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DUAL GRAPHS AND CELLULAR FOLDING OF 2-MANIFOLDS

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ABSTRACT

In this paper we have constructed dual graphs associated with the cellular folding in a natural way. Then we obtained the condition under which we may have a successive foldings of a CW-complex into itself. We also obtained the conditions for a cartesian product and a wedge sum of two cellular foldings to be a cellular folding. These conditions are obtained in terms of the dual graphs and by using the above results we also obtained some other results.

Keywords: *Cellular folding, Dual graph, Cartesian product, Wedge sum, Suspension.*

2001 Mathematics subject classification: *51H10. 57H10.*

1. INTRODUCTION

A cellular folding is a folding defined on regular *CW*-complexes first defined by, E. El-Kholy and H. Al-Khurasani, [1] and various properties of this type of folding are also studied by them.

By a cellular folding of regular *CW*-complexes, it is meant a cellular map $f: K \to L$ which maps *i*-cells of *K* to *i*-cells of *L* and such that $f \mid e^i$, for each *i*-cell *e*, is a homeomorphism onto its image.

The set of regular *CW*-complexes together with cellular foldings form a category denoted by $C(K, L)$. If $f \in C(K, L)$; then $x \in K$ is said to be a singularity of f iff f is not a local homeomorphism at *x*. The set of all singularities of f is denoted by $\sum f$. This set corresponds to the "folds" of the map. It is noticed that for a cellular folding *f*, the set $\sum f$ of singularities of f is a proper subset of the union of cells of dimension $\leq n-1$. Thus, when we consider any $f \in C(K, L)$, where *K* and *L* are connected regular *CW*-complexes of dimension 2, the set $\sum f$ will consists of 0-cells, and 1-cells, each 0-cell (vertex) has an even valency, [2, 3], of course, $\sum f$ need not to be connected. Thus in this case $\sum f$ has the structure of a locally finite graph Γ_f embedded in *K*, for which every vertex has an even valency. Note that if *K* is compact, then Γ_f is finite, also any compact connected 2-manifold without boundary (surface) *K* with a finite cell decomposition is a regular *CW*-complex, then the 0- and 1-cells of the decomposition *K* form a finite graph Γ_f without loops and f folds *K* along the edges or 1-cells of Γ_f.

Now a neat cellular folding $f: K \to L$ is a cellular folding such that $L^n - L^{n-1}$ consists of a single *n*-cell, $\operatorname{Int} L$, [2].

2. DEFINITIONS

Let $f \in C(K, L)$, then there is a cellular subdivision *S* on *K* by singularities of *f*. In the following we give the definition of the dual graph Γ_f^* associated to this stratification in a natural way.

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In fact the vertices of the graph Γ_f^* are just the *n*-cells of *S* and the edges are some $(n-1)$ cells. An edge $E \in S_{n-1}$ means that *E* lies in the frontiers of exactly two *n*-cells $\gamma, \gamma' \in S_n$ where $f(\gamma) = f(\gamma')$ such that $E, E' \in S_{n-1}$ are equivalent (the same) iff both *E* and E' lies in the frontier of γ and γ' and $\partial(\gamma) = \partial(\gamma')$ contains more than $(n-1)$-cells. We then say that *E* is an edge in Γ_f^* with end points γ, γ'. It is possible to realize Γ_f^* as a graph $\widetilde{\Gamma}_f^*$ embedded in *K* as follows:

For each *n*-cell $\gamma \in S_n$, we choose any point $u \in \gamma$. If $\gamma, \gamma' \in S_n$ are end points of $E \in S_{n-1}$, then we can join *u* to *v* by an arc $\widetilde{E}$ in *K* that runs from *u* through γ and γ' to *v* crossing *E* transversely at a single point. Trivially, the correspondence $\gamma \to u,\ E \to \widetilde{E}$ is a graph isomorphism from Γ_f^* to $\widetilde{\Gamma}_f^*$. It should be noted that the graph Γ_f^* has no multiple edges, no loops and generally disconnected. It will be connected in the case of neat cellular foldings, since all the *n*-cells will be send to the same *n*-cell. The graph Γ_f^* is the dual graph of Γ_f.

If (M, N) is a *CW*-pair consisting of a cell complex *M* and a subcomplex *N*, then the quotient space M / N inherits a natural cell complex structure from *M*. The cells of M / N are the cells of $M - N$ plus one new 0-cell, the image of *N* in M / N, [4].

The suspension SM of a complex *M* is the union of all line segments joining points of *M* to two external vertices, [4].

2-1 Examples

(a) Consider a complex *K* such that $|K|$ is a torus with cellular subdivision consisting of eight 0-cells, sixteen 1-cells and eight 2-cells, let $f: K \to K$ be a cellular folding given by

$$f(e_1^0, ..., e_8^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_2^0, e_1^0, e_3^0, e_4^0)$$
$$f(e_1^2, ..., e_8^2) = (e_1^2, e_2^2, e_1^2, e_2^2, e_1^2, e_2^2, e_1^2, e_2^2)$$

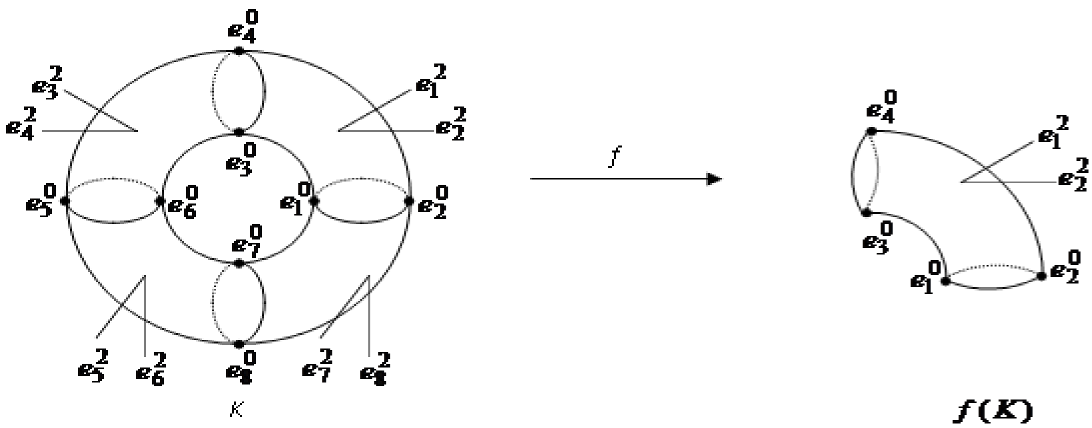

Fig. (1)

In this case the dual folding graph Γ_f^* is as shown in Fig. (2)

Fig. (2) : The dual folding graph Γ_f^*.

(b) Let K be a complex such that $|K|$ is a cylinder with cellular subdivision consisting of eight vertices, sixteen 1-cells and eight 2-cells, see Fig. (3). Let $f : K \to K$ be a cellular folding given by:

$$f(a, b, c, d, e, f, g, h) = (a, b, c, b, e, f, g, f)$$
$$f(\sigma_i) = \sigma_7, \qquad i = 1, \ldots, 7$$

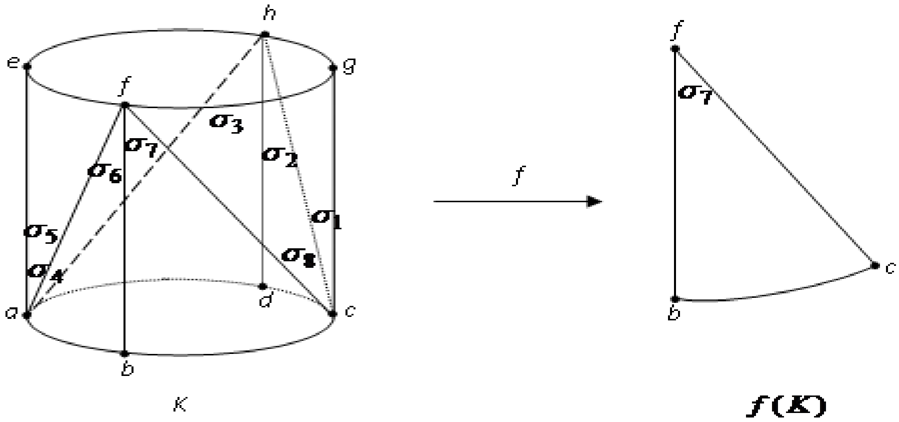

Fig. (3)

The dual folding graph Γ_f^* is as shown in Fig. (4), it is a connected graph.

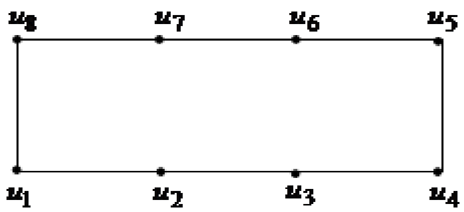

Fig. (4) : The dual folding graph Γ_f^*.

3. MAIN RESULTS

The following theorem gives the condition satisfied by the dual graphs in order to have a successive folding of *CW*-complex into itself.

Theorem 1: Let *K*, *L* and *M* be regular *CW*-complexes of the same dimension 2, such that $M \subset L \subset K$, let $f : K \to K,\ g : L \to L$ be cellular foldings such that $f(K) = L \neq K$ and $g(L) = M \neq L$ with dual folding graphs $\Gamma_f^* = (V_f, E_f)$ and $\Gamma_g^* = (V_g, E_g)$. Then $g \circ f$ is a cellular folding from *K* into *M* with dual folding graph $\Gamma_{g\circ f}^* = (V, E)$ such that $\Gamma_{g\circ f}^* = \Gamma_f^* \cup f^{-1}(\Gamma_g^*)$.

Proof: Let $f : K \to L,\ g : L \to M$ be cellular foldings, then *K* and *L* have stratifications S, S' respectively, such that $V_f = \{n - \text{cells of } S\}$ and $V_g = \{n - \text{cells of } S'\}$ where $S' \subset S$. But the composition map $g \circ f$ has the same stratification *S* on *K* i.e., $V_{g\circ f} = V_f$.

Also if $e \in E$, then $e \in S_{n-1}$ and lies in the frontier of exactly two *n*-cells $\gamma, \gamma' \in S_n$ such that $(g \circ f)(\gamma) = (g \circ f)(\gamma')$, thus $g(f(\gamma)) = g(f(\gamma'))$ where $f(\gamma)$ and $f(\gamma') \in S'_n$. Now there are three cases:

(1) $f(\gamma) = f(\gamma') = \sigma \in S'_n$, then there exists an edge belongs to E_f lies in the frontier of γ and γ'.

(2) $f(g) = \sigma \neq f(\gamma') = \sigma'$ and $g(\sigma) = g(\sigma') = \alpha$, then there exists an edge belongs two E_g lies in the frontier of σ and σ'.

(3) $f(\gamma) = \sigma \neq f(\gamma') = \sigma'$ and $g(\sigma) = g(\sigma') = \sigma$ or $g(\sigma) = g(\sigma') = \sigma'$, then there exists an edge belongs to E' where $E' \cup E_g = f^{-1}(E_g)$, lies in the frontier of σ and σ'.

We conclude from the above possibilities that $E = E_f \cup f^{-1}(E_g)$. Thus we have $\Gamma^*_{g \circ f} = \Gamma^*_f \cup f^{-1}(\Gamma^*_g)$.

The above theorem can be generalized for a finite series of cellular foldings.

3-1 Example

Let *K* be a complex such that $|K|$ is a torus with cellular subdivision consisting of eight 0-cells, twenty 1-cells and sixteen 2-cells. Let $f : K \to K$ be a cellular folding given by:

$$f(e_1^0, ..., e_8^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_6^0, e_3^0, e_4^0)$$

$$f(e_1^2, ..., e_{16}^2) = (e_1^2, e_2^2, e_3^2, e_4^2, e_5^2, e_6^2, e_7^2, e_8^2, e_1^2, e_2^2, e_3^2, e_4^2, e_7^2, e_8^2, e_5^2, e_6^2)$$

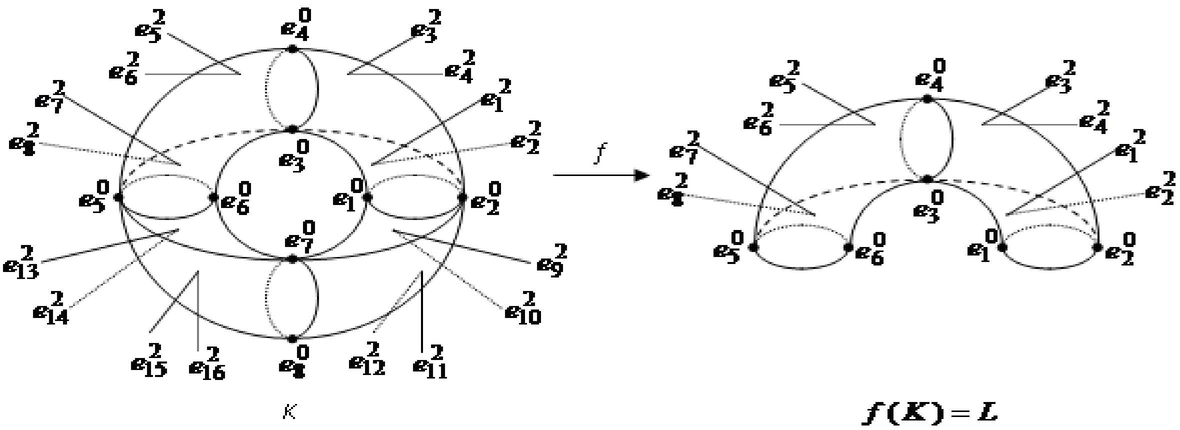

Fig. (5)

The dual folding graph Γ^*_f is as shown in Fig. (6)

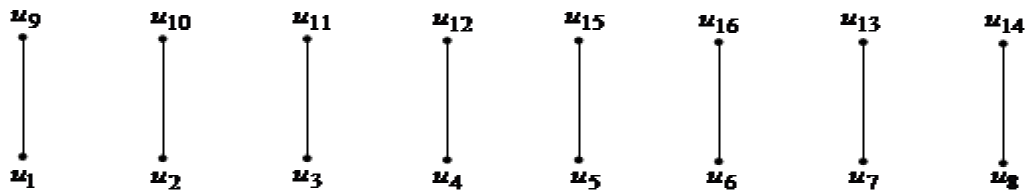

Fig. (6) : The dual folding graph Γ^*_f.

Now let $g : L \to L$ be a cellular folding where $L = f(K)$, defined by:

$$g(e_1^0, ..., e_6^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_2^0, e_1^0),$$

$$g(e_1^2, ..., e_8^2) = (e_1^2, e_2^2, e_3^2, e_4^2, e_3^2, e_4^2, e_1^2, e_2^2).$$

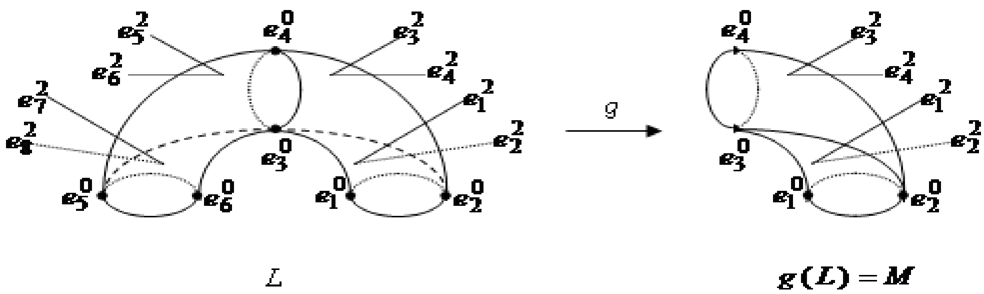

Fig. (7)

The dual folding graph Γ_g^* is as shown in Fig. (8)

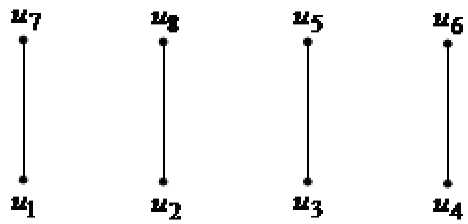

Fig. (8) : The dual folding graph Γ_g^*.

Then by theorem (1) $g \circ f$ is a cellular folding and the dual folding graph of $g \circ f : K \to M$ is as shown in Fig. (9).

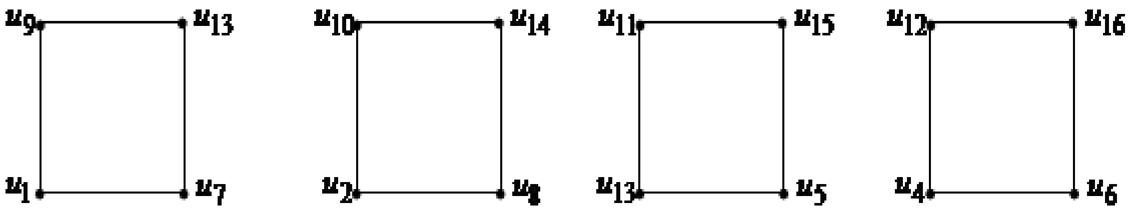

Fig. (9) : The dual folding graph $\Gamma_{g\circ f}^* = \Gamma_f^* \bigcup f^{-1}(\Gamma_g^*)$.

Again, let $h : M \to M$, $h(M) = N \neq M$ be a cellular folding defined by:

$$h(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_3^0, e_4^0),\ \ h(e_1^2, e_2^2, e_3^2, e_4^2) = (e_1^2, e_1^2, e_3^2, e_3^2).$$

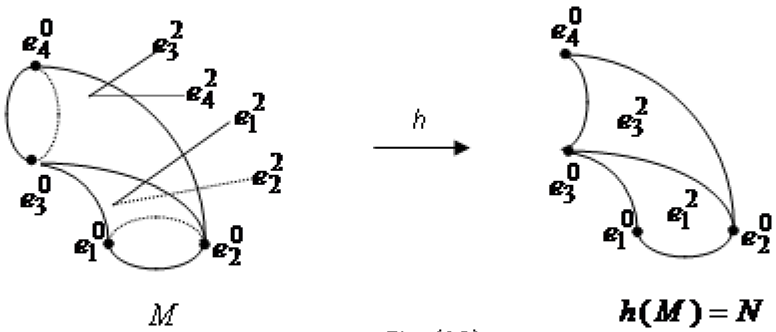

Fig. (10)

The dual folding graph Γ_h^* is as shown in Fig. (11)

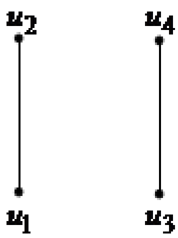

Fig. (11) : The dual folding graph Γ_h^*.

Then by theorem (1) $h \circ g \circ f$ is a cellular folding and the dual folding graph $\Gamma^*_{h \circ g \circ f}$ is as shown in Fig. (12).

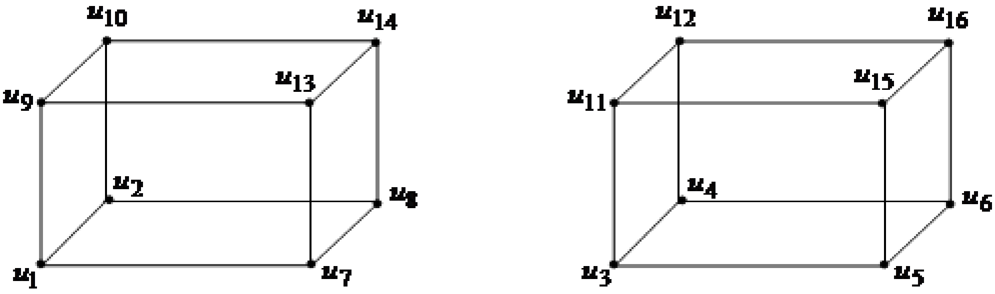

Fig. (12) : $\Gamma^*_{h \circ g \circ f} = \Gamma^*_f \cup f^{-1}(\Gamma^*_g) \cup (g \circ f)^{-1}(\Gamma^*_h)$.

Again, let $k : N \to N,\ k(N) = X \neq N$ be a cellular folding defined by:

$$k(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_3^0, e_1^0),$$
$$k(e_1^2, e_3^2) = (e_1^2, e_1^2).$$

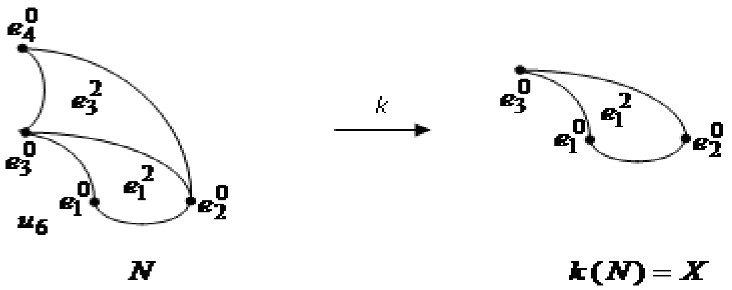

Fig. (13)

The dual folding graph Γ^*_k is as shown in Fig. (14).

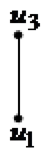

Fig. (14): The dual folding graph Γ^*_k

Again by theorem (1) $k \circ h \circ g \circ f$ is a cellular folding and the dual folding graph $\Gamma^*_{k \circ h \circ g \circ f}$ is as shown in Fig. (15).

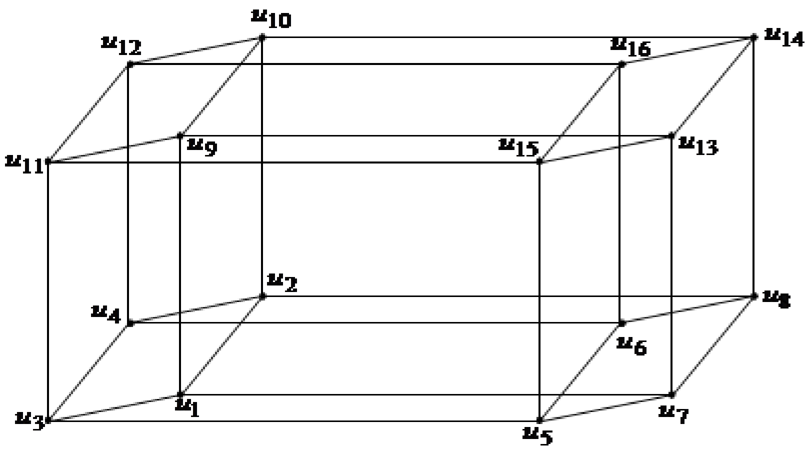

Fig. (15) : $\Gamma^*_{k \circ h \circ g \circ f} = \Gamma^*_f \cup f^{-1}(\Gamma^*_g) \cup (g \circ f)^{-1}(\Gamma^*_h) \cup (h \circ g \circ f)^{-1}(\Gamma^*_k)$.

Corollary 1: Let *K*, *L* and *M* be complexes of the same dimension 2 such that $M \subset L \subset K$, let $f : K \to L$ be a cellular folding with dual folding graph $\Gamma_f^* = (V_f, E_f)$, $g : L \to M$ be a cellular map and $h = g \circ f : K \to M$ be a cellular folding with dual folding graph $\Gamma_h^* = (V_h, E_h)$. Then $g : L \to M$ is a cellular folding with dual folding graph $\Gamma_g^* = (V_g, E_g)$ such that $\Gamma_g^* = f[\Gamma_h^* \setminus E_f]$.

3-2 Example

Consider a complex on $|K| = S^2$, with cellular subdivision consisting of six 0-cells, twelve 1-cells and eight 2-cells, let $f : K \to K$ be a cellular folding defined by:

$$f(e_1^0, ..., e_6^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_1^0),$$
$$f(e_1^1, ..., e_{12}^1) = (e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1, e_3^1, e_4^1, e_1^1, e_2^1),$$
$$f(e_1^2, ..., e_8^2) = (e_1^2, e_2^2, e_3^2, e_4^2, e_3^2, e_4^2, e_1^2, e_2^2).$$

In this case $f(K) = L$ is a complex with five 0-cells, eight 1-cells and four 2-cells, see Fig. (16).

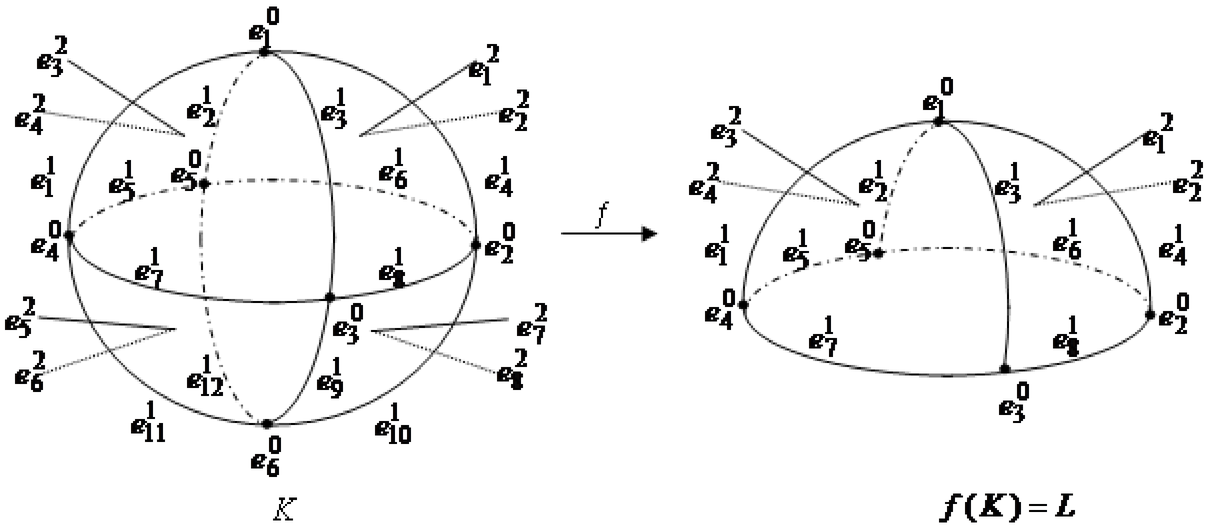

Fig. (16)

The dual folding graph Γ_f^* is as shown in Fig. (17).

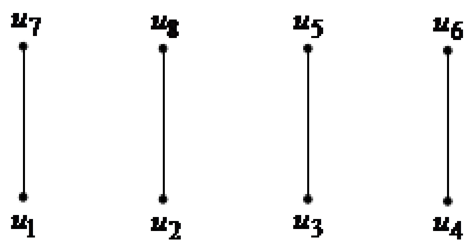

Fig. (17): The dual folding graph Γ_f^*.

Now, let $h : K \to K$ be a cellular folding defined by:

$$h(e_1^0, ..., e_6^0) = (e_1^0, e_2^0, e_3^0, e_2^0, e_3^0, e_1^0),$$
$$h(e_1^1, ..., e_{12}^1) = (e_4^1, e_3^1, e_3^1, e_4^1, e_8^1, e_8^1, e_8^1, e_8^1, e_3^1, e_4^1, e_4^1, e_3^1), \; h(e_i^2) = (e_1^2), \quad i = 1, ..., 8.$$

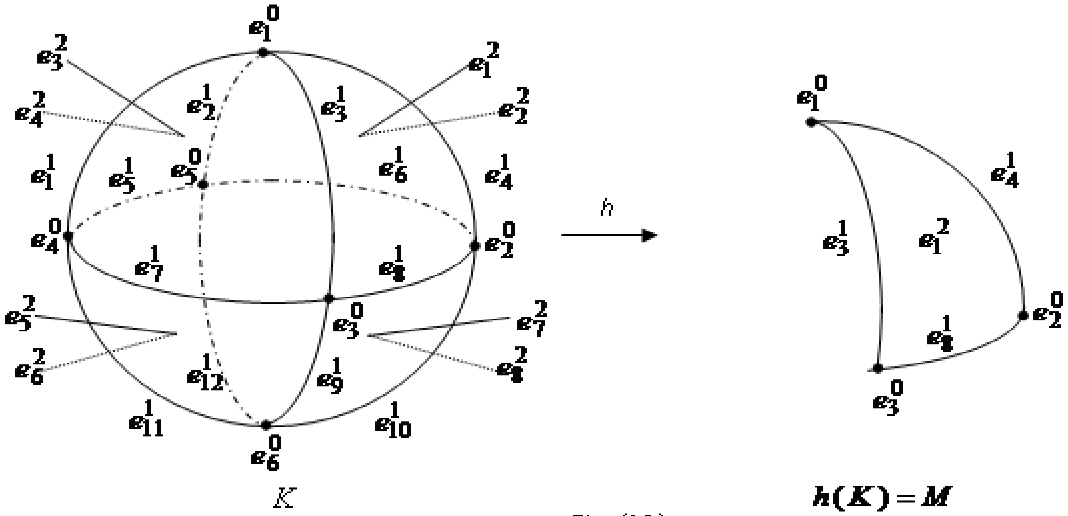

Fig. (18)

The dual folding graph Γ_h^* is as shown in Fig. (19).

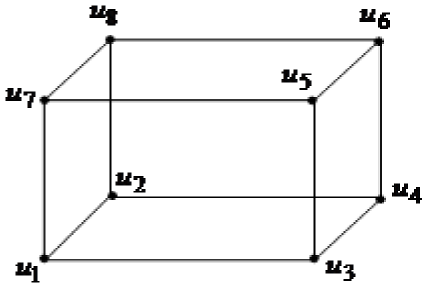

Fig. (19): The dual folding graph $\Gamma_h^* = \Gamma_{g \circ f}^*$.

Then by corollary (1) $g : L \to M$ is a cellular folding with dual folding graph is as shown in Fig. (20).

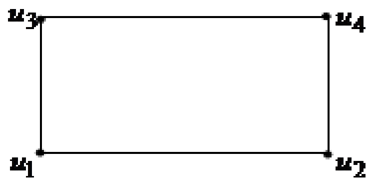

Fig. (20): $\Gamma_g^* = f[\Gamma_h^* \setminus E_f]$

The following theorem gives the condition satisfied by dual graphs in order to have the cartesian product of two cellular foldings is a cellular folding.

Theorem 2: Let *K*, *L*, *M*, and *N* be regular *CW*-complexes of the same dimension 2, let $f : K \to M$ and $g : L \to N$ be cellular foldings with dual folding graphs $\Gamma_f^* = (V_f, E_f)$ and $\Gamma_g^* = (V_g, E_g)$ respectively. Then $f \times g : K \times L \to M \times N$ is a cellular folding with dual folding graph $\Gamma_{f \times g}^* = (V, E)$ such that $\Gamma_{f \times g}^* = \Gamma_f^* \times \Gamma_g^*$. i.e., $V = V_f \times V_g$, $E = (V_f \times E_g) \bigcup (V_g \times E_f)$.

Proof: Let $f : K \to M$, $g : L \to N$ be cellular foldings, then *K* and *L* have stratifications *S*, S' respectively such that $V_f = \{n-\text{cells of } S\}$ and $V_g = \{n-\text{cells of } S'\}$. The map $f \times g : K \times L \to M \times N$ is a cellular folding iff there exists another stratification S'' on $K \times L$ such that $S'' = S_n \times S'_n$, thus $V_{f \times g} = V_f \times V_g$.

Now let $e \in E$, then $e \in S''_{n-1}$ and lies in the frontier of exactly two *n*-cells $\gamma, \gamma' \in S''_n$, $\gamma = (u, v)$, $\gamma' = (u', v')$ such that $(f \times g)(\gamma) = (f \times g)(\gamma')$ or $(f \times g)(u, v) = (f \times g)(u', v')$. Then there are two cases:

(1) $f(u) = f(u'),\ u,\ u' \in S_n$, then there exists an edge belongs to E_f lies in the frontier of *u* and u'.
(2) $g(v) = g(v'),\ v,\ v' \in S'_n$, then there exists an edge belongs to E_g lies in the frontier of *v* and v'.

Thus *u* and u' are incident with an edge belongs to E_f and *v*, v' are incident with an edge belongs to E_g.

Then *E* will be take the form $E = (V_f \times E_g) \bigcup (V_g \times E_f)$. Therefore

$$\Gamma_{f\times g}{}^* = \Gamma_f^* \times \Gamma_g^*.$$

The above theorem can be generalized for a finite number of cellular foldings.

3-3 Examples

(a) Consider *K* and *L* are complexes such that $|K\} = |L| = S^1$, with cellular subdivision given as shown in Fig. (21), and let $f : K \to K,\ g : L \to L$ be neat cellular foldings which squash *K* and *L* respectively to a 1-cell and two 0-cells.

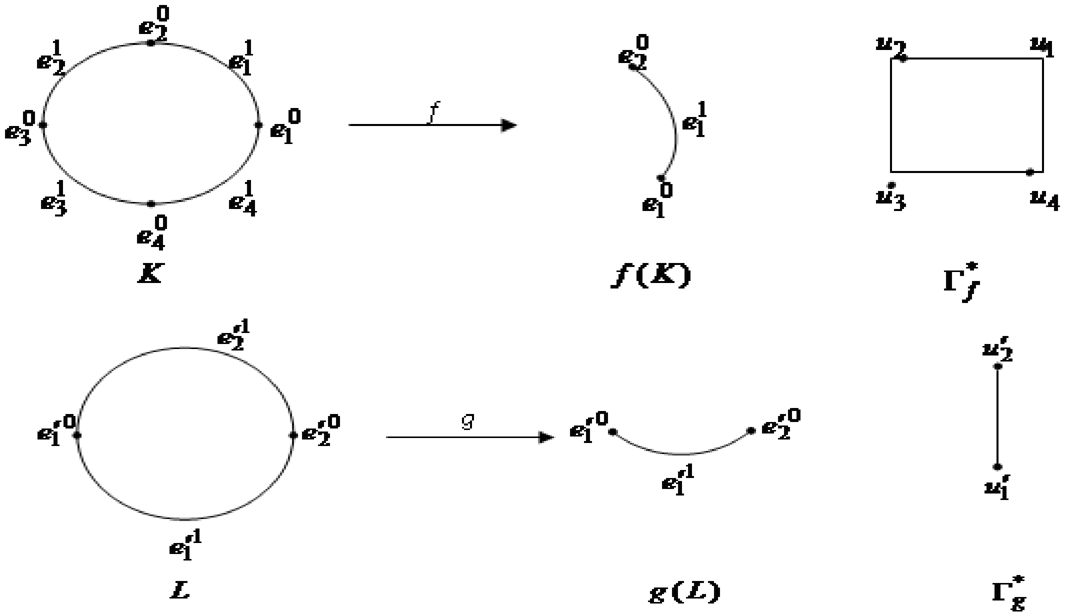

Fig. (21)

Now, $f \times g : K \times L \to K \times L$ is a neat cellular folding with dual folding graph $\Gamma_{f\times g}^*$ is as shown in Fig. (22).

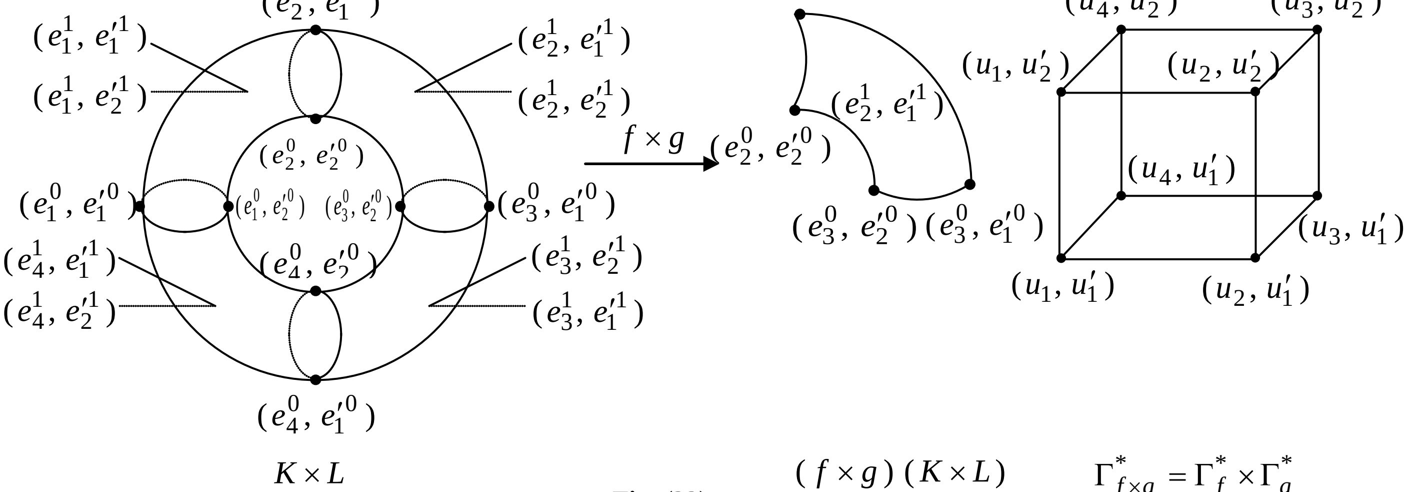

Fig. (22).

Corollary 2: Let M, N, M_1, M_2, N_1, N_2 be complexes of the same dimension and let $f: M \to M_1$, $g: N \to N_1$, $h: M_1 \to M_2$ and $k: N_1 \to N_2$ be cellular foldings with dual folding graphs $\Gamma^*_f, \Gamma^*_g, \Gamma^*_h$ and Γ^*_k respectively. Then $(h\times k)\circ(f\times g) = (h\circ f)\times(k\circ g)$ is a cellular folding with dual folding graphs $\Gamma^*_{(h\times k)\circ(f\times g)} = \Gamma^*_{f\times g} \cup (f\times g)^{-1}(\Gamma^*_{h\times k}) = \Gamma^*_{(h\circ f)\times(k\circ g)} = \Gamma^*_{h\circ f} \times \Gamma^*_{k\circ g}$.

3-4 Example

Suppose M, N, M_1, M_2, N_1, N_2 are complexes such that $|M| = S^1$, $|N| = |M_1| = |M_2| = |N_1| = |N_2| = I$, with cell decompositions shown in Fig. (23). Suppose $f: M \to M_1$, $h: M_1 \to M_2$, $g: N \to N_1$ and $k: N_1 \to N_2$ are cellular foldings.

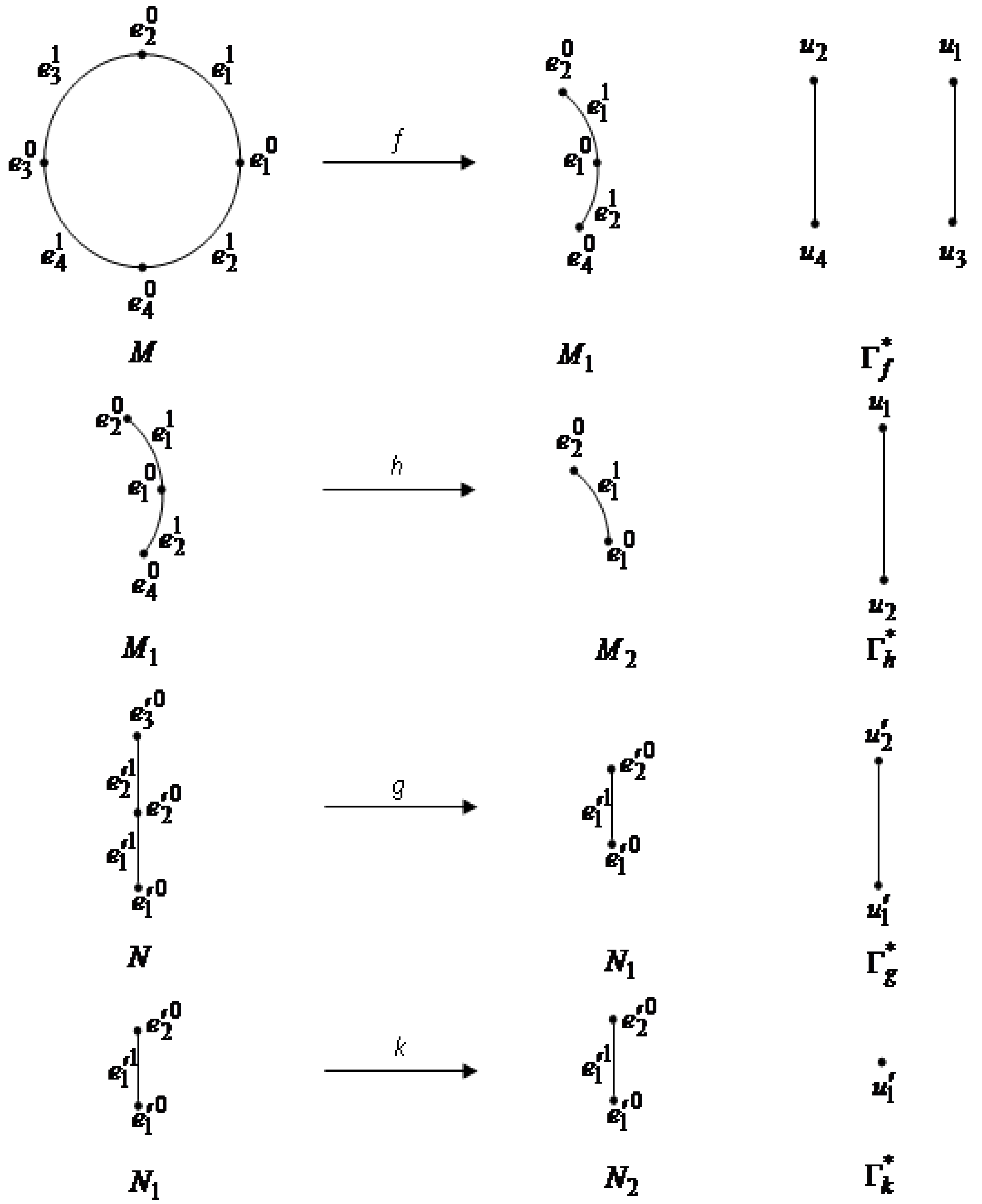

Fig. (23)

The cellular foldings $(f \times g)$, $(h \times k)$ and the dual folding graphs $\Gamma^*_{f\times g}$, $\Gamma^*_{h\times k}$, $\Gamma^*_{(h\times k)\circ(f\times g)}$ are as shown in Fig. (24).

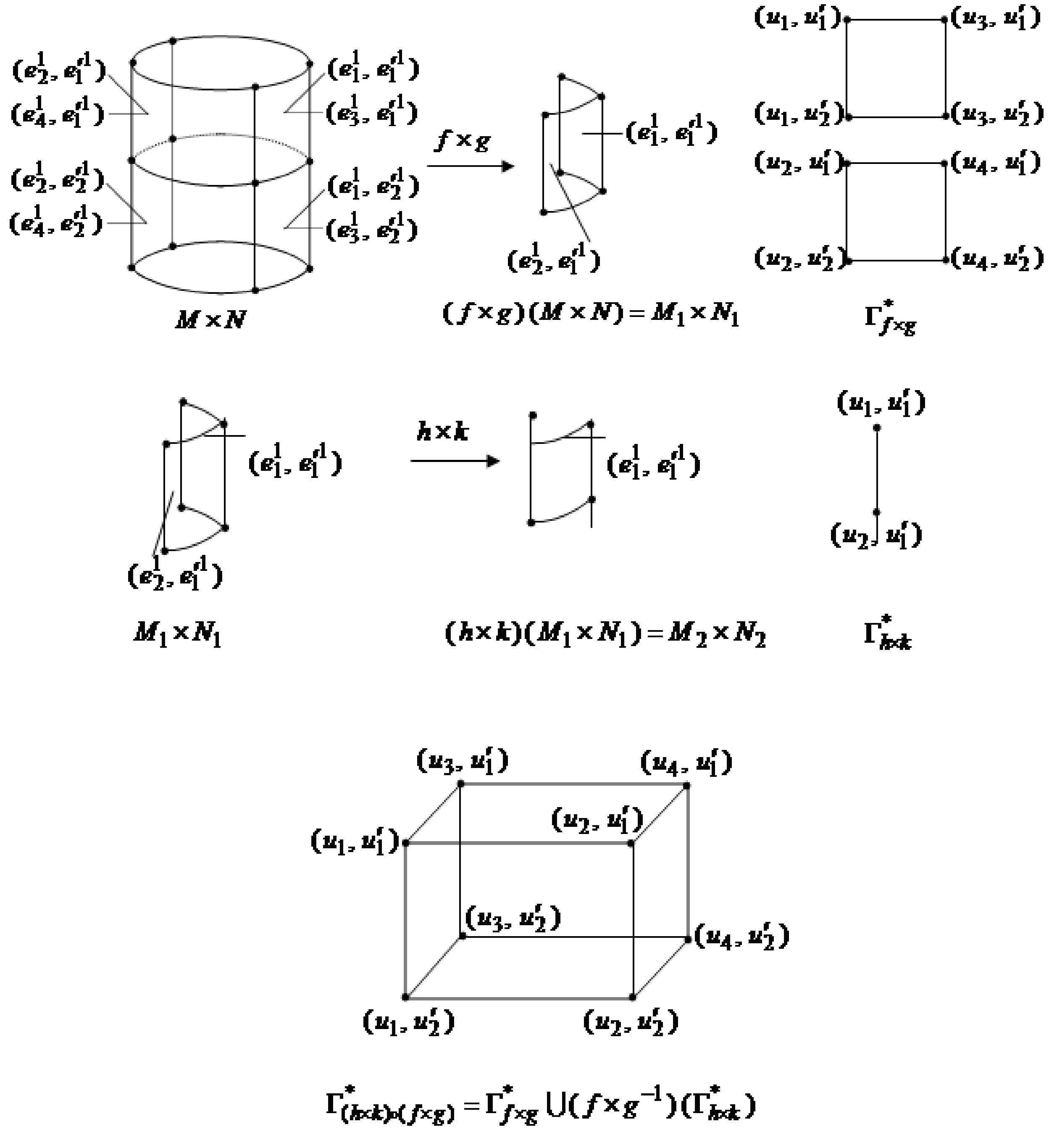

Fig. (24)

Also the cellular foldings $h \circ f$, $k \circ g$ and the dual folding graphs $\Gamma^*_{h\circ f}$, $\Gamma^*_{k\circ g}$, $\Gamma^*_{(h\circ f)\times(k\circ g)}$ are as shown in Fig. (25).

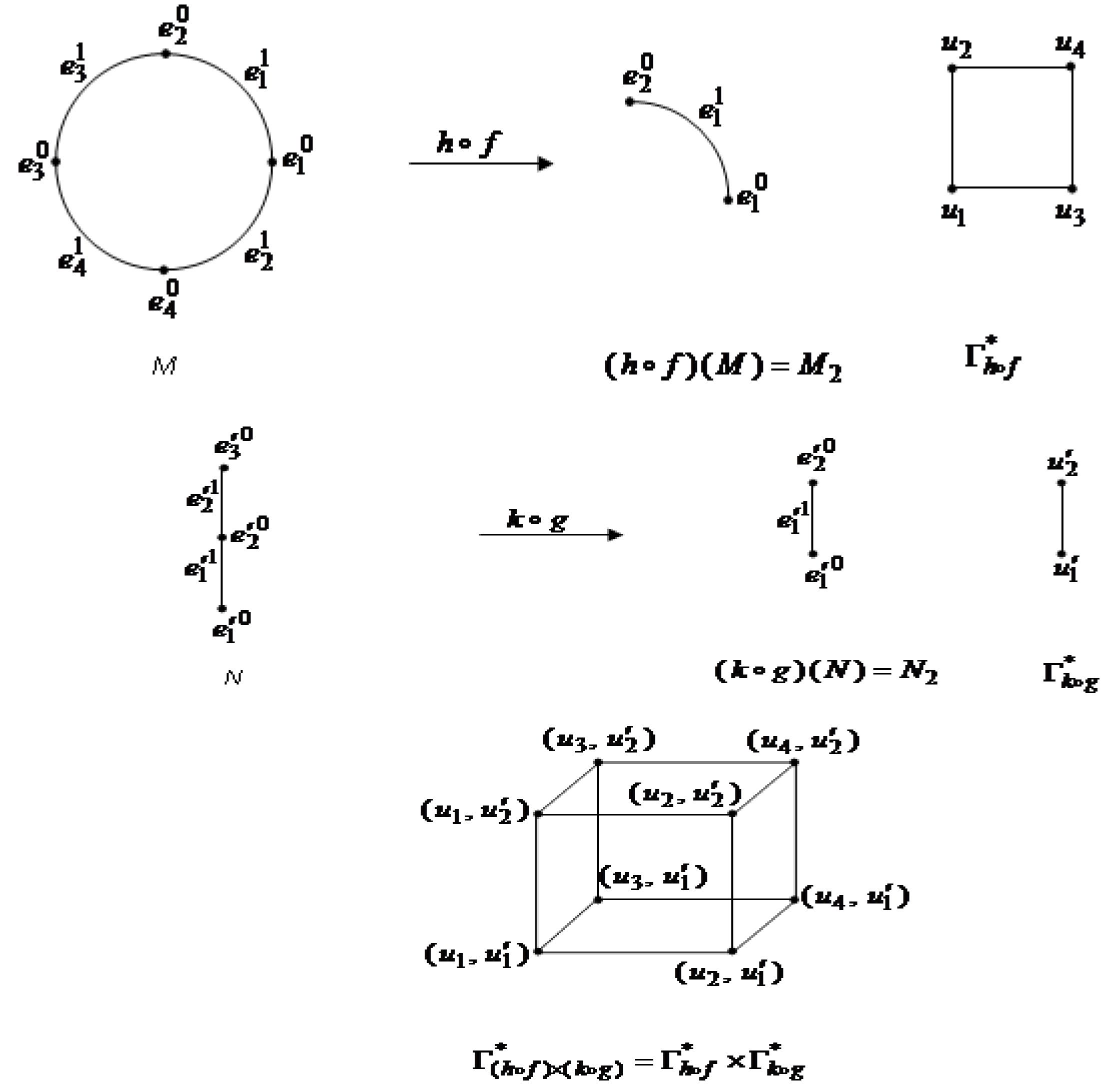

Fig. (25)

The following theorem gives the condition satisfied by the dual graphs in order to have the wedge sum of two cellular folding is a cellular folding.

Theorem 3: Let *M*, *N*, *K* and *L* be regular *CW*-complexes of the same dimension, let $f : M \to M,\ f(M) = N \neq M$ and $g : K \to K,\ g(K) = L \neq K$ be cellular foldings with dual folding graphs $\Gamma_f^* = (V_f, E_f)$ and $\Gamma_g^* = (V_g, E_g)$ respectively. Then the wedge sum $f \vee g : M \vee K \to N \vee L$ is a cellular folding with dual folding graph $\Gamma_{f \vee g}^* = (V, E)$ such that $\Gamma_{f \vee g}^* = \Gamma_f^* + \Gamma_g^*$.

Proof: Let $f : K \to M,\ g : L \to N$ be cellular foldings, then *K* and *L* have stratifications $S,\ S'$ respectively such that $V_f = \{n - \text{cells of } S\}$ and $V_g = \{n - \text{cells of } S'\}$ where $S \bigcap S' =$ one 0-cell.

The map $f \vee g : K \vee L \to M \vee N$ is a cellular folding iff there exists another stratification S'' on $K \vee L$ such that $S_n'' = S_n + S_n'$, thus $V_{f \vee g} = V_f + V_g$.

Now let $e \in E$, then $e \in S_{n-1}''$ and lies in the frontier of exactly two *n*-cells $\gamma, \gamma' \in S_n''$ such that $(f \vee g)(\gamma) = (f \vee g)(\gamma')$. Then there are two cases:

(1) $f(\gamma) = f(\gamma'),\ \gamma, \gamma' \in S_n$, then there exists an edges to E_f lies in the frontier of γ and γ'.

(2) $g(\gamma) = g(\gamma'),\ \gamma, \gamma' \in S_n'$, then there exists an edge belongs to E_g lies in the frontier of γ and γ'.

We conclude from the above possibilities that $E = E_f + E_g$. Thus $\Gamma^*_{f\vee g} = \Gamma^*_f + \Gamma^*_g$.

The above theorem can be generalized for a finite number of cellular foldings.

3-5 Example

Consider M and K are complexes such that $|M| = |K| = S^1$ with cell decompositions given in Fig. (26) and let $f : M \to M$, $g : K \to K$ be neat cellular foldings which squash M and K respectively to a 1-cell and two 0-cells.

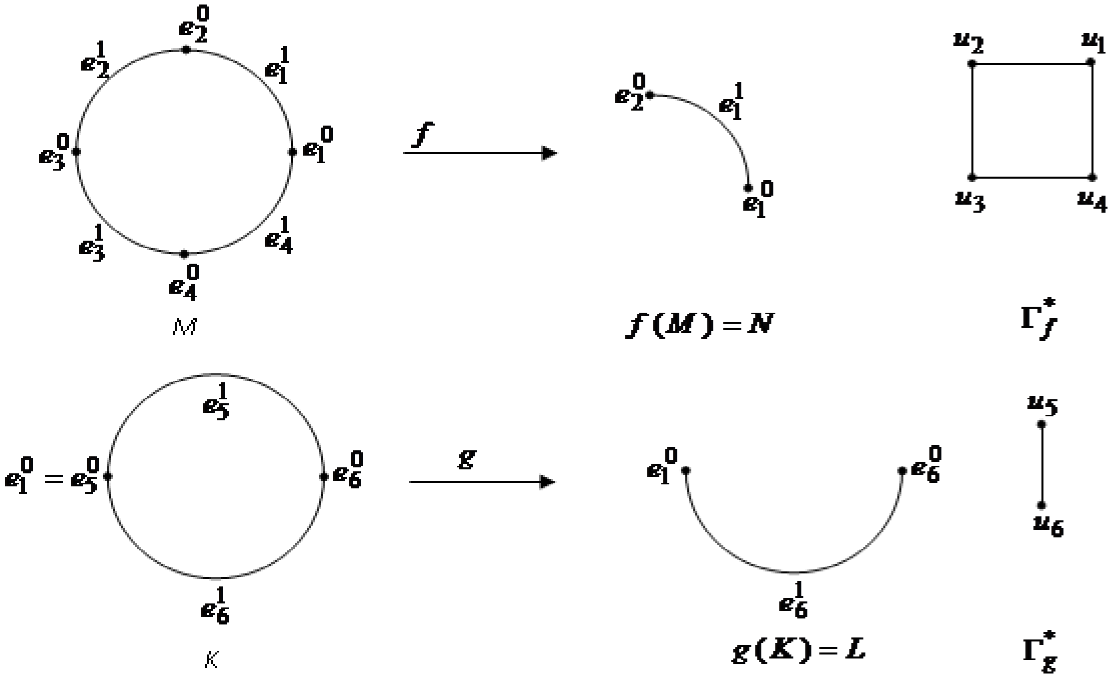

Fig. (26)

Then the wedge sum $f \vee g : M \vee K \to N \vee L$ is a cellular folding with dual folding graph $\Gamma^*_{f\vee g}$ is as shown in Fig. (27).

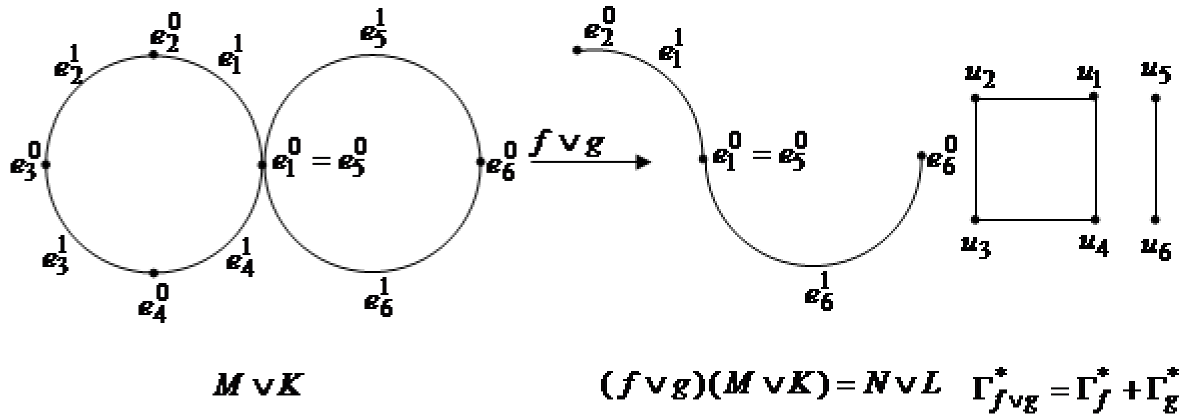

Fig. (27)

Lemma 1: Let M be a regular CW-complex of dimension 2, $N \subset M$, let $f : M \to M$ be a cellular folding with dual folding graph $\Gamma^*_f = (V_f, E_f)$. Then $g : M/N \to M/N$ is a cellular folding with dual folding graph $\Gamma^*_g = (V_g, E_g)$ such that $\Gamma^*_g = \Gamma^*_f$.

3-6 Example

Let $M = D^2$ be a disc with cellular subdivision consisting of five 0-cells, eight 1-cells and four 2-cells, and let $N = S^1 = \partial(D^2)$, $f : D^2 \to D^2$ be a cellular folding defined by

$$f(e_1^0, e_2^0, e_3^0, e_4^0, e_5^0) = (e_1^0, e_2^0, e_3^0, e_2^0, e_3^0),$$
$$f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1) = (e_1^1, e_1^1, e_3^1, e_3^1, e_5^1, e_5^1, e_5^1, e_5^1),$$
$$f(e_i^2) = (e_1^2), \qquad i = 1, 2, 3, 4.$$

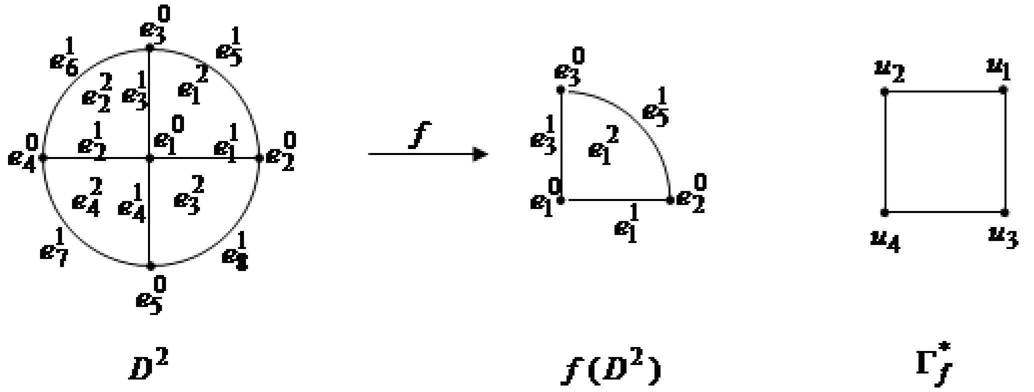

Fig. (28)

Then $g : D^2 / N \to D^2 / N$ is a cellular folding defined by:

$$g(e_1^0, e_6^0) = (e_1^0, e_6^0),$$
$$g(e_1^1, e_2^1, e_3^1, e_4^1) = (e_1^1, e_1^1, e_3^1, e_3^1),$$
$$g(e_i^2) = (e_1^2), \qquad i = 1, 2, 3, 4.$$

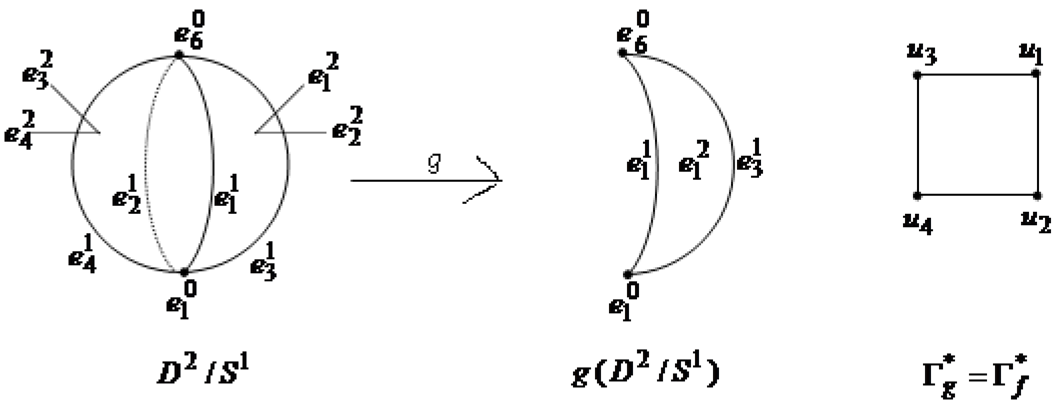

Fig. (29)

Lemma 2: Let *M* and *N* be regular *CW*-complexes of the same dimension, let $f : M \to N$ be cellular folding with dual folding graph $\Gamma_f^* = (V_f, E_f)$. Then the suspension map $g = S\,f : S\,M \to S\,N$ is a cellular folding with dual folding graph $\Gamma_g^* = (V_g, E_g)$ such that $\Gamma_g^* = \Gamma_f^* + \Gamma_f^*$.

3-7 Example

Let *M* be a complex such that $|M| = S^1$ with cellular subdivision consisting of four 0-cells and four 1-cells and let $f : M \to M$ be a cellular folding defined by:

$$f(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_4^0, e_1^0, e_4^0), \; f(e_1^1, e_2^1, e_3^1, e_4^1) = (e_4^1, e_4^1, e_4^1, e_4^1).$$

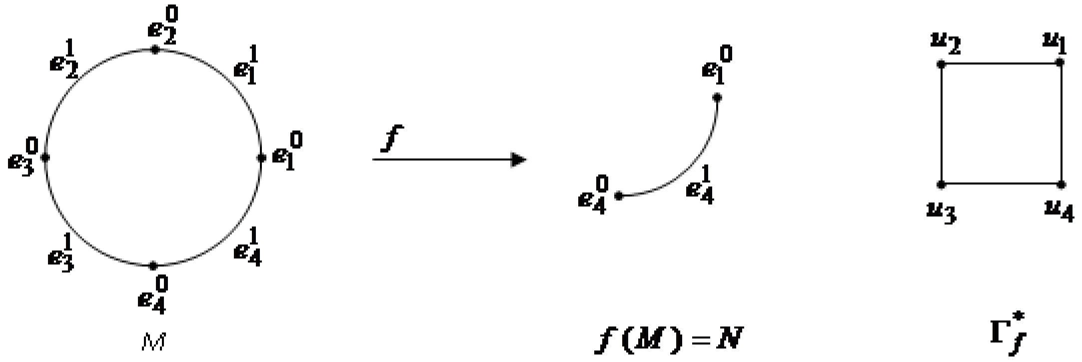

Fig. (30)

Then $g = S\,f : S\,M \to S\,N$ is a cellular folding defined by:

$$g\,(e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_6^0) = (e_1^0, e_4^0, e_1^0, e_4^0, e_5^0, e_6^0),$$

$$g\,(e_1^2, ..., e_8^2) = (e_1^2, e_1^2, e_1^2, e_1^2, e_5^2, e_5^2, e_5^2, e_5^2).$$

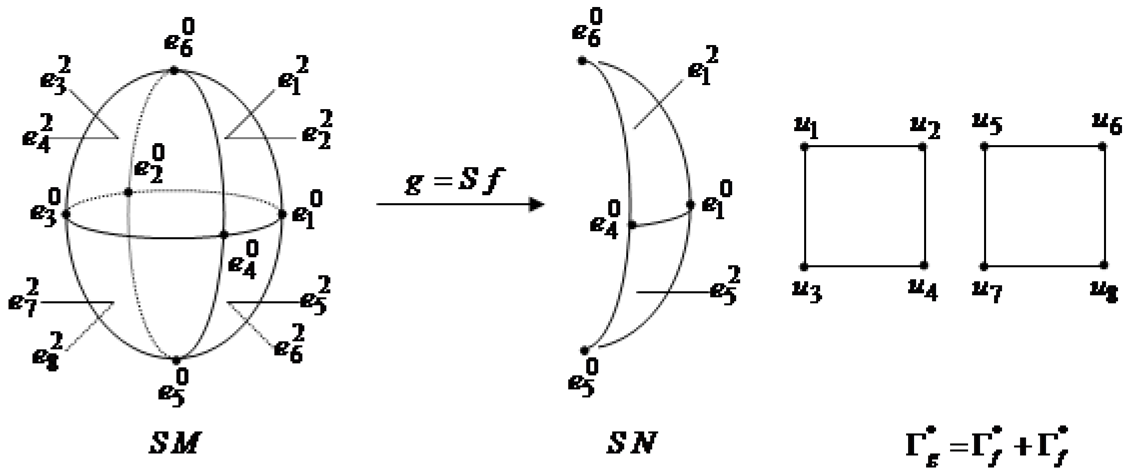

Fig. (31)

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