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ON THE HAUSDORFNESS AND FIRST COUNTABILITY OF CONE METRIC SPACES

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ABSTRACT

In this paper the Hausdorffness and first countability of cone metric spaces will be proved. Moreover the proofs given in the paper" Cone metric spaces and fixed point theorems in diametrically contractive mappings [Acta. Math. Sinica, English series. 26(3) (2010)489-496]" will be repaired.

Keywords: *Cone metric space; first countable.*

1. INTRODUCTION

In 2010, D. Turkoglu and M. Abuloha [1] proved that every cone metric space is Hausdorff and first countable. But their proofs have some gaps. In this note by giving a new proof their proofs will be repaired. All notations are considered as given in [1]. In what follows will be needed in the sequel. Let E be a topological vector space (t.v.s. for short) with its zero vector θ. By a cone $P \neq \{\theta\}$ we understand a closed convex subset of E such that $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P \bigcap -P = \{\theta\}$. Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ iff $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$ .if P has nonempty interior.

In the following we always suppose that E is a real t.v.s. with its zero vector θ, P is a cone with $\operatorname{int} P \neq \phi$, $e \in \operatorname{int} P$ and $\leq$ a partial ordering with respect to P.

By a cone metric space we mean an ordered pair (X, d) where, X is any nonempty set and $d : X \times X \to E$ is a mapping, called cone metric, satisfying the following conditions:

(i) $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for $x, y, z \in X$.

It is easy to see that if (X, d) is a cone metric space then the family $\{B(x, c)\}_{c \in \operatorname{int} P}$ is a basis for a topology on X where $B(x, c) = \{y \in X : d(x, y) << c\}$ and $c \in \operatorname{int} P$.

2. MAIN RESULTS

In this section two topological properties of the topology induced by a cone on cone metric space (X, d) will be considered and proved. More precise the Hausdorffness and first countability.

The authors in [1], for proving the Hausdorffness of a cone metric space, assumed that if (X, d) is a cone metric space and $x \neq y$ are two points in X then $d(x, y) = c > 0$ is a member of interior of P, that is $c \in \operatorname{int} P$ (see, line 10 from below, page 491, of [1]) which is not true in general. For example if one take $E = R^2$ and $P = \{(x, y) \in E : 0 \leq y \leq x\}$ and $x = (0,0)$, $y = (1,1)$ will understand this fact. Now we give a new proof for being Hausdorff of a cone metric space.

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Theorem 2.1: Every cone metric space (X,d) is Hausdorff.

Proof: Let $x \neq y$ be two arbitrary points in X. If $B(x,c) \bigcap B(y,c) \neq \phi$, for all $c \in \operatorname{int} P$, then there is $z_c \in B(x,c) \bigcap B(y,c)$ and so by triangle inequality we have $d(x,y) << 2c$, for all $c \in \operatorname{int} P$. Now fixed an element $c \in \operatorname{int} P$ and consider ε an arbitrary positive number.

Hence $2\varepsilon c - d(x,y) \in \operatorname{int} P \subset P$ and then by taking ε to zero and using the closedness of P we deduce that $-d(x,y) \in P$ and so $d(x,y) = \theta$ (note that $d(x,y) \in P$ and P is a pointed cone) and then $x = y$ which is a contradiction.

In Proposition 2 of [1], the authors proved the equality

$$clB(x,c) = cl\{y \in X : d(x,y) << c\} = \{y \in X : d(x,y) \leq c\}$$

where (X,d) is a cone metric space and P is a convex cone of a real Banach space E with $c \in \operatorname{int} P$. But the following example shows that it may fail. Let $X = E = R$ and $P = [0,\infty)$ and define $d : X \times X \to R$ as follows:

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$

Then

$$clB(x,c) = \{x\{ \neq \{y \in X : d(x,y) \leq 1\} = X = R.$$

We note that the set $\{x\{ \neq \{y \in X : d(x,y) \leq 1\} = X = R$ is sequentially closed with respect to the topology induced by meter d ((X,d) is a cone metric space). Because if $\{y_n\}$ is a sequence with the properties $d(y_n,x) \leq c$ and $y_n \to y$, then for positive number ε there exists n_ε such that $d(y_{n_\varepsilon},x) << \varepsilon c$. So

$$d(x,y) \leq d(y_{n_\varepsilon},y) + d(y_{n_\varepsilon},x) << \varepsilon c + c = (1-\varepsilon)c.$$

Hence $(1-\varepsilon)c - d(x,y) \in P$. By letting ε to zero and using the closedness of P we obtain $c - d(x,y) \in P$. This completes the proof of the assertion. Therefore

$$clB(x,c) = cl\{y \in X : d(x,y) << c\} \subseteq \{y \in X : d(x,y) \leq c\}.$$

Proposition 2.2: ([1]) Every cone metric space is first countable.

Proof: Let $c \in \operatorname{int} P$ be an arbitrary element. For each $x \in X$, the family $\left\{B(x,\frac{1}{n}e)\right\}_{n \in N}$ is a countable set of neighborhoods. If $B(x,c)$ is a neighborhood, then there is a natural number n such that $\frac{1}{n}e << c$ and so $B(x,\frac{1}{n}e) \subset B(x,c)$ and so the proof is finished.

Corollary 2.3: Let (X,d) be a cone metric space and A a subset of X. A is closed if and only if A is sequentially closed.

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