**International Journal of Mathematical Archive-3(5), 2012, 1779-1783**
**IJMA Available online through www.ijma.info** ISSN 2229 – 5046

# ON THE PARTIAL ORDERING OF RANGE HERMITIAN MATRICES

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***(Received on: 15-04-12; Accepted on: 30-04-12)***

***ABSTRACT***

*For a given range Hermitian matrix B, conditions are obtained for all matrices A that lie below B (above B) $A \leq B$ ($A \geq B$) to be range Hermitian under a given partial ordering on matrices. As an application, it is shown that the monotonicity of the constitutive operators in linear electro-mechanical systems having the same structure operator is preserved for the corresponding transfer impedances.*

***Key words:*** *Hermitian matrix, Almost definite matrix, quasi positive definite matrix, Positive semi definite matrix, Hermitian positive semi definite matrix, Range Hermitian matrix.*

***AMS Subject Classification:*** *15B57, 15A24.*

## 1 INTRODUCTION

Let $C_{nxn}$ be the set of all complex matrices of order n and $C_n$ be the set of all Complex vectors. For $A \in C_{nxn}$, let R(A), N(A), A*, $A^+$, $A^-$ and rk(A) be the range space, null space, conjugate transpose, Moore – Penrose inverse, generalized inverse ($A^-$ is a solution of the matrix equation A X A = A) and rank of A respectively. $A \in C_{nxn}$ is said to be almost definite (a.d [3]) if for $x \in C_n$, $x^*Ax = 0 \Rightarrow Ax = 0$. $A \in C_{nxn}$ is said to be positive semi definite (p.s.d [6]) if $Re(x^*Ax) \geq 0$ for $x \in C_n$. If A is also Hermitian, then A is Hermitian positive semi definite (h.p.s.d) and is denoted as $A \geq 0$. A matrix $A \in C_{nxn}$ is said to be Almost positive semi definite (a.p.d) if it is both a.d and p.s.d. Mitra and Puri [9] have introduced and developed the concept of quasi positive definite (q.p.d) matrix. $A \in C_{nxn}$ is said to be q.p.d if A is p.s.d and $Re(x^*Ax) = 0 \Rightarrow Ax = 0$ and they have proved that a q.p.d matrix is always a.p.d. For properties of a.d, a.p.d and q.p.d matrices, one may refer [3, 9]. These special types of matrices are widely used in the study of electrical networks and in linear electromechanical systems. It was pointed out by Duffin and Morley [3] that the unique transfer impedance in a general linear electromechanical system exists for every structure operator if and only if the constitutive operator is a.d. For terminology and representation of a general linear electromechanical system by a pair of equations, one may refer [3].

For A, B $\in C_{nxn}$, $A \geq B \Leftrightarrow A - B \geq 0 \Leftrightarrow$ A – B is Hermitian positive semi definite (h.p.s.d). It is well known that for nonsingular matrices A, B, if $A \geq B \geq 0$, then $B^{-1} \geq A^{-1} \geq 0$. This was extended to generalized inverses of certain types of pairs of singular matrices $A \geq B \geq 0$ by Hans J. Werner [4] and independently by Hartwig [5]. In [8], their results were extended for a wider class of a.p.d matrices. For a pair of Complex a.p.d matrices A and B such that $A \geq B$, conditions are obtained for $B^+ \geq A^+$. Here, we have extended our results found in [8] on partial orderings of almost definite matrices and some well known matrix inequalities on a pair of h.p.s.d matrices available in the literature [1, 2, 7, 9], for a wider class of range Hermitian matrices. $A \in C_{nxn}$ is said to be range Hermitian if R(A) = R(A*). The concept of range Hermitian matrices are introduced by Schwerdtfeger [11] for complex matrices. Since, for $A \in C_{nxn}$, $R(A^*) = N(A)^{\perp}$, R(A) = R(A*) is equivalent to N(A) = N(A*). Later, Pearl [10] has proved that $A \in C_{nxn}$ is range Hermitian $\Leftrightarrow$ $AA^+ = A^+A$, that is projectors are equal, hence Equi-projector matrix, that is, EP matrix in short. The class of EP matrices is a larger class that includes nonsingular matrices, Hermitian matrices. In [9] it is shown that the class of q.p.d matrices $\subseteq$ class of a.p.d matrices $\subseteq$ class of EP matrices.

## 2. PARTIAL ORDERING ON EP MATRICES

We are concerned with h.p.s.d partial ordering on EP matrices. $A \in C_{nxn}$ is EP means that A is an EP matrix. For A, B $\in C_{nxn}$, $A \geq B \Leftrightarrow A - B \geq 0 \Leftrightarrow$ A – B is h.p.s.d matrix. In this section; conditions for all those matrices that lie below (or) above a given EP matrix relative to h.p.s.d ordering to be EP are determined. First we shall prove certain lemmas, which will simplify the proof of the main result.

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**Lemma 2.1:** If $A \in C_{nxn}$ is EP, then $N(A) \subseteq N(\text{Sym } A)$, where Sym A = $\frac{1}{2}(A + A^*)$ is the symmetric part of A.

**Proof:** Since A is EP, N (A) = N (A*). For $x \in C_n$, $Ax=0 \Leftrightarrow A^*x=0$. Hence, (Sym A)x = 0. Thus $N(A) \subseteq N(\text{Sym } A)$.

**Lemma 2.2:** Let $A \in C_{nxn}$. Then A is EP and rk(A) = rk(Sym A) $\Leftrightarrow$ N(A) $=$ N(Sym A).

**Proof:** ($\Rightarrow$) Since A is EP, by Lemma (2.1) $N(A) \subseteq N(\text{Sym } A)$ and together with rk(A) = rk(Sym A) it follows that N(A) $=$ N(Sym A).

Conversly, if N(A) $=$ N(Sym A), then rk(A) = rk(Sym A) automatically holds. To prove A is EP,

if possible, let us assume the contrary, that is, for $0 \neq x \in C_n$, Ax=0 and $A^*x \neq 0$. Then, (Sym A)x $=\frac{1}{2}(Ax + A^*x) \neq 0$. Hence $x \notin N(\text{Sym } A)$. This contradicts that, N(A) $=$ N(Sym A). Hence A is EP.

**Remark 2.3:** In particular, for a q.p.d matrix A, by Lemma (2.8) in [9], the condition rk(A) = rk(Sym A) automatically holds. Further, by Lemma (2.1) in [9], the q.p.d matrix A is also EP.

Hence Lemma (2.2) reduces to the following:

**Lemma 2.4:** Let $A \in C_{nxn}$ be q.p.d. Then, N(A) = N(Sym A).

**Remark 2.5:** We observe that, in Lemma(2.2), both the conditions on A are essential. This is illustrated in the following example:

**Example 2.1:** Let us consider B = $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \in C_{2x2}$. B is EP, being nonsingular, N(B) = {0} = N(B*). For B, Sym B = $\frac{1}{2}(B + B^*) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Here, rk(Sym B) = $1 \neq$ rk B and N(Sym B) $\neq$ N(B). Hence, the Lemma (2.2) fails.

**Theorem 2.6:** Let $A, B \in C_{nxn}$ such that $A \geq B$, then the following hold:
(i) If B is EP and $N(A) \subseteq N(B)$, then A is EP.
(ii) If A is EP and $N(B) \subseteq N(A)$, then B is EP.

**Proof:**
(i) Since $A \geq B$, $A-B \geq 0$. Hence A – B is Hermitian. For any $x \in N(A)$, since B is EP and $N(A) \subseteq N(B)$; Ax = 0 $\Rightarrow$ Bx = B*x = 0 $\Rightarrow$ A*x = Ax – Bx + B*x =0. Hence $N(A) \subseteq N(A^*)$. Since rk(A) = rk(A*), N(A) = N(A*). Thus A is EP.

(ii) Can be proved in a similar manner.

Hence the Theorem.

**Corollary 2.7:** Let $A, B \in C_{nxn}$ such that $A \geq B$. If B is EP, then $N(A) \subseteq N(B) \Leftrightarrow R(B) \subseteq R(A)$.

**Proof:** If B is EP and $N(A) \subseteq N(B)$, then By Theorem (2.6) (i) A is EP. Hence, $N(A) \subseteq N(B) \Leftrightarrow N(A^*) \subseteq N(B^*) \Leftrightarrow R(B) \subseteq R(A)$. Hence the corollary.

**Remark 2.8:** In particular, if $A \geq B$ and B is a. p.d then $N(A) \subseteq N(B)$ automatically holds and corollary (2.7) reduces to Lemma (2) in [8].

**Theorem 2.9:** Let $A, B \in C_{nxn}$ such that $A \geq B$. If B is EP and $N(A) \subseteq N(B)$, then the following are equivalent:
(i) N(A) = N(Sym B)
(ii) R(A) = R(Sym B)
(iii) rk(A) = rk(Sym B)

 

**Proof:** Since B is EP and $N(A) \subseteq N(B)$, by Theorem (2.6)(i) A is EP. Further, by Lemma (2.1), $N(B) \subseteq N(\text{Sym } B)$. Hence $N(A) \subseteq N(B) \subseteq N(\text{Sym } B)$. Then by (iii), $N(A) = N(\text{Sym } B)$. Thus (i) holds. (i) $\Rightarrow$ (iii) is trivial. Thus (i) $\Leftrightarrow$ (iii). The equivalence of (i) and (ii) follows from the fact that A and sym B are EP matrices. Hence the Theorem.

**Theorem 2.10:** Let $A, B \in C_{nxn}$ such that $A \geq B$. If B is EP and $\text{Sym } B \geq 0$ then the following are equivalent:
(i) $R(\text{Sym } A) = R(\text{Sym } B)$
(ii) $(\text{Sym } B)^+ \geq (\text{Sym } A)^+$

**Proof:** (i) $\Rightarrow$ (ii): $A \geq B \Rightarrow A^* \geq B^* \Rightarrow A + A^* \geq B + B^* \Rightarrow \text{Sym } A \geq \text{Sym } B$. Thus $\text{Sym } A \geq \text{Sym } B \geq 0$. Now by Theorem (1) of [5], $R(\text{Sym } A) = R(\text{Sym } B) \Leftrightarrow (\text{Sym } B)^+ \geq (\text{Sym } A)^+$. Hence the Theorem.

**Remark 2.11:** In particular if B is q.p.d, then the condition $N(A) \subseteq N(B)$ in Theorem (2.9) and $\text{Sym } B \geq 0$ in Theorem (2.10) automatically hold. Further by Lemma (2.4), $N(\text{Sym } B) = N(B)$. Hence, Theorem (2.9) and Theorem (2.10), reduce the following:

**Corollary 2.12** (Theorem 2 in [8])**:** Let $A, B \in C_{nxn}$ such that $A \geq B$ and B is q.p.d. Then the following are equivalent:
(i) $R(A) = R(B)$
(ii) $rk(A) = rk(B)$
(iii) $(\text{Sym } B)^+ \geq (\text{Sym } A)^+$

**Remark 2.13:** In particular if $A \geq B \geq 0$, then Sym A = A and Sym B = B. Theorem (2.9) and Theorem (2.10) reduce to the following known results:

**Corollary 2.14** (Theorem 1 in [5])**:** Let $A, B \in C_{nxn}$ such that $A \geq B \geq 0$ then $B^+ \geq A^+ \Leftrightarrow R(A) = R(B)$.

**Corollary 2.15** (Theorem 1 in [4])**:** For $A, B \in C_{nxn}$ any two of the following conditions imply the other one.
(i) $A \geq B \geq 0$
(ii) $rk(A) = rk(B)$
(iii) $B^+ \geq A^+ \geq 0$.

**Proof:** (i) and (ii) $\Rightarrow$ (iii), (i) and (iii) $\Rightarrow$ (ii) follow from Theorem (2.10) using Sym A = A and Sym B = B. The proof for (ii) and (iii) $\Rightarrow$ (i) runs as follows:

Since $rkA = rkA^+$ and $rkB = rkB^+$; $B^+ \geq A^+ \geq 0$ and $rkA^+ = rkB^+ \Rightarrow (A^+)^+ \geq (B^+)^+ \geq 0 \Rightarrow A \geq B \geq 0$. Thus(i) holds. Hence the corollary.

**Remark 2.16:** We observe that in Theorem (2.9) the condition $N(A) \subseteq N(B)$ is essential. This is illustrated in the following:

**Example 2.2:** Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$. Here A is not EP and B is EP being nonsingular

$A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \Rightarrow A \geq B$

$N(A) = \{x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} / Ax = 0\}$. $N(A) \not\subset N(B)$.

$\text{Sym } B = 1/2(B^* + B) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. $rk(\text{Sym } B) = 1 = rkA$

$N(\text{Sym } B) = \{x = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} / (\text{Sym } B)x = 0\}$.

Hence $N(A) \neq N(\text{Sym } B)$ but $rk(\text{Sym } B) = rkA$.

Thus in Theorem (2.9) statement (i) fails and statement (iii) holds. Thus the condition $N(A) \subseteq N(B)$ is essential in Theorem (2.9).

 

## 3. PARTIAL ORDERING ON BLOCK EP MATRICES

In this section, we shall discuss the h.p.s.d. orderings on EP block matrices, involving Schur complements. For a Partitioned matrix M = $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the matrix denoted as M/A = D – CA$^+$ B is called generalized Schur complement of A in M [2]. In our earlier work [7], we have determined conditions for a Schur complement in an EP matrix to be EP for the case when rk(M) $\neq$ rk(A). When rk(M) = rk(A), M/A = 0.

**Lemma 3.1:** Let H, K $\in C_{n \times n}$ be p.s.d and EP such that H $\geq$-K. Let X and Y be n x m matrices satisfying.
(3.1) N(H) $\subseteq$ N(X*); N(K) $\subseteq$ N(Y*) and
(3.2) X*H$^+$ = (H$^+$ X)*; Y*K$^+$ = (K$^+$ Y)*.

Then the following hold:
(i) There exist matrices L, M $\in C_{(n+m)\ (n+m)}$ such that both are p.s.d and EP.
(ii) L + M $\geq$ 0.
(iii) L + M/H + K $\geq$ 0.

**Proof:** Let us consider L = $\begin{bmatrix} H & X \\ X^* & X^*H^+X \end{bmatrix}$ and M= $\begin{bmatrix} K & Y \\ Y^* & Y^*K^+Y \end{bmatrix}$

Since H is EP, N(H$^*$) = N (H) $\subseteq$ N(X$^*$). Further the generalized Schur complement of H in L, that is, L/H = X$^*$H$^+$X – X$^*$H$^+$X = 0. Hence, by Corollary under Theorem 1 of [2], rk(L) = rk(H). By applying Theorem 3 of [7], using H is EP and X$^*$H$^+$ = (H$^+$X$^*$), we get L is EP. Further, L can be factorized as L = P$\begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}$P$^*$, where P = $\begin{bmatrix} I & 0 \\ X^*H^+ & I \end{bmatrix}$.
Since H is p.s.d, L is also p.s.d. Thus L is EP and p.s.d. Similarly we can see that M is EP and p.s.d. Then (i) holds. Since L and M are p.s.d, L + M is p.s.d .L+M = $\begin{bmatrix} H+K & (X+Y) \\ (X+Y)^* & X^*H^+X+Y^*K^+Y \end{bmatrix}$. Since H $\geq$ -K, H + K $\geq$ 0, which implies H + K is Hermitian. By (3.2) X*H$^+$X + Y$^*$K$^+$Y is Hermitian. Hence L + M is Hermitian and together with p.s.d, it follows that L + M $\geq$ 0. Thus (ii) holds. Now, by a result of Albert [1], H + K $\geq$ 0 yields that L+M / H+M $\geq$ 0. Thus (iii) holds. Hence the Lemma.

**Theorem 3.2:** Let H and K be EP as well as p.s.d matrices of order n such that H$\geq$-K, partitioned in the form
H = $\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ and K = $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ Satisfying

(3.3) $N(H_{11}) \subseteq N(H_{21})$ ; $N(H/H_{11}) \subseteq N(H_{12})$

(3.4) $N(K_{11}) \subseteq N(K_{21})$ ; $N(K/K_{11}) \subseteq N(K_{12})$

(3.5) $H_{21}H_{11}^+ = (H_{11}^+H_{12})^* = (H_{11}^+H_{21}^*)^*$

(3.6) $K_{21}K_{11}^+ = (K_{11}^+K_{12})^* = (K_{11}^+K_{21}^*)^*$.

Then $H + K/ H_{11}+K_{11} \geq H/H_{11} + K/K_{11} \geq 0$.

**Proof:** Since H is EP satisfying (3.3), by Theorem 1 and Remark 2 of [7], $H_{11}$ and $H/H_{11}$ are EP. Similarly K is EP satisfying (3.4) implies $K_{11}$ and $K/K_{11}$ are EP. Since H $\geq$ -K, $H_{11} \geq -K_{11}$ and $H+K/H_{11}+K_{11} \geq 0$ by result Albert[1]. By definition of generalized Schur complement [2], we have $H+K/H_{11}+K_{11} = H_{22} + K_{22} – (H_{21} +K_{21}) (H_{11} + K_{11})^+ (H_{12} + K_{12})$. By using (3.5), (3.6) and $H+K/H_{11} + K_{11}$ is Hermitian we get

$$H+K/H_{11}+K_{11} \geq H_{22}+K_{22} - (H_{21}H_{11}^+H_{21}^*+ K_{21}K_{11}^+K_{21}^*)$$

$$= H_{22}+K_{22} - H_{21}H_{11}^+H_{21}^* - K_{21}K_{11}^+K_{21}^*$$

$$= (H_{22} - H_{21}H_{11}^+H_{12}) + (K_{22} - K_{21}K_{11}^+K_{12})$$

$$= H/H_{11} + K/K_{11}.$$

Thus $H+K/H_{11}+K_{11} \geq H/H_{11} + K/K_1$. Since $H_{21}H_{11}^+ = (H_{11}^+H_{12})^*$ and $K_{21}K_{11}^+ = (K_{11}^+K_{12})^*$ by applying Theorem 3 of [7] for the p.s.d and EP matrices H and K, we see that $H/H_{11}$ and $K/K_{11}$ are both EP and p.s.d. Since $H \geq -K$, H+K is Hermitian. By using (3.5), $H_{21}H_{11}^+H_{12}$ is Hermitian and by using (3.6), $K_{21}k_{11}^+k_{12}$ is Hermitian. Hence, $H/H_{11} + K/K_{11} = H_{22}+K_{22} - H_{21}H_{11}^+H_{12} - K_{21}K_{11}^+K_{22}$ is Hermitian. Since $H/H_{11}$ and $K/K_{11}$ are p.s.d. $H/H_{11} + K/K_{11}$ is p.s.d. Hence, $H/H_{11} + K/K_{11}$ is Hermitian and p.s.d. Therefore $H/H_{11} + K/K_{11} \geq 0$. Thus $H+K/ H_{11}+ K_{11} \geq H/H_{11} + K/K_{11} \geq 0$. Hence the Theorem.

## 4. APPLICATION TO LINEAR ELECTROMECHANICAL SYSTEMS:

Let us consider two linear electromechanical system with constitutive operators H and K having the same structure operator A. For terminology and notation one may refer [3]. The transfer impendence $\psi$ (H) and $\psi$ (K) exist by Theorem 7 of [3].

$$\psi(H) = (A^+)^*(H_{22} - H_{21}H_{11}^+H_{12})A^+ = (A^+)^*(H/H_{11})A^+$$

$$\psi(K) = (A^+)^*(K_{22} - K_{21}K_{11}^+K_{12})A^+ = (A^+)^*(K/K_{11})A^+.$$

If we assume that the constitutive operators H and K satisfies (3.5) and (3.6) respectively and $H \geq -K$, then by Theroem (3.2), $H/H_{11} \geq -K/K_{11} \Rightarrow \psi(H) \geq -\psi(K)$. Thus the monotonicity of the constitutive operators is preserved for the corresponding transfer impedances.

## CONCLUSION

We have extended matrix inequalities on a pair of h.p.s.d matrices in the references [1, 2, 4, 5] and on a.p.d matrices in [8,9] for a wider class of range Hermitian matrices.


## ACKNOWLEDGEMENT

This work was done when the author visited the Department of Mathematics, Avinashilingam University for women, Coimbatore, India.


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**Source of support: Under UGC Grant, Conflict of interest: None Declared**