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ON PARTIAL SUMS OF CERTAIN NEW CLASS OF ANALYTIC AND UNIVALENT FUNCTIONS

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ABSTRACT

Let ω be an arbitrary fixed point in the open unit disk $U=\{z:|z|<1\}$. Let $\Psi(z)$ be a fixed analytic and univalent functions of the form $\psi(z)=(z-\omega)+\sum_{k=2}^{\infty} b_k(z-\omega)^k$ and $H\psi(\omega,b_k,\delta)$ be the subclass consisting of analytic and univalent functions of the form $f(z)=(z-\omega)+\sum_{k=2}^{\infty} a_k(z-\omega)^k$ which satisfy the condition $\sum_{k=2}^{\infty}(r+d)^{k-1}b_k|a_k|\le\delta$.

In the present investigation the author determines the sharp lower bounds for $\Re\left\{\frac{I_{\omega}^{m}(\lambda,l)f(z)}{I_{\omega}^{m}(\lambda,l)f_n(z)}\right\}$ and $\Re\frac{I_{\omega}^{m}(\lambda,l)f_n(z)}{I_{\omega}^{m}(\lambda,l)f(z)}$ where $f_n(z)=(z-\omega)+\sum_{k=2}^{n} a_k(z-\omega)^k$ be the sequence of the partial sums of a function $f(z)=(z-\omega)+\sum_{k=2}^{n} a_k(z-\omega)^k$ belonging to the class $H_{\Psi}(\omega,b_k,\delta)$ and $I_{\omega}^{m}(\lambda,l)$ denotes the Aouf derivative operator [2]. This investigation does not only extends the results in [4.5.12.15] but also provides some conditions as remedy for the results of Frasin in [4] and [5]. Our present investigations also give rise to many new classes with new results.

Keywords and Phrases: *Analytic, univalent, partial sums, sequence, Aouf derivative operator.*

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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \tag{1}$$

where are analytic in the open unit disk $U=\{z:|z|<1\}$ and normalized with {0} = 0 and $f'(0)-1=0$. Furthermore, we denote by S the class of functions is A which are univalent in U. A function $f(z)$ in S is said to be starlike of order $\alpha(0\le\alpha<1)$, denoted by $S^*(\alpha)$ if it satisfies $R\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\ (z\in U)$. A function $f(z)$ in S is said to be convex of order $\alpha(0\le\alpha<1)$, denoted by $K(\alpha)$ if it satisfies $R\left\{1+\frac{zf'(z)}{f(z)}\right\}>\alpha,\ (z\in U)$.

Several authors have discussed these aforementioned classes as we can see in many existing literatures.

Now, let ω be an arbitrary fixed point in U. Let $A(\omega)\subset A$ denotes the class of functions of the form

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$$f(z)=(z-\omega)+\sum_{k=2}^{\infty} a_k (z-\omega)^k \quad (2)$$

which are analytic in the open unit disk U and normalized with $f(\omega)=0$ *and* $f'(\omega)-1=0$ [6]. We denote by $S(\omega)\subset S$ the class of functions which are univalent in U. A function $f(z)\in S(\omega)$ is said to be $\omega-$starlike of order α $(0\le\alpha<1)$, denoted by $S^*(\omega,\alpha)$ if it satisfies $R\left\{\frac{(z-\omega)f'(z)}{f(z)}\right\}>\alpha,\ (z\in U)$ and a function $f(z)\in S(\omega)$ is said to be convex of order α $(0\le\alpha<1)$, denoted by $S^c(\omega,\alpha)$, if it satisfies $R\left\{1+\frac{(z-\omega)f''(z)}{f'(z)}\right\}>\alpha,\ (z\in U)$ where ω is an arbitrary fixed point in U. This is deduce able in [8, 10, 11]

Let $T(\omega)$ denote the subclass of $S(\omega)$ whose elements can be represented in the form

$$f(z)=(z-\omega)-\sum_{k=2}^{\infty} a_k (z-\omega)^k,\ a_k\ge 0,\ (z\in U), \quad (3)$$

and ω is arbitrary fixed point in U [9,11].

Here we denote by $H(\omega,\alpha)$ and $K(\omega,\alpha)$ respectively the subfamilies of $S^*(\omega,\alpha)$ *and* $S^c(\omega,\alpha)$ obtained by taking the intersection of $S^*(\omega,\alpha)$ and $S^*(\omega,\alpha)$ with $T(\omega)$, [9, 11]

A sufficient condition for a function of the form (2) to be in $S^*(\omega,\alpha)$ and $S^c(\omega,\alpha)$ are respectively given by

$$\sum_{k=2}^{\infty}(r+d)^{k-1}(k-\alpha)|a_k|\le 1-\alpha \quad (4)$$

and

$$\sum_{k=2}^{\infty}(r+d)^{k-1}k(k-\alpha)|a_k|\le 1-\alpha \quad (5)$$

which is deduceable in [8]. Furthermore, for the functions of the form (3), the above conditions are also necessary [11]. At $d=0\Rightarrow\omega=0$ that is, if f is of the form (1) we have the results of Silverman [14]

Now, let $\Psi(z)\in S(\omega)$ be a fixed function of the form

$$\Psi(z)=(z-\omega)+\sum_{k=2}^{\infty} b_k (z-\omega)^k,\ (b_k\ge b_2\ge 0,\ k\ge 2). \quad (6)$$

Here, we define the class $H_\Psi(\omega,b_k\ \delta)$ consisting of function of the form (2) which satisfies the inequality

$$\sum_{k=2}^{\infty}(r+d)^{k-1} b_k|a_k|\le\delta,\ \ |z|=r, |\omega|=d. \quad (7)$$

where $\delta>0$. This class of functions is the analogue by extension of the one defined by Frasin in [5].

In the present paper, the author wishes to determine sharp lower bounds for $\Re\left\{\frac{I_\omega^m(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f_n(z)}\right\}$ *and* $\Re\frac{I_\omega^m(\lambda,l)f_n(z)}{I_\omega^m(\lambda,l)f(z)}$

where

$$f_n(z)=(z-\omega)+\sum_{k=2}^{\infty} a_k (z-\omega)^k \quad (8)$$

be the sequence of partial sums of a function $f(z)=(z-\omega)-\sum_{k=2}^{\infty} a_k (z-\omega)^k$ belonging to the class $H_\psi[\omega,b_k,\delta]$ and the operator $I_\omega^m(\lambda,l)$ denote the Aouf et al derivative operator introduced in [2], and it is defined as follows $I_\omega^m(\lambda,l): A(\omega)\to A(\omega)$ *such that* $I_\omega^0(\lambda,l)f(z)=f(z)$

$$I_{\omega}^{I}(\lambda,l)\ f(z)=I_{\omega}(\lambda,l)\,f(z)=I_{\omega}^{0}(\lambda,l)\,f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\omega}^{0}(\lambda,l)\,f(z)\right)'\frac{\lambda(z-\omega)}{1+l}$$

$$=(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)a_k\,(z-\omega)^k$$

And

$$I_{\omega}^{2}(\lambda,l)\ f(z)=I_{\omega}^{1}(\lambda,l)\,f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\omega}^{1}(\lambda,l)\,f(z)\right)'\frac{\lambda(z-\omega)}{1+l}$$

$$=(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^2 a_k\,(z-\omega)^k$$

and in general

$$I_{\omega}^{m}(\lambda,l)\ f(z)=I_{\omega}(\lambda,l)\left(I_{\omega}^{m-1}(\lambda,l)\,f(z)\right)=(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k\,(z-\omega)^k$$

$m\in N\ U\ \{0\}=0,1,2,3,\ldots\ \lambda\geq 0,\ l\geq -0,$ and ω is an arbitrary fixed point in U.

Remark A: At $\omega=0$ we have Catas et al derivative operator [3], if $\omega=0$ *and* $l=0$ we obtain A1-Oboundi operator [1]. setting $\omega=0, l=0$ *and* $\lambda=1$ we obtain Salagean derivative operator [13].

The present investigation does not only extends the results of Frasin [4] and [5]. Rossy et al [12] and Silverman [15], but also pointed out some conditions that are must for the result of Frasin [4] and [5], but which are neglected, not only these, the present investigation also give rise to new classes of analytic and univalent functions with new results.

2. MAIN RESULTS

Theorem 2.1: If $f\ (z)\in H_{\psi}\ (\omega,b_k,\delta),$ then

$$(i)\qquad \Re\left\{\frac{I_{\omega}^{m}(\lambda,l)\,f(z)}{I_{\omega}^{m}(\lambda,l)\,f_n(z)}\right\}\geq\frac{b_{n+1}-(r+d)^n\sigma^m\delta}{b_{n+1}}\qquad(9)$$

and

$$(ii)\qquad \Re\left\{\frac{I_{\omega}^{m}(\lambda,l)\,f_n(z)}{I_{\omega}^{m}(\lambda,l)\,f(z)}\right\}\geq\frac{b_{n+1}}{b_{n+1}+(r+d)^n\sigma^m\delta}\qquad(10)$$

where

$$b_k\geq\begin{cases}(r+d)^{k-1}\gamma^m\delta & if\ \ k=2,3,\ldots,n\\ \dfrac{(r+d)^{k-1}\gamma^m b_{n+1}}{(r+d)^n\sigma^m} & if\ \ k=n+1,\ n+2,\ldots\end{cases}$$

and

$$\gamma^m=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m,\quad \sigma^m=\left(\frac{1+\lambda n+l}{1+l}\right)^m$$

The results (9) and (10) are sharp with the function given by

$$f\,(z)=(z-w)+\frac{\delta}{(r+d)^n\,b_{n+1}}\,(z-w)^{n+1}\qquad(11)$$

where

$$0<\delta\leq\frac{b_{n+1}}{(r+d)^n\sigma^m}\quad \sigma^m=\left(\frac{1+\lambda n+l}{1+l}\right)^m$$

Proof: To prove (i) we define the function ◊ (z) by

$$\frac{1+\Phi(z)}{1+\Phi(z)}=\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\left[\frac{l_{\omega}^{m}(\lambda,l)f(z)}{l_{\omega}^{m}(\lambda,l)f_n(z)}-\left(\frac{b_{n+1}-(r+d)^n\sigma^m\delta}{b_{n+1}}\right)\right]$$

$$=\frac{1+\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}+\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^m a_k(z-\omega)^{k-1}}{1+\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^m a_k(z-\omega)^{k-1}}. \tag{12}$$

It suffices to show that $|\Phi(z)|\le 1$, from (12) we can write

$$\Phi(z)=\frac{\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}}{2+2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}+\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^{k-1}}.$$

Hence,

$$\Phi(z)\le\frac{\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|}{2-2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|-\frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|}$$

$\Phi(z)\le 1$ if

$$2\frac{b_{n+i}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|\le 2-2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|$$

Or equivalently,

$$\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|+\frac{b_{n+i}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m (r+d)^{k-1}|a_k|\le 1 \tag{13}$$

It is sufficient to show that the L.H.S of (13) is bounded above by

$$\sum_{k=2}^{\infty}\frac{(r+d)^{k-1}b_k}{\delta}\ |a_k|$$

which is equivalent to

$$\sum_{k=2}^{\infty}\frac{(r+d)^{k-1}b_k-(r+d)^{k-1}\gamma^m\delta}{\delta}+\sum_{k=n+1}^{\infty}\frac{(r+d)^n(r+d)^{k-1}b_k-b_{n+1}\gamma^m(r+d)^{k-1}}{(r+d)^n\sigma^m\delta}\ \ge 0$$

$$\gamma^m=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m \ and\ \sigma^m=\left(\frac{1+\lambda n+l}{1+l}\right)^m$$

To see that the function given by (11) gives the sharp results, we observed that for $(z-w)=(r+d)e^{\frac{i\pi}{n}}$

$$\frac{I_w^m(\lambda,l)f(z)}{I_w^m(\lambda,l)f_n(z)} = 1+\frac{\delta}{b_{n+1}}\sigma^m(r+d)^n \to 1-\frac{\delta}{b_{n+1}}\sigma^m(r+d)^n = \frac{b_{n+1}-\delta\sigma^m(r+d)^n}{b_{n+1}}$$

To prove (ii) of our theorem, we write

$$\frac{1+\Phi(z)}{1+\Phi(z)} = \frac{b_{n+1}+\delta\sigma^m(r+d)^n}{(r+d)^n\sigma^m\delta}\left[\frac{I_w^m(\lambda,l)f_n(z)}{I_w^m(\lambda,l)f_n(z)} - \frac{b_{n+1}}{b_{n+1}+\delta\sigma^m(r+d)^n}\right] =$$

$$\frac{1+\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+1}{1+l}\right)^m a_k(z-w)^{k-1} - \frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-w)^{k-1}}{1+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-w)^{k-1}}$$

where

$$|\Phi(z)| \le \frac{\frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|}{2+2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k| - \frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|},$$

$$\frac{\frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|}{2+2\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k| - \frac{b_{n+1}-\sigma^m\delta(r+d)^n}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k|},$$

Equality is equivalent to

$$\sum_{k=2}^{n}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k| + \frac{b_{n+1}}{(r+d)^n\sigma^m\delta}\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m(r+d)^{k-1}|a_k| \le 1.$$

Making use of (7) to get (14). Equality holds in (10) for the function $f(z)$ given by (11) and the proof of Theorem 2.1 is complete.

If we choose d = 0 which implies that $\omega=0,\ r\to 1-$ (i. e for $f(z)$ defined as in (1)), then we obtain the following:

Corollary A: If $f\in H_\Psi(0,b_k\delta)$, and $f(z)$ is of the form (1), then

$$(i)\ R_e\left\{\frac{I_0^m(\lambda,l)f(z)}{I_0^m(\lambda,l)f_n(z)}\right\} \ge \frac{b_{n+1}-\sigma^m\delta}{b_{n+1}} \tag{15}$$

and

$$(ii)\ R_e\left\{\frac{I_0^m(\lambda,l)f_n(z)}{I_0^m(\lambda,l)f(z)}\right\} \ge \frac{b_{n+1}}{b_{n+1}+\sigma^m\delta} \tag{16}$$

where

$$b_k \geq \begin{cases} \gamma^m \delta, \ if \ \ k = 2,3,\dots,n \\ \dfrac{\gamma^m b_{n+1}}{\sigma^m}, \ if \ \ k = n+1, n+2, \dots \end{cases}$$

and

$$\gamma^m = \left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m, \ \sigma^m = \left(\frac{1+\lambda n + l}{1+l}\right)^m$$

with $0 < \delta \leq \dfrac{b_{n+1}}{\sigma^m}$ and the results (15) and (16) are sharp for functions given by (11).

This result is completely new and the operator $l^m(\lambda, l)$ the same as Catas et al derivative operator [3].

Putting $\omega = 0, l = 0$ in Theorem 2.1, we have

Corollary B: If $f \in H_\Psi(0, b_k\delta)$, and $f(z)$ is of the form (1), then

$$(i) \ R_e\left\{\frac{I_0^m(\lambda,0)f(z)}{I_0^m(\lambda,0)f_n(z)}\right\} \geq \frac{b_{n+1} - (1+\lambda n)^m \delta}{b_{n+1}}$$

and

$$(ii) \ R_e\left\{\frac{I_0^m(\lambda,0)f_n(z)}{I_0^m(\lambda,0)f(z)}\right\} \geq \frac{b_{n+1}}{b_{n+1} + (1+\lambda n)^m \delta}$$

where

$$b_k \geq \begin{cases} [1+\lambda(k-1)]^m \delta, \ \ if \ k = 2,3,\dots,n \\ \left[\dfrac{1+\lambda(k-1)}{1+\lambda n}\right]^m b_{n+1} \ if \ \ k = n+1, n+2, \dots \end{cases}$$

The result are sharp with functions given by (11) with $0 < \delta \leq \dfrac{b_{n+1}}{(1+\lambda n)^m}$, *and the* $l_0^m(\lambda,0)$ is the same as AL-Oboudi operator [1], the result is new.

Putting $\lambda = 1$ in corollary B we have

Corollary C: If $f \in H_\Psi(0, b_k\delta)$ then

$$(i) \ R_e\left\{\frac{I_0^m(1,0)f(z)}{I_0^m(1,0)f_n(z)}\right\} \geq \frac{b_{n+1} - (1+n)^m \delta}{b_{n+1}}$$

and

$$(ii) \ R_e\left\{\frac{I_0^m(1,0)f_n(z)}{I_0^m(1,0)f(z)}\right\} \geq \frac{b_{n+1}}{b_{n+1} + (1+n)^m \delta}$$

where

$$b_k \geq \begin{cases} k^m \delta, \ if \ \ k = 2,3,\dots,n \\ \dfrac{k^m b_{n+1}}{(n+1)^m}, \ if \ \ k = n+1, n+2, \dots \end{cases}$$

The results are sharp with functions given by (11) with $0<\delta\le\frac{b_{n+1}}{(n+1)^m}$, and the $I_0^m(1,0)$ is the same as Salagean operator [3], this result is new.

Taking m = 0 in corollary C we obtain the result given by Frasin [5]

Corollary D: If $f\in H_\psi(0,b_k\delta)$, then

$$\frac{f(z)}{f_n(z)}\ge\frac{b_{n+1}-\delta}{b_{n+1}}$$

and

$$\frac{f_n(z)}{f(z)}\ge\frac{b_{n+1}}{b_{n+1}+\delta}$$

where

$$b_k\ge\begin{cases}\delta,\ if\ \ k=2,3,\dots,n\\ b_{n+1},\ if\ k=n+1,\ n+2,\dots\end{cases}$$

The results are sharp with the function given by (11).

If we choose $m=1,\lambda=1,l=0,\omega=0$ in Theorem 2.1 we have

Corollary E: If $f\in H_\psi(0,b_k\delta)$ and for f of the form (1), then

$$\frac{f'(z)}{f_n'(z)}\ge\frac{b_{n+1}-(n+1)\delta}{b_{n+1}}$$

and

$$\frac{f_n'(z)}{f'(z)}\ge\frac{b_{n+1}}{b_{n+1}+(n+1)\delta}$$

where

$$b_k\ge\begin{cases}k\delta,\ if\ \ k=2,3,\dots,n\\ \dfrac{k(b_{n+1})}{n+1}if\ k=n+1,\ n+2,\dots\end{cases}$$

The results in corollary E are sharp with function given by (11).

Remark B: Frasin in [5] showed in his Theorem 2.7 that for $f\in H_\psi(0,b_k\delta)$, inequalities in Corollary E hold with the condition that

$$b_k\ge\begin{cases}k\delta,\ if\ \ k=2,3,\dots,n\\ k\delta\left(1+\dfrac{b_{n+1}}{n+1}\right)if\ k=n+1,\ n+2,\dots\end{cases}\tag{17}$$

But it is can easily be seen that condition (17), for $k=n+1$ gives $b_{n+1}\ge(n+1)\delta\left(1+\frac{b_{n+1}}{(n+1)\delta}\right)$ or simply as $\delta\le0$, which surely contradicts the initial assumption that $\delta>0$. Therefore, Theorem 2.7 of [5] seems not suitable with the condition (17) but we have conditions on b_k in Corollary E as a remedy for Frasin Theorem 2.7 of [5].

If we take $m=0,b_k=\frac{[(1+\rho)k-(\alpha+\rho)]}{1-\alpha}\binom{k+\tau-1}{k}$, where $\tau\ge0,\rho\ge0,-1\le\alpha<1,l=0,\lambda=1$ and $\delta=1$ in Theorem 2.1, we obtain the following results given by Rosy et al. in [12].

Corollary F: If $f \in A$ is of the form (1) and the condition $\sum_{k=2}^{\infty} b_k |a_k| \le 1$ is satisfied, where

$$b_k = \frac{[(1+\rho)k - (\alpha+\rho)]}{1-\alpha} \binom{k+\tau-1}{k}$$

and $\tau \ge 0, \rho \ge 0, -1 \le \alpha < 1, l = 0, \lambda = 1, l = 0.$ Then

$$R_e\left\{\frac{f(z)}{f_n(z)}\right\} \ge \frac{b_{n+1}-1}{b_{n+1}}, \qquad (z \in U)$$

and

$$R_e\left\{\frac{f_n(z)}{f(z)}\right\} \ge \frac{b_{n+1}}{b_{n+1}+1}, \qquad (z \in U)$$

The results are sharp with function given by

$$f(z) = z + \frac{1}{b_{n+1}} z^{n+1} \tag{18}$$

where

$m = 1, w = 0, \lambda = 1, l = 0, \delta = 1,$ *and* $b_k = \dfrac{\left[(1+\rho)k - (\alpha+\rho)\right]}{1-\alpha} \dbinom{k+\tau-1}{k}, \ \tau \ge 0, \rho \ge 0, -1 \le \alpha < 1,$ in Theorem 2.1, we have

Corollary G: If f of the form (1) and satisfies $\sum_{k=2}^{\infty} b_k |a_k| \le 1$, then

$$R_e\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{b_{n+1}-(n+1)}{b_{n+1}}$$

and

$$R_e\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{b_{n+1}}{b_{n+1}+(n+1)}$$

$$b_k \ge \begin{cases} k, \ \ if \ \ k = 2,3,\dots,n \\ \dfrac{kb_{n+1}}{n+1} if \ k = n+1, \ n+2,\dots \end{cases}$$

The results are sharp with the function given by (18). With $m = 0, b_k = \tau_k - \alpha\mu_k, \delta = 1-\alpha$ where $0 \le \alpha < 1, \tau_k \ge 0, \mu_k \ge 0,$ and $\tau_k \ge \mu_k (k \ge 2), l = 0, \lambda = 1$ in Theorem 2.1 we have the following by Frasin [4].

Corollary H: If f is of the form (1) with and satisfies $\sum_{k=2}^{\infty} (\tau_k - \alpha\mu_k)|a_k| \le 1-\alpha,$ then

$$R_e\left\{\frac{f(z)}{f_n(z)}\right\} \ge \frac{\tau_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\tau_{n+1} - \alpha\mu_{n+1}}, \quad (z \in U)$$

and

$$R_e\left\{\frac{f_n(z)}{f(z)}\right\} \ge \frac{\tau_{n+1} - \alpha\mu_{n+1}}{\tau_{n+1} - \alpha\mu_{n+1} + 1 - \alpha} \qquad (z \in U)$$

where

$$\tau_k - \alpha\mu_k \ \ge \begin{cases} 1-\alpha, \ if \ \ k = 2,3,\dots,n \\ \tau_{n+1} - \alpha\mu_{n+1}, \ if \ k = n+1, \ n+2,\dots \end{cases}$$

The results are sharp with the function given by

$$f(z)=z+\frac{1-\alpha}{\tau_{n+1}-\alpha\mu_{n+1}}\,z_{n+1} \tag{19}$$

If we take $m=1,\omega=0,b_k=\tau_k-\alpha\mu_k,\delta=1-\alpha,0\le\alpha<1,\tau_k\ge 0,\mu_k\ge 0,\lambda=1,l=0$ and $\tau_k\ge\mu_k\ (k\ge 2)$ in Theorem 2.1 we have

Corollary I: If f is of the form (1) and satisfy $\sum_{k=2}^{\infty}(\tau_k-\alpha\mu_k)|a_k|\le 1-\alpha,$ then

$$R_e\left\{\frac{f'(z)}{f_n'(z)}\right\}\ge\frac{\tau_{n+1}-\alpha\mu_{n+1}-(n+1)(1-\alpha)}{\tau_{n+1}-\alpha\mu_{n+1}}\qquad (z\in U)$$

and

$$R_e\left\{\frac{f_n'(z)}{f'(z)}\right\}\ge\frac{\tau_{n+1}-\alpha\mu_{n+1}}{\tau_{n+1}-\alpha\mu_{n+1}+(n+1)(1-\alpha)},\qquad (z\in U)$$

where

$$\tau_k-\alpha\mu_k\ \ge\begin{cases}k(1-\alpha), & if\ \ k=2,3,\ldots,n\\[4pt] \dfrac{k(\tau_n+1-\alpha\mu_{n+1})}{n+1}, & if\ k=n+1,\ n+2,\ldots\end{cases}$$

The results are sharp with function given by (19).

Remark C: Frasin obtained the inequalities in Corollary I in his Theorem 2 of [4] under the condition that

$$\tau_{k+1}-\alpha\mu_k+1\ \ge\begin{cases}k(1-\alpha), & if\ \ k=2,3,\ldots,n\\[4pt] k(1-\alpha)+\dfrac{k(\tau_n+1-\alpha\mu_{n+1})}{n+1}, & if\ k=n+1,\ n+2,\ldots\end{cases}$$

But this paper critically looked at the proof of his Theorem 2 of [4] and find out that the last inequality of the theorem,

$$\sum_{k=2}^{n}\left(\frac{\tau_k-\alpha\mu_k}{1-\alpha}\right)|a_k|+\sum_{k=2}^{\infty}\left(\frac{\tau_k-\alpha\mu_k}{1-\alpha}-\left(1+\frac{\tau_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)k\right)|a_k|\ge 0. \tag{20}$$

It is seen that the inequality (20) of [4] Theorem 2) cannot hold with function given by (19) to support the sharpness of the results in Corollary I. This paper provides remedy in our corollary I for the condition (2.25) of Theorem 2 in [4]. Additionally, with $m=0,\omega=0,b_k=(k-\alpha),\ \lambda=1,l=0,\ b_k=k(k-\alpha),\delta=1-\alpha,0\le\alpha<1,$ in our Theorem 2.1, we have Theorem 1-3 given by Silverman in [15], also, if $m=1$ and other parameters remain as in this paragraph, we would have Theorem 4-5 given by Silverman in [15].

The second parts of the corollaries are the ones which give rise to the new classes and new results. Putting $l=0$ in Theorem 2.1 then we have

Corollary J: If $f\in H_\psi(w,b_k\delta_0)$, then

$$(i)\ \ R\left\{\frac{I_w^m(\lambda,0)f(z)}{I_w^m(\lambda,0)f_n(z)}\right\}\ge\frac{b_{n+1}-(r+d)^n\sigma_0^m\delta_0}{b_{n+1}}$$

and

$$(ii)\ \ R\left\{\frac{I_w^m(\lambda,0)f_n(z)}{I_w^m(\lambda,0)f(z)}\right\}\ge\frac{b_{n+1}}{b_{n+1}+(r+d)^n\sigma_0^m\delta_0}$$

where

$$b_k \geq \begin{cases} (r+d)^{k-1}\gamma_0^m \delta_0 \;\; if\; k=2,3...,n \\ \dfrac{(r+d)^{k-1}\gamma_0^m b_{n+1}}{(r+d)^n \delta_0^m} \;\; if\;\; k=n+1,n+2 \end{cases}$$

and

$$\gamma_0^m = [1+\lambda(k-1)]^m,\; \sigma_0^m = [1+\lambda n]^m$$

The results are sharp with the function given by (11) where $0<\delta_0 \leq \dfrac{b_{n+1}}{(r+d)^n \sigma_0^m}$

If we let $\lambda=1,\; l=0$ in Theorem 2.1 we have

Corollary K: If $f \in H_{\psi}(w, b_k \delta_1)$, then

$$(i)\;\; R\left\{\frac{I_w^m(1,0)f(z)}{I_w^m(1,0)f_n(z)}\right\} \geq \frac{b_{n+1}-(r+d)^n(1+n)^m \delta_1}{b_{n+1}}$$

And

$$(ii)\;\; R\left\{\frac{I_w^m(1,0)f_n(z)}{I_w^m(1,0)f(z)}\right\} \geq \frac{b_{n+1}}{b_{n+1}+(r+d)^n(1+n)^m \delta_1}$$

The results are sharp with the function given in (11) where $0<\delta_1 \leq \dfrac{b_{n+1}}{(r+d)^n(1+n)^m}$ with

$$b_k \geq \begin{cases} (r+d)^{k-1}k^m \delta \;\; if\; k=2,3...,n \\ \dfrac{(r+d)^{k-1}k^m b_{n+1}}{(r+d)^n(1+n)^m} \;\; if\;\; k=n+1,n+2,... \end{cases}$$

If we continue with various special choices of the parameters involved, many new results shall be obtained.

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