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# APPLICATIONS OF δ-PRECONTINUOUS MAPS IN TOPOLOGICAL VECTOR SPACES

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***ABSTRACT***

*It is shown that linear functional on topological vector spaces are $\delta-$precontinuous. Also we gave some application for $\delta-$precontinuous on topological vector spaces, our results can be viewed as a generalization to the results in [12].*

***Keywords:*** *$\delta-$Preopen sets, $\delta-$precompact sets, $\delta-$precontinuous maps*

## 1. INTRODUCTION

N. Levine [7] introduced the theory of semi-open sets and the theory of $\alpha$-sets for topological spaces. For a systematic development of semi-open sets and the theory of $\alpha$-sets one may refer to [1, 2, 4, 5, 9]. Mashhour et al. in [8] introduced preopen sets and precontinuous functions in topological spaces. On the other hand, Velicko [15] introduced the notion of $\delta$-open sets which are stronger than open sets. Since then, $\delta$-open sets have been widely used in order to introduce new spaces and functions. Recently, Raychaudhuri and Mukherjee [13] have introduced the notions of $\delta$-preopen sets and $\delta$-almost continuity in topological spaces. The class of $\delta$-preopen sets is larger than that of preopen sets. By using $\delta$-preopen sets, in [14], they introduced and investigated $\delta$-pclosed spaces. These concepts above are closely related. It is known that, in a topological space, a set is preopen and semi-open if and only if it is an $\alpha$-set [10, 11]. In section 2, we show that every linear functional on a topological vector space is $\delta-$precontinuous . In section 3, we define a $\delta-$prebounded set, totally $\delta-$prebounded set, and $\delta-$precompact set in a topological vector space and find the relations between them. In section 4, we show that every topological vector space is a $\delta-$prehausdorff space, and also identify totally $\delta-$prebounded and $\delta-$precompact subset of any topological vector space. Finally our result are extended to the results which found in [12].

## 2. $\delta-$PRECONTINUOUS MAPS

**Definition 2.1 (12):** *Let $X$ be a topological space. A subset $S$ of $X$ is said to be preopen if $S \subset \operatorname{int}(\operatorname{cl}(S))$. A preneighbourhood of the point $x \in X$ is any preopen set containing $x$ .*

**Definition 2.2 (14):** *Let $X$ be a topological space. A point $x \in X$ is said to be a $\delta-$cluster point of a set $S$ if $S \cap U \neq \emptyset$ for every regular preopen set $U$ containing $x$.*

**Definition 2.3: (14)** *.Let $X$ be a topological space. The set of all $\delta-$cluster points of $S$ forms the $\delta-$preclosure , denoted by $precl_{\delta}(S)$.*

**Definition 2.4:** *Let $X$ be a topological vector space. A subset $S$ of $X$ is said to be $\delta-$preopen if $S \subseteq \operatorname{int}(\operatorname{cl}_{\delta}(S))$ The set of all $\delta-$cluster points of $A$ forms the $\delta-$preclosure , denoted by $precl_{\delta}(A)$. A $\delta-$preneighbourhood of the point $x \in X$ is any $\delta-$preopen set containing $x$ .*

**Definition 2.5:** *Let $X$ and $Y$ be topological vector spaces and $f: X \to Y$ . The function $f$ is said to be $\delta-$precontinuous if the inverse image $f^{-1}(B)$ of each open set $B$ in $Y$ is a $\delta-$preopen set in $X$ . The function $f$ is said to be $\delta-$preopen if the image $f(A)$ of every open set $A$ in $X$ is $\delta-$preopen in $Y$ .*

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**Lemma 2.1:** *Let $X$ and $Y$ be topological vector spaces and $f : X \to Y$ linear. The function $f$ is $\delta-$preopen if and only if, for every open set $U$ containing $0 \in X$ , $0 \in Y$ is an interior point of* $\mathrm{cl}_\delta(f(U))$.

**Proof**: Trivial

**Theorem 2.1:** *Let $X, Y$ be topological vector spaces and let $Y$ have the Baire property, that is, whenever $Y = \cup_{n=1}^{\infty} B_n$ with closed sets $B_n$, there is is $N$ such that $\mathrm{int}_\delta(B_N)$ is nonempty. Let $f : X \to Y$ be linear and $f(X) = Y$ . Then $f$ is $\delta-$preopen.*

**Proof:** Let $U \subset X$ be a neighborhood of $0$. There is a neighborhood $V$ of $0$ such that $V - V \subset U$. Since $V$ is a neighborhood of $0$ we have $X = \cup_{n=1}^{\infty} nV$ . It follows from linearity and surjectivity of $f$ that $Y = \cup_{n=1}^{\infty} nf(V)$. Since $Y$ has the Baire property, there is $N$ such that $\mathrm{cl}_\delta(Nf(V)) = N\mathrm{cl}_\delta(f(V))$ contains an open set $S$ which is not empty. Then $\mathrm{cl}_\delta(f(V))$ contains the open set $T = \frac{1}{N}S$ . It follows that

$$T - T \subset \mathrm{cl}_\delta(f(V)) - \mathrm{cl}_\delta(f(V)) \subset \mathrm{cl}_\delta(f(V) - f(V)) = \mathrm{cl}_\delta(f(V - V)) \subset \mathrm{cl}_\delta(f(U)).$$

The set $T - T$ is open and contains $0$. Therefore, $0 \in Y$ is an interior point of $\mathrm{cl}_\delta(f(U))$. From Lemma 2.1 we conclude that $f$ is $\delta-$preopen.

Note that $f$ can be any linear surjective map. It is not necessary to assume that $f$ is continuous or $\delta-$precontinuous.

**Theorem 2.2:** *Let $X, Y$ be topological vector spaces, and let $X$ have the Baire property. Then every linear map $f : X \to Y$ is $\delta-$precontinuous.*

**Proof:** Let $G = \{(x, f(x)) : x \in X\}$ be the graph of $f$ . The projections $\pi_1 : G \to X$ and $\pi_2 : G \to Y$ are continuous. The projection $\pi_1 : G \to X$ is bijective. It follows from Theorem 2.1 that $\pi_1$ is $\delta-$preopen. Therefore, the inverse mapping $\pi_1^{-1}$ is $\delta-$precontinuous. Then $f = \pi_2 \circ \pi_1^{-1}$ is $\delta-$precontinuous.

Theorem 2.2 shows that many linear maps are automatically precontinuous. Therefore, it is natural to ask for an example of a linear map which is not $\delta-$precontinuous.

Let $X = C[0,1]$ be the vector space of real-valued continuous functions on $[0,1]$ equipped with the norm

$$\| f \|_1 = \int_0^1 | f(x) | dx .$$

Let $Y = C[0,1]$ be equipped with the norm

$$\| f \|_\infty = \max_{x \in [0,1]} | f(x) | .$$

**Lemma 2.2:** *The identity operator $T : X \to Y$ is not $\delta-$precontinuous.*

**Proof:** Let $U = \{f \in C[0,1] : \| f \|_\infty < 1\}$ which is an open subset of $Y$ . Let $\mathrm{cl}_\delta(U)$ be the closure of $U$ in $X$ . We claim that

$$\mathrm{cl}_\delta(U) \subset \{f \in C[0,1] : \| f \|_\infty \le 1\}. \quad (1)$$

For the proof, consider a sequence $f_n \in U$ and a function $f \in C[0,1]$ such that $\{f_n\}$ converges to $f$ in $X$ .

Suppose that there is $x_0 \in 0,1]$ such that $f(x_0) > 1$. By continuity of $f$, there are $a < b$ and $\delta > 0$ such that $0 \le a \le x_0 \le b \le 1$ and $f(x) > 1+\delta$ for $x \in (a,b)$. Then, as $n \to \infty$,

$$(b-a)\delta \le \int_a^b |f_n(x) - f(x)| dx \le \int_0^1 |f_n(x) - f(x)| dx \to 0$$

which is a contradiction. Therefore, $f(x) \le 1$ for all $x \in 0,1]$. Similarly, we show that $f(x) \ge -1$ for all $x \in 0,1]$. Now $0 \in U = T^{-1}(U)$ but $U$ is not $\delta-$preopen in $X$. We see this as follows. Suppose that $U$ is $\delta-$preopen in $X$. The sequence $g_n(x) = 2x^n$ converges to $0$ in $X$. Therefore, $g_n \in \mathrm{cl}_\delta(U)$ for some $n$ and (1) implies $2 = \| g_n \|_\infty \le 1$ which is a contradiction.

We can improve Theorem 2.2 for linear functionals.

**Theorem 2.3:** *Let $f$ be a linear functional on a topological vector space $X$. If $V$ is a $\delta-$preopen subset of $\mathbb{R}$ then $f^{-1}(V)$ is a $\delta-$preopen subset of $X$. In particular, $f$ is $\delta-$precontinuous.*

**Proof:** We distinguish the cases that $f$ is continuous or discontinuous.

Suppose that $f$ is continuous. If $f(x) = 0$ for all $x \in X$ the statement of the theorem is true. Suppose that $f$ is not zero. We choose $u \in X$ such that $f(u) = 1$. Let $V$ be a $\delta-$preopen subset of $\mathbb{R}$, and set $U := f^{-1}(V)$. Let $x \in U$ so $f(x) \in V$. Since $V$ is $\delta-$preopen, there is $\delta > 0$ such that

$$I := (f(x) - \delta, f(x) + \delta) \subset \mathrm{cl}_\delta(V). \tag{2}$$

Since $f$ is continuous, $f^{-1}(I)$ is an open subset of $X$ containing $x$. We claim that

$$f^{-1}(I) \subset \mathrm{cl}_\delta(U). \tag{3}$$

In order to prove (3), let $y \in f^{-1}(I)$ so $f(y) \in I$. By (2), there is a sequence $\{t_n\}$ in $V$ converging to $f(y)$. Set

$$y_n := y + (t_n - f(y))u.$$

We have $f(y_n) = t_n \in V$ so $y_n \in U$. Since $X$ is a topological vector space, $y_n$ converges to $y$. This establishes (3). It follows that $U$ is $\delta-$preopen.

Suppose now that $f$ is not continuous. By [3, Corollary 22.1], $N(f) = \{x \in X : f(x) = 0\}$ is not closed. Therefore, there is $y \in \mathrm{cl}(N(f))$ such that $y \not\in N(f)$ so $f(y) \ne 0$. Let $x$ be any vector in $X$. There is $t \in \mathbb{R}$ such that $f(x) = tf(y)$ and so $x - ty \in N(f)$. It follows that $x \in \mathrm{cl}_\delta(N(f))$. We have shown that $N(f)$ is dense in $X$. Let $a \in \mathbb{R}$. There is $y \in X$ such that $f(y) = a$. Then $f^{-1}(\{a\}) = y + N(f)$ and so the closure of $f^{-1}(\{a\})$ is $y + \mathrm{cl}_\delta(N(f)) = X$. Therefore, $f^{-1}(\{a\})$ is dense for every $a \in \mathbb{R}$. Let $V$ be a $\delta-$preopen set in $\mathbb{R}$. If $V$ is empty then $f^{-1}(V)$ is empty and so is $\delta-$preopen. If $V$ is not empty choose $a \in V$. Then $f^{-1}(V) \supset f^{-1}(\{a\})$ and so $f^{-1}(V)$ is dense. Therefore, $f^{-1}(V)$ is $\delta-$preopen.

## 3. MAIN RESULTS

We need the following known lemma.

**Lemma 3.1:** *If $U$, $V$ are two vector spaces, and $W$ is a linear subspace of $U$ and $f : W \to V$ is a linear map. then there is a linear map $g : U \to V$ such that $f(x) = g(x)$ for all $x \in W$.*

 

**Proof:** We choose a basis $A$ in $W$ and then extend to a basis $B \supset A$ in $U$. We define $h(a) = f(a)$ for $a \in A$ and $h(b)$ arbitrary in $V$ for $b \in B - A$. There is a unique linear map $g: U \to V$ such that $g(b) = h(b)$ for $b \in B$. Then $g(x) = f(x)$ for all $x \in W$.

We obtain the following result.

**Theorem 3.1:** *Every topological vector space $X$ is a $\delta-$prehausdorff space, that is, for each $x, y \in X$, $x \neq y$, there exists a $\delta-$preneighbourhood $U$ of $x$ and a $\delta-$preneighbourhood $V$ of $y$ such that $U \cap V = \varnothing$.*

**Proof:** Let $x, y \in X$ and $x \neq y$. If $x, y$ are linearly dependent we choose a linear functional on the span of $\{x, y\}$ such that $f(x) < f(y)$. If $x, y$ are linearly independent we set $f(sx + ty) = t$. By Lemma 3.1 we extend $f$ to a linear functional $g$ with $g(x) < g(y)$. Choose $c \in (g(x), g(y))$ and define $U = g^{-1}((-\infty, c))$, and $V = g^{-1}((c, \infty))$. Then, using Theorem 2.3, $U$, $V$ are $\delta-$preopen. Also $U$ and $V$ are disjoint and $x \in U$, $y \in V$.

We now determine totally $\delta-$prebounded subsets in $\mathbb{R}$. The result may not be surprising but the proof requires some care.

**Lemma 3.2:** *A subset of $\mathbb{R}$ is totally $\delta-$prebounded if and only if it is finite.*

**Proof:** It is clear that a finite set is totally $\delta-$prebounded. Let $E$ be a countable (finite or infinite) subset of $\mathbb{R}$ which is totally $\delta-$prebounded. Let $A := \{x - y : x, y \in E\}$. The set $A$ is countable. We define a sequence $\{u_n\}$ of real numbers inductively as follows. We set $u_1 = 0$. Then we choose $u_2 \in (-1, 0)$ such that $u_2 - u_1 \not\in A$. Then we choose $u_3 \in (0,1)$ such that $u_3 - u_i \not\in A$ for $i = 1, 2$. Then we choose $u_4 \in (-1, -\frac{1}{2})$ such that $u_4 - u_i \not\in A$ for $i = 1, 2, 3$. Continuing in this way we construct a set $U = \{u_n : n \in \mathbb{N}\} \subset (-1,1)$ such that every interval of the form $(m2^{-k}, (m+1\ 2)^k)$ with $-2^k \le m < 2^k$, $k \in \mathbb{N}$, contains at least one element of $U$, and such that $0 \in U$ and $u - v \not\in A$ for all $u, v \in U$, $u \neq v$. Then $\mathrm{cl}_\delta(U) = [-1,1]$ so $U$ is a $\delta-$preneighborhood of $0$. Since $E$ is totally $\delta-$prebounded, there is a finite set $F$ such that $E \subset F + U$. If $z \in F$ and $x, y \in E$ lie in $z + U$ then $x = z + u$, $y = z + v$ with $u, v \in U$. It follows that $u - v = x - y \in A$ and, by construction of $U$, $u = v$. Therefore, $x = y$ and so each set $z + U$, $z \in F$, contains at most one element of $E$. Therefore, $E$ is finite. We have shown that every countable set which is totally $\delta-$prebounded is finite. It follows hat every totally $\delta-$prebounded set is finite.

Combining several of our results we can now identify totally $\delta-$prebounded and precompact subset of any topological vector space.

**Theorem 3.2:** *Let $X$ be a topological vector space. A subset of $X$ is totally $\delta-$prebounded if and only if it is finite. Similarly, a subset of $X$ is $\delta-$precompact if and only it is finite.*

**Proof:** Every finite set is totally $\delta-$prebounded. Conversely, suppose that $E$ is a totally $\delta-$prebounded subset of $X$. Let $f$ be a linear functional on $X$. It follows easily from Theorem 2.3 that $f(E)$ is a totally $\delta-$prebounded subset of $\mathbb{R}$. By Lemma 3.2, $f(E)$ is finite. It follows that $E$ is finite as we see as follows. Suppose that $E$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ which is linearly independent. Then, using Lemma 3.1, we can construct a linear functional $f$ on $X$ such that $f(x_n) \neq f(x_m)$ if $n \neq m$. This is a contradiction so $E$ must lie in a finite dimensional subspace $Y$ of $X$. We choose a basis $y_1, \ldots, y_k$ in $Y$, and represent each $x \in E$ in this basis

$$x = f_1(x) y_1 + \ldots + f_k(x) y_k.$$

Every $f_j$ is a linear functional on $Y$ so $f_j(E)$ is a finite set for each $j = 1, 2, \ldots, k$. It follows that $E$ is finite.

Clearly, every finite set is $\delta-$precompact. Conversely, by Lemma ??, a $\delta-$precompact subset of $X$ is totally $\delta-$prebounded, so it is finite.

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