{I}$,is the set of elements x $\in$N such that x satisfies an equation of the form $x^n + a_1 x^{n-1} + ... + a_n = 0$ where $a_j \in I^j$ for $1 \le j \le n$.If x$\in \overline{I}$ we say x is integral over I. It is not difficult to prove that $\overline{I}$ is an ideal. An ideal I of a commutative Noetherian near ring N is said to be normal if all of its powers are integrally closed. If Nis an integrally closed domain then the normality of I is equivalent to the normality of the Rees algebra of N with respect to I, N[It]= $\bigoplus_{n\ge 0} I^n t^n$ .This paper was originally motivated by trying to prove that normality of the Rees algebra implied the Cohen-Macaulay property of the Rees algebra for a particular class of ideals in regular local Near rings.

2. PRELIMINARIES

In this section we shall give the definitions and required examples related to the next section topics.

2.1Definition: A non empty set R with two binary operations '+'and '.' Is said to be a ring if i)(R,+)is a commutative ring ; ii)(R,.)is a semi group
iii) Distributive laws hold good.

2.2 Example: The set off all integers modulo m under addition and multiplication modulo m is a ring

2.3 Definition: A near ring N is said to be Regular ring if for each element x∈N then there exists an element y∈N such that x= xyx.

2.4 Example:
(i) M (Γ) and $M_0(\Gamma)$ are regular rings (Beidleman (10) NR Text)
(ii) Constant rings
(iii) Direct sum and product of fields.

2.5. Definition: A nonempty set N is said to be a Right near-ring with two binary operations '+'and '.' If
i) (N, +) is a group (not necessarily abelian)
ii) (N, .) is a semi group and
iii) (x + y)z= xz + yz for all x, y, z ∈ N

Corresponding author: P. Jyothi*
Asst.Prof. in Mathematics, Laqshya Institute of Technology and Sciences, Khammam (AP)

2.6 Example: Let Z be the set of positive, negative integers with '0', then (z, + , .) is a near ring with usual addition and multiplication.

2.7 Definition: A near ring N is said to be Regular near ring if for each element x∈ N then there exists an element y∈ N such that x= xyx.

2.8 Example: (i) M (Γ) and $M_0(\Gamma)$ are regular near rings (Beidleman (10) NR Text)

2.9 Definition: Let (*R*, *M, k*) be a Noetherian local ring. (The notation means that the maximal ideal is *M* and the residue field is *k* = *R/M*.) If *d* is the dimension of *R*, then by the dimension theorem every generating set of *M* has at least *d* elements. If *M* does in fact have a generating set *S* of *d* elements, we say that *R* is *regular* and that *S* is a *regular system of parameters*

2.10 Example: If *R* has dimension 0, then *R* is regular iff {0}is a maximal ideal, in other words, iff *R*is a field.

2.11 Definition: A Local near ring which satisfy regular property then it is called a regular local near ring.

2. 12 Definition: The Rees algebra of an ideal $I \subset R$ by definition a graded algebra satisfying R(I)=R$\oplus I \oplus I^2 \oplus \ldots$..

3. NON COHEN – MACAULAY REES ALGEBRAS

Given a three dimensional regular local Near ring (N , m) and normal four – generated height two unmixed ideal I ofN, it is natural to ask about the Cohen – Macaulayness of the Rees algebra NIt]. Assume further that I is generically a complete intersection (that is, Ip is a complete intersection for every prime ideal P minimal over I). Our main result of this section will characterize, in terms of a presentation matrix of I, when N[It] is Cohen – Macaulay for such an ideal. It builds on work of Vasconcelos and Aberbach – Huneke where techniques for studying Rees algebras via presentation matrices were staged. The assumption that I is normal is not needed for the proof of our theorem however. Instead we need only the weaker condition that $\left(mI^2 : m\right) = I^2$. If I^2 is integrally closed then it satisfies this condition. For if $w \in \left(mI^2 : m\right)$ then ω m$\subset mI^2$, thus $\omega \in \overline{I^2} = I^2$ by the determinant trick.

Theorem 3.1: Let (N,m) be a d-dimensional regular local near ring containing a field and I a height d – 1 unmixed ideal of N. Assume that I is generically a complete intersection, I has a d – generated reduction, $\mu(I) = d+1, and\ I^2 m : m = I^2$, but I^2 is not unmixed. Then there is a generating set $\{x_1, ..., x_d\}$ for m , positive integers $m \geq n$, and a presentation matrix ϕ for I, such that $I_1\phi = (x_2, ..., x_d, x^n{}_1)$ and

$$\phi = \begin{pmatrix} x_2 \ \ldots \ x_d & x_1^m & 0 \ldots 0 \\ . \ . \ . & . & . \ldots . \\ . \ . \ . & . & . \ldots . \\ . \ . \ . & . & . \ldots . \end{pmatrix}$$

Proof: By our assumption, $m \in Ass(N / I^2)$. In particular, $(I^2 : m) \neq I^2$. Choose an element $e \in (I^2 : m) \setminus I^2$. Because $(mI^2 : m) = I^2$, $em \not\subseteq mI^2$. Choose $x \in m \setminus m^2$ such that $ex \notin mI^2$. Expand $x = x_1$ to a minimal generating set $\{x_1, ..., x_d\}$ for m. Using that $ex \in I^2 \setminus mI^2$ we may assume that $I = (p_1..., p_{d+1})$ is a minimal generating set for I, and that $ex = ap^2{}_1 + p$ for some $p \in (p_2, ..., p_{d+1})I$ and unit α . Set $ex_i = g_i \in I^2$. Consider the homomorphism

$$\psi : R[T_1, ..., T_{d+1}] \rightarrow R[It]$$

Given by $T_i \rightarrow p_i t$. Let F, $G_i \in R[T_1...., T_{d+1}]$ be homogeneous of degree 2 (in the $T^{'}{}_i$ s) such that $\psi(F) = pt^2$, $\psi(G_i) = g_i t^2$. By our choice of p we can assume that $F \in (T_2, ...T_{d+1})R[T_1, ..., T_{d+1}]$. Let Q be the kernel of ψ , and let Q_j denote the ideal generated by the homogeneous elements of Q having degree at most j. The trivial relation $(ex)x_i = (ex_i)x$ forces $x_i(\alpha T_1^2 + P) - xG_i \in Q_2$. But $Q_2 = Q_1$ because I is syzygetic [8], thus there are linear homogeneous polynomials $A_i \in R[T_1,...,T_{d+1}]$ such that $x_i(\alpha T_1^2 + P) - xG_i = A_1L_1 + ... + A_sL_s \; where\, Q_1 = (L_1...L_s)$. The coefficient of T_1^2 must be $x_i - xb_{1i}$ for some $b_{1i} \in R$ and the coefficients of L_i all lie in m, therefore Q_1 contains polynomials of the form $(x_i - xb_{1i})T_1 + b_{2i}T_2 + ...b_{si}T_s$. The existence of these linear polynomials implies that $(x_i - xb_{1i}x \in ((p_2,..., p_{d+1}) : p_1)$. Replacing x_i with $x_i - xb_{1i}x$ yields that $x_i \in ((p_2,..., p_{d+1}) : p_1)$. Therefore invertible row and column operations yield that ϕ may be reduced to the form described in (1). In particular, $I_1(\phi) = (x_2,..., x_d, x^n)$ for some $n \le m$.

We are particularly interested in applying (3.1) to curves in 3 – space, where several of our assumptions automatically are valid.

Corollary 3.2. Let (N,m) be a 3 – dimensional regular local near ring containing a field and I a height 2 unmixed ideal ofN. Assume that I is generically a complete intersection, I has a 3 – generated reduction, $\mu(I) = 4$, and $I^2m : m = I^2$. There is a generating set {x, y, z} for m, positive integers $m \ge n$, and a presentation matrix ϕ for I,

such that $I_1(\phi) = (y, z, x^n)$ and $\phi = \begin{pmatrix} y & z & x^m \\ . & . & . \\ . & . & . \\ . & . & . \end{pmatrix}$.

Proof: This follows at once from Proposition 3.1 as soon as we observe that the assumption that I^2 is not unmixed is automatic in this case we first analyze to have precisely three generators by the Hilbert – Burch theorem.

We recall Vasconcelos' construction. That is, let ϕ be a 4 X 3 presentation matrix of $I = (p_1, p_2, p_3, p_4)$ and assume $I_1(\phi)$ (the ideal generated by the entries of ϕ is a complete intersection, generated by {a,b,c}. Consider t he homomorphism

$$\psi : N[T_1, T_2, T_3, T_4] \rightarrow N[It]$$

given by $T_i \rightarrow p_i t.$ The symmetric algebra of I is a complete intersection whose defining ideal is generated by three elements $L_1, L_2 \; and \; L_3$. These elements satisfy the matrix equation

$$\begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix} = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 \end{pmatrix} . \phi$$

Build an associated matrix B(ϕ) via the matrix equation

$$3.3 (T_1 \;\; T_2 \; T_3 \; T_4). \emptyset = (L_1 \quad L_2 \quad L_3) = (a \; b \; c). B(\emptyset)$$

We also define C($\emptyset$) to be the image of B($\emptyset$) in k(T_1 T_2 T_3 T_4),i.e the matrix B($\emptyset$) reduced modulo the maximal ideal of N

Remark 3.4: We will need to perform various changes on the generations of I, the choice of presentation ϕ, and the matrix $B(\phi)$. It is convenient to record exactly what changes we will use and how they affect the rest of the data.

First, consider column operations upon $B(\phi)$. Let ϕ be a 3 by 3 invertible matrix with coefficients in R. Column operations upon $B(\phi)$ coming from the base ring R are obtained by replacing $B(\phi)$ by $B(\phi)\theta$. To preserve equation (3.3) we must also multiply ϕ by θ and then the corresponding equation (2.3) is valid. This only changes the generators for the syzygies of I, and not the chosen generating set. Otherwise stated, $B(\phi\theta) = B(\phi)\theta$.

Row operations on $B(\phi)$ coming from ~Correspond to multiplying $B(\phi)$ by an invertible 3 by 3 matrix θ with coefficients in R on the left side of $B(\phi)$. To insure that (3.3) continues to hold, we must then multiply the matrix (a,b,c) by θ^{-1} on the right, basically changing the choice of generators for the ideal (a, b, c).

Finally, we are free to change the chosen generators of I. In this case we replace ϕ by $\theta\phi$, with θ in this case a 4 by 4 invertible matrix with coefficients inN. If we change the corresponding T_i by multiplying on the right by θ^{-1}, then (3.3) is still valid without change to the complete intersection a,b,c or the matrix $B(\phi)$.

Vasconcelos proved that if I is a prime ideal such that $\det\left(C(\phi)\right) \neq 0$ then N [It] is Cohen – Macaulay. Implicit in the work of Aberbach and Huneke is an improvement of this statement.

Proposition 3.5: Let (N, m) be three – dimensional regular local Near ring containing a field of characteristic not 2 and I a four – generated height tow unmixed ideal of Nwhich is generically a complete intersection. Assume further that I has a three – generated minimal reduction (automatic if N/m is infinite) and $I_1(\phi)$ is a complete intersection. Then N[It] is Cohen – Macaulay if and only if $\det\left(C(\phi)\right) \neq 0$.

Proof: The proof of the forward direction is contained in the proof of the (1) implies (3) part of the argument given in {2}. Conversely, if $\det\left(B(\phi)\right) \notin mN[T_1,T_2,T_3,T_4]$ then we may assume (after changing the generators of I if necessary) that $\det(B(\phi)) = T_1^3 + A$ for some homogeneous (in the $T^{'}$s) degree 3 polynomial A contained in $(T_2,T_3,T_4)N[T_1,T_2,T_3,T_4]$. Because the elements $\psi(f), \psi(g)\, and\, \psi(h)$ vanish, $\psi(\det(B(\phi)) = 0$. Hence $p_1^3 \in (p_2, p_3, p_4)I^2$ which means I has reduction number two, therefore N[It] is Cohen – Macaulay by [3].

The following theorem is the key theorem of this paper upon which the examples are based.

Theorem 3.6: Let (N,m) be a 3 – dimensional regular local ring containing a field of characteristic not 2 and I a height 2 un mixed ideal of N. Assume that I is generically a complete intersection, I has a 3 – generated reduction, $\mu(I)$=4, and $I^2m : m = I^2$. Further assume that a generating set {x,y,z} for m and a presentation matrix ϕ for I have been chosen as in (3.2) . Then R[It] is not C – M if and only if $I_1\phi = (y,z,x^n)$ =(u,v,w) and there is a presentation matrix θ for I of the form

$$\theta = \begin{pmatrix} v & w & 0 \\ u & 0 & w \\ 0 & u & -v \\ 0 & 0 & 0 \end{pmatrix} \mod\ mI_1(\theta)$$

Proof: If θ has the form described in (3.6) then it easy to see that $\det(C(\theta)) = 0$. Therefore R[It] is not Cohen – Macaulay by Proposition 3.5.

Assume R [It] is not Cohen – Macaulay and that ϕ is a presentation matrix for I having the form prescribed in (3.2), that is

$$\phi = \begin{pmatrix} y & z & x^m \\ . & . & . \\ . & . & . \end{pmatrix},$$

Where $m \geq n$ and $I_1(\phi) = (y,z,x^n)$. By using invertible row operations we may also assume that m is the least power of x appearing in the last column of ϕ. We will analyze the matrix $B(\phi) = \begin{pmatrix} T_1 + A & B & C \\ D & T_1 + E & F \\ G & H & x^{m-n}(T_1 + K) \end{pmatrix}$, Where A, B,…K $\in N[T_2, T_3, T_4]$. If m = n the ϕ satisfies the row condition, leading to the contradiction that N[It] is Cohen – Macaulay . Therefore we assume m>n. Let C = C(ϕ) denote the image of B modulo $mN[T_1, T_2, T_3, T_4]$, and use lower-case letters to denote images modulo $mN[T_1, T_2, T_3, T_4]$. Then

(3.7) $$C(\phi) = \begin{pmatrix} t_1 + a & b & c \\ d & t_1 + e & f \\ g & h & 0 \end{pmatrix}.$$

Here a, b, c…,h $\in k[t_1, t_2, t_3, t_4]$ are liner forms. In addition, the fact that $I_1(\phi) = (y, z, x^n)$ implies that either $g \neq 0$ or $h \neq 0$ (else the least pure power of x in $I_1(\phi)$ would be greater than n).

Since R[It] is not Cohen – Macaulay the determinant of C(ϕ) is zero by Proposition 3.5, therefore

(3.8) $$g(bf - ct_1 - ce) - h(ft_1 + af - cd) = 0$$

We consider two cases; either g and h are relatively prime or not. First suppose that g and h are relatively prime. Because a,b,….h $\in k[t_2, t_3, t_4]$, (2.8) implies that gc + hf = 0. Therefore there exists an $\alpha \in k$ such that $c = -\alpha h$ and $f = \alpha g$. Substituting into (3.7) yields

$$C(\phi) = \begin{pmatrix} t_1 + a & b & -\alpha h \\ d & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}$$

Thus,

$$0 = \det(C) = \alpha(bg^2 + ghe - gha - dh^2)$$

Using that g and h are relatively prime we obtain that h divides b, hence b = β h for some $\beta \in k$.

Substituting above and factoring out h implies that

$$\beta g^2 + ge - ga - dh = 0$$

Therefore g divides d, hence d= γg for some $\gamma \in k$.

Substituting and factoring our g implies that $\beta g \;\; + e - a - \gamma h$, therefore

$$e - \gamma h = a - \beta g$$

Further, after making the substitutions $b = \beta h$ and $d = \gamma g, C(\phi)$ takes the form

$$C(\phi) = \begin{pmatrix} t_1 + a & \beta h & -\alpha h \\ \gamma g & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}$$

By using row operations $C(\phi)$ may be reduced to

$$C(\phi) = \begin{pmatrix} t_1 + a - \beta g & 0 & -h \\ 0 & t_1 + e - \gamma h & g \\ g & h & 0 \end{pmatrix}$$

As in Remark 2.4, these row operations will change the choice of generators of the ideal (y, z, x^n). Let us call the new generations u, v, w. Set $q = a - \beta g = e - \gamma h$. By the above calculation

$$C(\phi) = \begin{pmatrix} t_1 + q & 0 & -h \\ 0 & t_1 + q & g \\ g & h & 0 \end{pmatrix}.$$

Note that $\{t_1 + q, g, h\}$ are independent linear forms in $k[t_1, t_2, t_3, t_4]$ because g and h are relatively prime linear forms and do not involve t_1. Therefore by changing variables we may replace $t_1 + q, g$ *and* h with t_1, t_2 *and* t_3 (respectively replace $T_1 + Q$, G and H with T_1, T_2 *and* T_3). As in Remark 2.4 this change also will change our original choice of generators for I. Substituting this variable – change yields that

$$C(\phi) = \begin{pmatrix} t_1 & 0 & -t_3 \\ 0 & t_1 & t_2 \\ t_2 & t_3 & 0 \end{pmatrix}$$

And lifting back to N $N[T_1, T_2 \ \ , T_3, T_4]$

$$B(\phi) = \begin{pmatrix} T_1 & 0 & -T_3 \\ 0 & T_1 & T_2 \\ T_2 & T_3 & 0 \end{pmatrix} \ \text{mod} \ \ mN[T_1, ..., T_4].$$

Finally, to see that the matrix ϕ may be chose to have the form described in the statement of the theorem, use the matrix equation.

$B(\phi) = (T_1 \quad T_2 \quad T_3 \quad T_4)\phi$

to re build $\emptyset$.

This completes the proof of theorem3.6 in the case that g and h are relatively prime.

REFERENCES

[1] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Springer – Verlag, New York – 1995.

[2] I. M. Aberbach and C. Huneke, An improved Braincon – Skoda theorem with applicationsto the Cohen – Macaulayaness of Rees algebras, Math. Annalen 297 (1993), 343 – 369

[3] J.C. Beidleman "A Note on Regular Near – Rings" J.Indian Math. Soc 33 (1969), 207 – 210.

[4] Pilz, G. (1983). Near – rings. Amsterdam: North – Holland / American Elsevier.

[5] O. Zariski and P.Samuel, Commulative Algebra, vol.2, Springer – Verlag, New York, *1960.*

[6] SAM HUCKABA AND CRAIG HUNEKE, Normal ideals in Regular rings.

[7] S. Itoh, Integral closures of ideals generated by regular sequences, J. Algebra 117 (1988), 390 – 401.

[8] S. Morey and B. Ulrich, Rees algebras of ideals with low codimension, Proc. Amer. Math. Soc.124 (1996) 3653 – 3661.

[9] T. Marley, The coefficients of the Hilbert polynomial and the reduction number of an ideal J. London Math. Soc.40 (1989), 1- 8.

Source of support: Nil, Conflict of interest: None Declared