International Journal of Mathematical Archive-3(4), 2012, Page: 1743-1753
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON PARTIAL SUMS OF CERTAIN NEW CLASS OF ANALYTIC AND UNIVALENT FUNCTIONS

***Oladipo, A. T.**

Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, Ogbomoso P. M. B. 4000, Ogbomoso, Nigeria, E-mail: atlab_3@yahoo.com

(Received on: 21-03-12; Revised & Accepted on: 09-04-12)

ABSTRACT

Let ω be an arbitrary fixed point in the open unit disk $U=\{z:|z|<1\}$. Let $\Psi(z)$ be a fixed analytic and univalent functions of the form $\psi(z)=(z-\omega)+\sum_{k=2}^{\infty} b_k (z-\omega)^k$ and $H\psi(\omega,b_k,\delta)$ be the subclass consisting of analytic and univalent functions of the form $f(z)=(z-\omega)+\sum_{k=2}^{\infty} a_k (z-\omega)^k$ which satisfy the condition $\sum_{k=2}^{\infty} (r+d)^{k-1} b_k |a_k| \le \delta$.

In the present investigation the author determines the sharp lower bounds for $\Re\left\{\frac{I_\omega^m(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f_n(z)}\right\}$ and $\Re\frac{I_\omega^m(\lambda,l)f_n(z)}{I_\omega^m(\lambda,l)f(z)}$ where $f_n(z)=(z-\omega)+\sum_{k=2}^{n} a_k (z-\omega)^k$ be the sequence of the partial sums of a function $f(z)=(z-\omega)+\sum_{k=2}^{n} a_k (z-\omega)^k$ belonging to the class $H_\Psi(\omega,b_k,\delta)$ and $I_\omega^m(\lambda,l)$ denotes the Aouf derivative operator [2]. This investigation does not only extends the results in [4.5.12.15] but also provides some conditions as remedy for the results of Frasin in [4] and [5]. Our present investigations also give rise to many new classes with new results.

Keywords and Phrases: *Analytic, univalent, partial sums, sequence, Aouf derivative operator.*

2010 Subject classification: *primary 30C45.*

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \tag{1}$$

where are analytic in the open unit disk $U=\{z:|z|<1\}$ and normalized with {0} = 0 and $f'(0)-1=0$. Furthermore, we denote by S the class of functions is A which are univalent in U. A function $f(z)$ in S is said to be starlike of order $\alpha\,(0\le\alpha<1)$, denoted by $S^*(\alpha)$ if it satisfies $R\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\ (z\in U)$. A function $f(z)$ in S is said to be convex of order $\alpha(0\le\alpha<1)$, denoted by $K(\alpha)$ if it satisfies $R\left\{1+\frac{zf'(z)}{f(z)}\right\}>\alpha,\ (z\in U)$.

Several authors have discussed these aforementioned classes as we can see in many existing literatures.

Now, let ω be an arbitrary fixed point in U. Let $A(\omega)\subset A$ denotes the class of functions of the form

****Corresponding author:*Oladipo A. T., *E-mail: atlab_3@yahoo.com***

$$f(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k \tag{2}$$

which are analytic in the open unit disk U and normalized with $f(\omega)=0$ *and* $f'(\omega)-1=0$ [6]. We denote by $S(\omega)\subset S$ the class of functions which are univalent in U. A function $f(z)\in S(\omega)$ is said to be $\omega-$starlike of order α $(0\le\alpha<1)$, denoted by $S^*(\omega,\alpha)$ if it satisfies $R\left\{\frac{(z-\omega)f'(z)}{f(z)}\right\}>\alpha,\ (z\in U)$ and a function $f(z)\in S(\omega)$ is said to be convex of order α $(0\le\alpha<1)$, denoted by $S^c(\omega,\alpha)$, if it satisfies $R\left\{1+\frac{(z-\omega)f''(z)}{f'(z)}\right\}>\alpha,\ (z\in U)$ where ω is an arbitrary fixed point in U. This is deduce able in [8, 10, 11]

Let $T(\omega)$ denote the subclass of $S(\omega)$ whose elements can be represented in the form

$$f(z)=(z-\omega)-\sum_{k=2}^{\infty}a_k(z-\omega)^k,\ a_k\ge 0,\ (z\in U), \tag{3}$$

and ω is arbitrary fixed point in U [9,11].

Here we denote by $H(\omega,\alpha)$ and $K(\omega,\alpha)$ respectively the subfamilies of $S^*(\omega,\alpha)$ *and* $S^c(\omega,\alpha)$ obtained by taking the intersection of $S^*(\omega,\alpha)$ and $S^*(\omega,\alpha)$ with $T(\omega)$, [9, 11]

A sufficient condition for a function of the form (2) to be in $S^*(\omega,\alpha)$ and $S^c(\omega,\alpha)$ are respectively given by

$$\sum_{k=2}^{\infty}(r+d)^{k-1}(k-\alpha)|a_k|\le 1-\alpha \tag{4}$$

and

$$\sum_{k=2}^{\infty}(r+d)^{k-1}k(k-\alpha)|a_k|\le 1-\alpha \tag{5}$$

which is deduceable in [8]. Furthermore, for the functions of the form (3), the above conditions are also necessary [11]. At $d=0\Rightarrow\omega=0$ that is, if f is of the form (1) we have the results of Silverman [14]

Now, let $\Psi(z)\in S(\omega)$ be a fixed function of the form

$$\Psi(z)=(z-\omega)+\sum_{k=2}^{\infty}b_k(z-\omega)^k,\ (b_k\ge b_2\ge 0,\ k\ge 2). \tag{6}$$

Here, we define the class $H_\Psi(\omega,b_k\ \delta)$ consisting of function of the form (2) which satisfies the inequality

$$\sum_{k=2}^{\infty}(r+d)^{k-1}b_k|a_k|\le\delta,\ \ |z|=r,|\omega|=d. \tag{7}$$

where $\delta>0$. This class of functions is the analogue by extension of the one defined by Frasin in [5].

In the present paper, the author wishes to determine sharp lower bounds for $\Re\left\{\frac{I_\omega^m(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f_n(z)}\right\}$ *and* $\Re\frac{I_\omega^m(\lambda,l)f_n(z)}{I_\omega^m(\lambda,l)f(z)}$

where

$$f_n(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k \tag{8}$$

be the sequence of partial sums of a function $f(z)=(z-\omega)-\sum_{k=2}^{\infty}a_k(z-\omega)^k$ belonging to the class $H_\psi[\omega,b_k,\delta]$ and the operator $I_\omega^m(\lambda,l)$ denote the Aouf et al derivative operator introduced in [2], and it is defined as follows $I_\omega^m(\lambda,l):A(\omega)\to A(\omega)$ *such that* $I_\omega^0(\lambda,l)f(z)=f(z)$

$$I_{\omega}^{I}(\lambda,l)\ f(z)=I_{\omega}(\lambda,l)\,f(z)=I_{\omega}^{0}(\lambda,l)\,f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\omega}^{0}(\lambda,l)\,f(z)\right)'\frac{\lambda(z-\omega)}{1+l}$$

$$=(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)a_k\,(z-\omega)^k$$

And

$$I_{\omega}^{2}(\lambda,l)\ f(z)=I_{\omega}^{1}(\lambda,l)\,f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\omega}^{1}(\lambda,l)\,f(z)\right)'\frac{\lambda(z-\omega)}{1+l}$$

$$=(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^2 a_k\,(z-\omega)^k$$

and in general

$$I_{\omega}^{m}(\lambda,l)\ f(z)=I_{\omega}(\lambda,l)\left(I_{\omega}^{m-1}(\lambda,l)\,f(z)\right)=(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k\,(z-\omega)^k$$

$m\in N\ U\ \{0\}=0,1,2,3,\ldots\ \lambda\geq 0,\ l\geq -0,$ and ω is an arbitrary fixed point in U.

Remark A: At $\omega=0$ we have Catas et al derivative operator [3], if $\omega=0$ *and* $l=0$ we obtain A1-Oboundi operator [1]. setting $\omega=0, l=0$ *and* $\lambda=1$ we obtain Salagean derivative operator [13].

The present investigation does not only extends the results of Frasin [4] and [5]. Rossy et al [12] and Silverman [15], but also pointed out some conditions that are must for the result of Frasin [4] and [5], but which are neglected, not only these, the present investigation also give rise to new classes of analytic and univalent functions with new results.

2. MAIN RESULTS

Theorem 2.1: If $f(z)\in H_{\psi}(\omega,b_k,\delta)$, then

$$(i)\qquad \Re\left\{\frac{I_{\omega}^{m}(\lambda,l)\,f(z)}{I_{\omega}^{m}(\lambda,l)\,f_n(z)}\right\}\geq\frac{b_{n+1}-(r+d)^n\sigma^m\delta}{b_{n+1}} \tag{9}$$

and

$$(ii)\qquad \Re\left\{\frac{I_{\omega}^{m}(\lambda,l)\,f_n(z)}{I_{\omega}^{m}(\lambda,l)\,f(z)}\right\}\geq\frac{b_{n+1}}{b_{n+1}+(r+d)^n\sigma^m\delta} \tag{10}$$

where

$$b_k\geq\begin{cases}(r+d)^{k-1}\gamma^m\delta & if\ \ k=2,3,\ldots.,n\\[4pt] \dfrac{(r+d)^{k-1}\gamma^m b_{n+1}}{(r+d)^n\sigma^m} & if\ \ k=n+1,\ n+2,\ldots.\end{cases}$$

and

$$\gamma^m=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m,\quad \sigma^m=\left(\frac{1+\lambda n+l}{1+l}\right)^m$$

The results (9) and (10) are sharp with the function given by

$$f(z)=(z-w)+\frac{\delta}{(r+d)^n b_{n+1}}(z-w)^{n+1} \tag{11}$$

where

$$0<\delta\leq\frac{b_{n+1}}{(r+d)^n\sigma^m}\quad \sigma^m=\left(\frac{1+\lambda n+l}{1+l}\right)^m$$

Proof: To prove (i) we define the function ◊ (z) by

$$\frac{1+\Phi(z)}{1+\Phi(z)}=\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\left[\frac{l_\omega^m(\lambda,l)f(z)}{l_\omega^m(\lambda,l)f_n(z)}-\left(\frac{b_{n+1}-(r+d)^n\sigma^m\delta}{b_{n+1}}\right)\right]$$

$$=\frac{1+\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}+\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^m a_k(z-\omega)^{k-1}}{1+\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^m a_k(z-\omega)^{k-1}}. \tag{12}$$

It suffices to show that $|\Phi(z)|\leq 1,$ from (12) we can write

$$\Phi(z)=\frac{\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}}{2+2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}+\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}}.$$

Hence,

$$\Phi(z)\leq\frac{\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|}{2-2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|-\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|}$$

$\Phi(z)\leq 1$ if

$$2\frac{b_{n+i}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|\leq 2-2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|$$

Or equivalently,

$$\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|+\frac{b_{n+i}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|\leq 1 \tag{13}$$

It is sufficient to show that the L.H.S of (13) is bounded above by

$$\sum_{k=2}^{\infty}\frac{(r+d)^{k-1}b_k}{\delta}\;|a_k|$$

which is equivalent to

$$\sum_{k=2}^{\infty}\frac{(r+d)^{k-1}b_k-(r+d)^{k-1}\gamma^m\delta}{\delta}+\sum_{k=n+1}^{\infty}\frac{(r+d)^n(r+d)^{k-1}b_k-b_{n+1}\gamma^m(r+d)^{k-1}}{(r+d)^n\sigma^m\delta}\geq 0$$

$$\gamma^m=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m \; and \; \sigma^m=\left(\frac{1+\lambda n+l}{1+l}\right)^m$$

To see that the function given by (11) gives the sharp results, we observed that for $(z-w)=(r+d)e^{\frac{i\pi}{n}}$

$$\frac{I_w^m(\lambda,l)f(z)}{I_w^m(\lambda,l)f_n(z)}=1+\frac{\delta}{b_{n+1}}\sigma^m(r+d)^n\to 1-\frac{\delta}{b_{n+1}}\sigma^m(r+d)^n=\frac{b_{n+1}-\delta\sigma^m(r+d)^n}{b_{n+1}}$$

To prove (ii) of our theorem, we write

$$\frac{1+\Phi(z)}{1+\Phi(z)}=\frac{b_{n+1}+\delta\sigma^m(r+d)^n}{(r+d)^n\sigma^m\delta}\left[\frac{I_w^m(\lambda,l)f_n(z)}{I_w^m(\lambda,l)f_n(z)}-\frac{b_{n+1}}{b_{n+1}+\delta\sigma^m(r+d)^n}\right]=$$

$$\frac{1+\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^m a_k(z-w)^{k-1}-\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-w)^{k-1}}{1+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-w)^{k-1}}$$

where

$$|\Phi(z)|\le\frac{\frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|}{2+2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|-\frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|},$$

$$\frac{\frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|}{2+2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|-\frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|},$$

Equality is equivalent to

$$\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|+\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|\le 1.$$

Making use of (7) to get (14). Equality holds in (10) for the function $f(z)$ given by (11) and the proof of Theorem 2.1 is complete.

If we choose d = 0 which implies that $\omega=0,\ r\to 1-$ (i. e for $f(z)$ defined as in (1)), then we obtain the following:

Corollary A: If $f\in H_\Psi(0,b_k\delta)$, and $f(z)$ is of the form (1), then

$$(i)\ R_e\left\{\frac{I_0^m(\lambda,l)f(z)}{I_0^m(\lambda,l)f_n(z)}\right\}\ge\frac{b_{n+1}-\sigma^m\delta}{b_{n+1}} \tag{15}$$

and

$$(ii)\ R_e\left\{\frac{I_0^m(\lambda,l)f_n(z)}{I_0^m(\lambda,l)f(z)}\right\}\ge\frac{b_{n+1}}{b_{n+1}+\sigma^m\delta} \tag{16}$$

where

$$b_k \geq \begin{cases} \gamma^m \delta, \ if \ \ k = 2,3,\ldots,n \\ \dfrac{\gamma^m b_{n+1}}{\sigma^m}, \ if \ \ k = n+1, n+2, \ldots. \end{cases}$$

and

$$\gamma^m = \left(\frac{1+\lambda(k-1)+l}{1+l} \right)^m, \ \sigma^m = \left(\frac{1+\lambda n + l}{1+l} \right)^m$$

with $0 < \delta \leq \dfrac{b_{n+1}}{\sigma^m}$ and the results (15) and (16) are sharp for functions given by (11).

This result is completely new and the operator $l^m(\lambda, l)$ the same as Catas et al derivative operator [3].

Putting $\omega = 0, l = 0$ in Theorem 2.1, we have

Corollary B: If $f \in H_\Psi(0, b_k \delta)$, and $f(z)$ is of the form (1), then

$$(i) \ R_e \left\{ \frac{I_0^m(\lambda,0) f(z)}{I_0^m(\lambda,0) f_n(z)} \right\} \geq \frac{b_{n+1} - (1+\lambda n)^m \delta}{b_{n+1}}$$

and

$$(ii) \ R_e \left\{ \frac{I_0^m(\lambda,0) f_n(z)}{I_0^m(\lambda,0) f(z)} \right\} \geq \frac{b_{n+1}}{b_{n+1} + (1+\lambda n)^m \delta}$$

where

$$b_k \geq \begin{cases} [1+\lambda(k-1)]^m \delta, \ \ if \ k = 2,3,\ldots,n \\ \left[\dfrac{1+\lambda(k-1)}{1+\lambda n} \right]^m b_{n+1} \ if \ \ k = n+1, n+2, \ldots. \end{cases}$$

The result are sharp with functions given by (11) with $0 < \delta \leq \dfrac{b_{n+1}}{(1+\lambda n)^m}$, *and the* $l_0^m(\lambda, 0)$ is the same as AL-Oboudi operator [1], the result is new.

Putting $\lambda = 1$ in corollary B we have

Corollary C: If $f \in H_\Psi(0, b_k \delta)$ then

$$(i) \ R_e \left\{ \frac{I_0^m(1,0) f(z)}{I_0^m(1,0) f_n(z)} \right\} \geq \frac{b_{n+1} - (1+n)^m \delta}{b_{n+1}}$$

and

$$(ii) \ R_e \left\{ \frac{I_0^m(1,0) f_n(z)}{I_0^m(1,0) f(z)} \right\} \geq \frac{b_{n+1}}{b_{n+1} + (1+n)^m \delta}$$

where

$$b_k \geq \begin{cases} k^m \delta, \ if \ \ k = 2,3,\ldots,n \\ \dfrac{k^m b_{n+1}}{(n+1)^m}, \ if \ \ k = n+1, n+2, \ldots. \end{cases}$$

The results are sharp with functions given by (11) with $0<\delta\leq\frac{b_{n+1}}{(n+1)^m}$, and the $I_0^m(1,0)$ is the same as Salagean operator [3], this result is new.

Taking m = 0 in corollary C we obtain the result given by Frasin [5]

Corollary D: If $f\in H_\psi(0,b_k\delta)$, then

$$\frac{f(z)}{f_n(z)}\geq\frac{b_{n+1}-\delta}{b_{n+1}}$$

and

$$\frac{f_n(z)}{f(z)}\geq\frac{b_{n+1}}{b_{n+1}+\delta}$$

where

$$b_k\geq\begin{cases}\delta,\ if\ \ k=2,3,...,n\\ b_{n+1},\ if\ k=n+1,\ n+2,...\end{cases}$$

The results are sharp with the function given by (11).

If we choose $m=1,\lambda=1,l=0,\omega=0$ in Theorem 2.1 we have

Corollary E: If $f\in H_\psi(0,b_k\delta)$ and for f of the form (1), then

$$\frac{f'(z)}{f_n'(z)}\geq\frac{b_{n+1}-(n+1)\delta}{b_{n+1}}$$

and

$$\frac{f_n'(z)}{f'(z)}\geq\frac{b_{n+1}}{b_{n+1}+(n+1)\delta}$$

where

$$b_k\geq\begin{cases}k\delta,\ if\ \ k=2,3,...,n\\ \frac{k(b_{n+1})}{n+1}if\ k=n+1,\ n+2,...\end{cases}$$

The results in corollary E are sharp with function given by (11).

Remark B: Frasin in [5] showed in his Theorem 2.7 that for $f\in H_\psi(0,b_k\delta)$, inequalities in Corollary E hold with the condition that

$$b_k\geq\begin{cases}k\delta,\ if\ \ k=2,3,...,n\\ k\delta\left(1+\frac{b_{n+1}}{n+1}\right)if\ k=n+1,\ n+2,...\end{cases}\tag{17}$$

But it is can easily be seen that condition (17), for $k=n+1$ gives $b_{n+1}\geq(n+1)\delta\left(1+\frac{b_{n+1}}{(n+1)\delta}\right)$ or simply as $\delta\leq0$, which surely contradicts the initial assumption that $\delta>0$. Therefore, Theorem 2.7 of [5] seems not suitable with the condition (17) but we have conditions on b_k in Corollary E as a remedy for Frasin Theorem 2.7 of [5].

If we take $m=0,b_k=\frac{[(1+\rho)k-(\alpha+\rho)]}{1-\alpha}\binom{k+\tau-1}{k}$, where $\tau\geq0,\rho\geq0,-1\leq\alpha<1,l=0,\lambda=1$ and $\delta=1$ in Theorem 2.1, we obtain the following results given by Rosy et al. in [12].

Corollary F: If $f \in A$ is of the form (1) and the condition $\sum_{k=2}^{\infty} b_k |a_k| \le 1$ is satisfied, where

$$b_k = \frac{[(1+\rho)k - (\alpha+\rho)]}{1-\alpha}\binom{k+\tau-1}{k}$$

and $\tau \ge 0, \rho \ge 0, -1 \le \alpha < 1, l = 0, \lambda = 1, l = 0.$ Then

$$R_e\left\{\frac{f(z)}{f_n(z)}\right\} \ge \frac{b_{n+1}-1}{b_{n+1}}, \qquad (z \in U)$$

and

$$R_e\left\{\frac{f_n(z)}{f(z)}\right\} \ge \frac{b_{n+1}}{b_{n+1}+1}, \qquad (z \in U)$$

The results are sharp with function given by

$$f(z) = z + \frac{1}{b_{n+1}} z^{n+1} \tag{18}$$

where

$m = 1, w = 0, \lambda = 1, l = 0, \delta = 1,$ *and* $b_k = \frac{[(1+\rho)k - (\alpha+\rho)]}{1-\alpha}\binom{k+\tau-1}{k}, \ \tau \ge 0, \rho \ge 0, -1 \le \alpha < 1,$ in Theorem 2.1, we have

Corollary G: If f of the form (1) and satisfies $\sum_{k=2}^{\infty} b_k |a_k| \le 1$, then

$$R_e\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{b_{n+1}-(n+1)}{b_{n+1}}$$

and

$$R_e\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{b_{n+1}}{b_{n+1}+(n+1)}$$

$$b_k \ge \begin{cases} k, \ \ if \ \ k = 2,3,\ldots,n \\ \dfrac{kb_{n+1}}{n+1} if \ k = n+1, \ n+2,\ldots \end{cases}$$

The results are sharp with the function given by (18). With $m = 0, b_k = \tau_k - \alpha\mu_k, \delta = 1-\alpha$ where $0 \le \alpha < 1, \tau_k \ge 0, \mu_k \ge 0,$ and $\tau_k \ge \mu_k (k \ge 2), l = 0, \lambda = 1$ in Theorem 2.1 we have the following by Frasin [4].

Corollary H: If f is of the form (1) with and satisfies $\sum_{k=2}^{\infty} (\tau_k - \alpha\mu_k)|a_k| \le 1-\alpha,$ then

$$R_e\left\{\frac{f(z)}{f_n(z)}\right\} \ge \frac{\tau_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\tau_{n+1} - \alpha\mu_{n+1}}, \quad (z \in U)$$

and

$$R_e\left\{\frac{f_n(z)}{f(z)}\right\} \ge \frac{\tau_{n+1} - \alpha\mu_{n+1}}{\tau_{n+1} - \alpha\mu_{n+1} + 1 - \alpha} \quad (z \in U)$$

where

$$\tau_k - \alpha\mu_k \ \ge \begin{cases} 1-\alpha, \ if \ \ k = 2,3,\ldots,n \\ \tau_{n+1} - \alpha\mu_{n+1}, \ if \ k = n+1, \ n+2,\ldots \end{cases}$$

The results are sharp with the function given by

$$f(z)=z+\frac{1-\alpha}{\tau_{n+1}-\alpha\mu_{n+1}}\, z_{n+1} \tag{19}$$

If we take $m=1, \omega=0, b_k=\tau_k-\alpha\mu_k, \delta=1-\alpha, 0\le\alpha<1, \tau_k\ge 0, \mu_k\ge 0, \lambda=1, l=0$ and $\tau_k\ge\mu_k\ (k\ge 2)$ in Theorem 2.1 we have

Corollary I: If f is of the form (1) and satisfy $\sum_{k=2}^{\infty}(\tau_k-\alpha\mu_k)|a_k|\le 1-\alpha,$ then

$$R_e\left\{\frac{f'(z)}{f_n'(z)}\right\}\ge\frac{\tau_{n+1}-\alpha\mu_{n+1}-(n+1)(1-\alpha)}{\tau_{n+1}-\alpha\mu_{n+1}}\qquad (z\in U)$$

and

$$R_e\left\{\frac{f_n'(z)}{f'(z)}\right\}\ge\frac{\tau_{n+1}-\alpha\mu_{n+1}}{\tau_{n+1}-\alpha\mu_{n+1}+(n+1)(1-\alpha)},\qquad (z\in U)$$

where

$$\tau_k-\alpha\mu_k\ \ge\begin{cases} k(1-\alpha), & if\ \ k=2,3,\dots,n \\ \dfrac{k(\tau_n+1-\alpha\mu_{n+1})}{n+1}, & if\ k=n+1,\ n+2,\dots\end{cases}$$

The results are sharp with function given by (19).

Remark C: Frasin obtained the inequalities in Corollary I in his Theorem 2 of [4] under the condition that

$$\tau_{k+1}-\alpha\mu_k+1\ \ge\begin{cases} k(1-\alpha), & if\ \ k=2,3,\dots,n \\ k(1-\alpha)+\dfrac{k(\tau_n+1-\alpha\mu_{n+1})}{n+1}, & if\ k=n+1,\ n+2,\dots\end{cases}$$

But this paper critically looked at the proof of his Theorem 2 of [4] and find out that the last inequality of the theorem,

$$\sum_{k=2}^{n}\left(\frac{\tau_k-\alpha\mu_k}{1-\alpha}\right)|a_k|+\sum_{k=2}^{\infty}\left(\frac{\tau_k-\alpha\mu_k}{1-\alpha}-\left(1+\frac{\tau_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)k\right)|a_k|\ge 0. \tag{20}$$

It is seen that the inequality (20) of [4] Theorem 2) cannot hold with function given by (19) to support the sharpness of the results in Corollary I. This paper provides remedy in our corollary I for the condition (2.25) of Theorem 2 in [4]. Additionally, with $m=0, \omega=0, b_k=(k-\alpha), \lambda=1, l=0, b_k=k(k-\alpha), \delta=1-\alpha, 0\le\alpha<1,$ in our Theorem 2.1, we have Theorem 1-3 given by Silverman in [15], also, if $m=1$ and other parameters remain as in this paragraph, we would have Theorem 4-5 given by Silverman in [15].

The second parts of the corollaries are the ones which give rise to the new classes and new results. Putting $l=0$ in Theorem 2.1 then we have

Corollary J: If $f\in H_\psi(w, b_k\delta_0)$, then

$$(i)\ \ R\left\{\frac{I_w^m(\lambda,0)f(z)}{I_w^m(\lambda,0)f_n(z)}\right\}\ge\frac{b_{n+1}-(r+d)^n\sigma_0^m\delta_0}{b_{n+1}}$$

and

$$(ii)\ \ R\left\{\frac{I_w^m(\lambda,0)f_n(z)}{I_w^m(\lambda,0)f(z)}\right\}\ge\frac{b_{n+1}}{b_{n+1}+(r+d)^n\sigma_0^m\delta_0}$$

where

$$b_k \geq \begin{cases} (r+d)^{k-1}\gamma_0^m \delta_0 \ \ if\ k=2,3...,n \\ \dfrac{(r+d)^{k-1}\gamma_0^m b_{n+1}}{(r+d)^n \delta_0^m} \ \ if\ \ k=n+1,n+2 \end{cases}$$

and

$$\gamma_0^m = [1+\lambda(k-1)]^m,\ \sigma_0^m = [1+\lambda n]^m$$

The results are sharp with the function given by (11) where $0 < \delta_0 \leq \dfrac{b_{n+1}}{(r+d)^n \sigma_0^m}$

If we let $\lambda = 1,\ l = 0$ in Theorem 2.1 we have

Corollary K: If $f \in H_\psi(w, b_k\delta_1)$, then

$$(i)\ \ R\left\{\frac{I_w^m(1,0)f(z)}{I_w^m(1,0)f_n(z)}\right\} \geq \frac{b_{n+1}-(r+d)^n(1+n)^m\delta_1}{b_{n+1}}$$

And

$$(ii)\ \ R\left\{\frac{I_w^m(1,0)f_n(z)}{I_w^m(1,0)f(z)}\right\} \geq \frac{b_{n+1}}{b_{n+1}+(r+d)^n(1+n)^m\delta_1}$$

The results are sharp with the function given in (11) where $0 < \delta_1 \leq \dfrac{b_{n+1}}{(r+d)^n(1+n)^m}$ with

$$b_k \geq \begin{cases} (r+d)^{k-1} k^m \delta \ \ if\ k=2,3...,n \\ \dfrac{(r+d)^{k-1} k^m b_{n+1}}{(r+d)^n(1+n)^m} \ \ if\ \ k=n+1,n+2,... \end{cases}$$

If we continue with various special choices of the parameters involved, many new results shall be obtained.

ACKNOWLEDGMENT

The author wish to thank the organizer of CIMPA school on Real and Complex Analysis and their Applications to other fields which took place in Buea in Cameroonm between 2nd – 14th May 2011, and also for sponsoring me to attend the school. Also, I wish to specially thank Professor (Emeritus) Alline Bonnami of Olean University in France for her role in bringing more light to my understanding in my field.

REFERENCE

[1] F.M. Al-Oboudi. On univalent functions defined by a generalized Salagean derivative operator, Internet. J. Math. Sci., (2004) 1429 – 1436.

[2] M.K. Aouf, A Shamandy, A.O. Mostafa and S.M. Madian. A subclass of M-W starlike functions, Acta Universitatis Apulensis No. 21, 135-142.

[3] A. Catas, G.I. Oros and G.Oro. Differential subordinations associated with multiplier transformations. Abstract Appli Anal. ID 84 5724, 1-11.

[4] B.A. Frasin: Partial sums of certain analytic and univalent function, Acta Math. Acad, Paed. Nyir., 21, (2005), 135 – 145.

[5] B.A. Frasin: Generalization of partial sums of certain analytic and univalent functions Appl. Math Lett, 21 (7), (2008), 735 – 741

[6] S. Kanas and F. Ronning: Uniformily starlike and convex functions and other related classes of univalent functions Ann. Univ. Mariac Curie-Sklodowska section A, 53 (1999), 95 – 105.

[7] A. Mugur and S. Owa: On some subclasses of univalent functions. J. of Inequal. In pure and Appl. Math., Vol. 6, Issue 3 (2005) Article 70, 1-14.

[8] A.T. Oladipo: Certain subclasses of analytic and univalent functions, Advances in Applied Mathematical analysis vol 4 No. 1-2 (2009) 87-93.

[9] A.T. Oladipo: On a subclass of univalent functions defined by Salagean derivative operator. Pioneer Journal of Mathematics and Mathematical Sciences. Vol. 2 No. 2 (2011) 89-100.

[10] A.T. Oladipo: On analytic and univalent functions with negative coefficients. Pioneer Journal of Mathematics and Mathematical Sciences (to appear).

[11] A.T. Oladipo: A short note on certain new classes of analytic and univalent functions in the unit disk. Theory and Applications of Math and Compt. Sc., 1, (2011), 68-73.

[12] T. Rosy, K.G. Subramanian and G. Murugundaramoorthy. Neighbourhoods and partial sums of starlike based on Ruscheweyh derivative, J. Ineq. Pure and Appl. Math; 4, (4) (2003), 1-8, art 64.

[13] G.S. Salagean: Subclasses of univalent functions Complex Analysis-Fifth Romanian Finish Seminar, Bucharest, 1, (1983), 362-372.

[14] H. Silverman: Univalent functions with negative coefficients. Proc. Amer. Math. Soc., 51 (1975), 109 – 116.

[15] H. Silverman: Partial sums of starlike and convex functions J. Math Anal. Appl. 209 (1997), 221 – 227.

Source of support: CIMPA school on Real and Complex Analysis, Conflict of interest: None Declared