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SECOND ORDER FUZZY PRODUCT TOPOLOGY

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ABSTRACT

In this paper second order fuzzy product topology is defined and analysed.

INTRODUCTION

A **fuzzy set** on a set X is a map defined on X with values in I, where I is the closed unit interval [0, 1]. Equivalently fuzzy sets which are named as first order fuzzy sets in this paper deal with crisply defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [7] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. This leads to the following definition of a second order fuzzy set or a fuzzy set of type 2.

A second order fuzzy set on a nonempty set X is a map from X to I^I.

First order fuzzy sets are denoted by f, g, h, . . . and second order fuzzy sets are denoted by $\hat{f}$, $\hat{g}$, $\hat{h}$, ...

In this paper the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously.

Whenever a fuzzy set is considered without mentioning the order, it always refers to a first order fuzzy set.

Similar terminology applies to all concepts related to first order fuzzy sets.

Definition and examples of second order fuzzy topological spaces are given in [4].

Six important and interesting connections $¢_1$, $¢_2$, $¢_3$, $¢_4$, $¢_5$ and $¢_6$ between first order and second order fuzzy topological spaces are also discussed in [4].

Every first order fuzzy topology $\delta = \{f_\lambda / \lambda \in \Lambda\}$ on a non-empty set X defines a second order fuzzy topology

$$\hat{\delta} = \{\hat{f}_\lambda / f_\lambda \in \delta\} \text{ on X where } \hat{f}_\lambda(x)(\alpha) = f_\lambda(x), \text{ for every } x \in X \text{ and for every } \alpha \in I.$$

The correspondence $\delta \to \hat{\delta}$ is denoted as $¢_1$.

Let $\hat{\delta} = \{\hat{f}_\lambda / \lambda \in \Lambda\}$ be a second order fuzzy topology on a nonempty set X.

Fix $x \in X$. The collection δ_x = distinct elements of the collection $\{\hat{f}_\lambda(x) / \hat{f}_\lambda \in \hat{\delta}\}$ defines a first order fuzzy topology on I. $¢_2$ denotes the correspondence $\hat{\delta} \to \delta_x$.

Fix $\alpha \in I$. The collection δ_α = the distinct elements of the collection $\{(f_\lambda)_\alpha / \hat{f}_\lambda \in \hat{\delta}\}$ defines a first order fuzzy topology on X where $(f_\lambda)_\alpha(x) = \hat{f}_\lambda(x)(\alpha)$, for every $x \in X$. $¢_3$ denotes the correspondence $\hat{\delta} \to \delta_\alpha$.

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The collection $S^{\Diamond} = \{(f_\lambda)^{\Diamond} / \hat{f}_\lambda \in \hat{\delta}\}$ is a subbase for a first order fuzzy topology $\delta^{\Diamond}$ on X where $(f_\lambda)^{\Diamond}(x) = \bigvee_{\alpha \in I} \hat{f}_\lambda(x)(\alpha)$, for every $x \in X$. $¢_4$ denotes the correspondence $\hat{\delta} \to \delta^{\Diamond}$.

Every first order fuzzy topology $\delta = \{f_\lambda / \lambda \in \Lambda\}$ on I defines a second order fuzzy topology $\hat{\delta}_I = \{(\hat{f}_\lambda)_I / f_\lambda \in \delta\}$ on a nonempty set X where $(\hat{f}_\lambda)_I(x) = f_\lambda$, for every $x \in X$. The correspondence $\delta \to \hat{\delta}_I$ is denoted as $¢_5$.

Given a second order fuzzy topology on a nonempty set X, the association $¢_6$, gives a way of getting another second order fuzzy topology on the same set X. That is, given a second order fuzzy topology $\hat{\delta} = \{\hat{f}_\lambda / \lambda \in \Lambda\}$ on a nonempty set X, the collection $\hat{\delta}_c = \{(\hat{f}_\lambda)_c / \hat{f}_\lambda \in \hat{\delta}\}$ is also a second order fuzzy topology on X.

In this paper second order fuzzy product topology is defined and it is proved that the associations $¢_1$, $¢_3$, $¢_5$ and $¢_6$ preserve product.

SECOND ORDER FUZZY PRODUCT TOPOLOGY

Definition: 1 Let $(X, \hat{\delta}_1)$, $(Y, \hat{\delta}_2)$ be two second order fuzzy topological spaces. If $\hat{f}_1 \in \hat{\delta}_1$ and $\hat{f}_2 \in \hat{\delta}_2$ then the **star product** $\hat{f}_1 * \hat{f}_2$ on X x Y is defined as follows:

$(\hat{f}_1 * \hat{f}_2)(x, y)(\alpha) = \hat{f}_1(x)(\alpha) \wedge \hat{f}_2(y)(\alpha)$, for every $(x, y) \in X \times Y$ and for every $\alpha \in I$.

The **product topology** $\hat{\delta}_1 \times \hat{\delta}_2$ on X x Y is the second order fuzzy topology having the collection

$$\{\hat{f}_1 * \hat{f}_2 / \hat{f}_1 \in \hat{\delta}_1, \hat{f}_2 \in \hat{\delta}_2\} \text{ as a basis.}$$

Theorem: 2 Let (X, δ_1) and (Y, δ_2) be two first order fuzzy topological spaces. Let $(X, \hat{\delta}_1)$ and $(Y, \hat{\delta}_2)$ be the second order fuzzy topological spaces got from (X, δ_1) and (Y, δ_2), respectively, through the association $¢_1$. Then

$$\widehat{\delta_1 \times \delta_2} = \hat{\delta}_1 \times \hat{\delta}_2$$

Proof: $\widehat{\delta_1 \times \delta_2} = \{\hat{f} / f \in \delta_1 \times \delta_2\}$ where

$\hat{f}(x, y)(\alpha) = f(x, y)$, $\forall (x, y) \in X \times Y$ and $\forall \alpha \in I$.

Consider a basis element $\hat{f}_1 * \hat{f}_2$ of $\hat{\delta}_1 \times \hat{\delta}_2$.

For $(x, y) \in X \times Y$ and $\alpha \in I$, consider

$$\begin{aligned}(\hat{f}_1 * \hat{f}_2)(x, y)(\alpha) &= \hat{f}_1(x)(\alpha) \wedge \hat{f}_2(y)(\alpha) \\ &= f_1(x) \wedge f_2(y) \\ &= (f_1 * f_2)(x, y) \\ &= \widehat{f_1 * f_2}(x, y)(\alpha)\end{aligned}$$

$$\therefore \hat{f}_1 * \hat{f}_2 = \widehat{f_1 * f_2}$$

Since $f_1 * f_2$ is a basis element of $\delta_1 \times \delta_2$,

$$\hat{\delta}_1 \times \hat{\delta}_2 \subseteq \widehat{\delta_1 \times \delta_2}$$

Consider $\hat{g} \in \widehat{\delta_1 \times \delta_2}$

where $g \in \delta_1 \times \delta_2$

$g = V\{f_1 * f_2 / f_1 \in \delta_1, f_2 \in \delta_2 \text{ and } f_1 * f_2 < g\}$

$\therefore \hat{g} = V\{f_1 * f_2 / f_1 \in \delta_1, f_2 \in \delta_2 \text{ and } f_1 * f_2 < g\}$

$\therefore \hat{g} = V\{\hat{f}_1 * \hat{f}_2 / \hat{f}_1 \in \hat{\delta}_1, \hat{f}_2 \in \hat{\delta}_2, \hat{f}_1 * \hat{f}_2 < \hat{g}\}$

$\therefore \hat{g} \in \hat{\delta}_1 \times \hat{\delta}_2$

$\therefore \widehat{\delta_1 \times \delta_2} \subseteq \hat{\delta}_1 \times \hat{\delta}_2$

Hence $\widehat{\delta_1 \times \delta_2} = \hat{\delta}_1 \times \hat{\delta}_2$.

Theorem: 3 Let $(X, \hat{\delta}_1)$ and $(Y, \hat{\delta}_2)$ be two second order fuzzy topological spaces. Let $\hat{\delta}_3 = \hat{\delta}_1 \times \hat{\delta}_2$ and $\alpha \in I$. Let $(\delta_1)_\alpha$, $(\delta_2)_\alpha$ and $(\delta_3)_\alpha$ be the first order fuzzy topologies got from $\hat{\delta}_1$, $\hat{\delta}_2$ and $\hat{\delta}_3$, respectively, through the association $¢_3$. Then $(\delta_3)_\alpha = (\delta_1)_\alpha \times (\delta_2)_\alpha$.

Proof: Consider a basis element $(f_1)_\alpha * (f_2)_\alpha$ of $(\delta_1)_\alpha \times (\delta_2)_\alpha$. Given $(f_1)_\alpha$ and $(f_2)_\alpha$, let $\hat{f}_1$, $\hat{f}_2$ be such that $(\hat{f}_1)(x)(\alpha) = (f_1)_\alpha(x)$ and $(\hat{f}_2)(x)(\alpha) = (f_2)_\alpha(x)$.

For $(x, y) \in X \times Y$, consider

$$\begin{aligned}((f_1)_\alpha * (f_2)_\alpha)(x, y) &= (f_1)_\alpha(x) \Lambda (f_2)_\alpha(y)\\ &= (\hat{f}_1)(x)(\alpha) \Lambda (\hat{f}_2)(y)(\alpha)\\ &= (\hat{f}_1 * \hat{f}_2)(x, y)(\alpha)\\ &= (\hat{f}_3)(x, y)(\alpha) \text{ where } \hat{f}_3 = \hat{f}_1 * \hat{f}_2 \in \hat{\delta}_3\\ &= (f_3)_\alpha(x, y)\end{aligned}$$

$\therefore (f_1)_\alpha * (f_2)_\alpha = (f_3)_\alpha \in (\delta_3)_\alpha$

$\therefore (\delta_1)_\alpha \times (\delta_2)_\alpha \subseteq (\delta_3)_\alpha$ (1)

Consider $g_\alpha \in (\delta_3)_\alpha$

There exists a $\hat{g} \in \hat{\delta}_3$ such that $\hat{g}(x)(\alpha) = g_\alpha(x), \forall x \in X$. Then

$\hat{g} = V\{\hat{f}_1 * \hat{f}_2 / \hat{f}_1 \in \hat{\delta}_1, \hat{f}_2 \in \hat{\delta}_2 \text{ and } \hat{f}_1 * \hat{f}_2 < \hat{g}\}$

For $(x, y) \in X \times Y$, consider

$$\begin{aligned}g_\alpha(x, y) &= \hat{g}(x, y)(\alpha)\\ &= (V(\hat{f}_1 * \hat{f}_2))(x, y)(\alpha)\end{aligned}$$

$= V(\hat{f}_1(x)(\alpha) \wedge \hat{f}_2(y)(\alpha))$

$= V((f_1)_\alpha(x) \wedge (f_2)_\alpha(y))$

$= (V(f_1)_\alpha * (f_2)_\alpha)(x, y)$

$\therefore \quad g_\alpha = V((f_1)_\alpha * (f_2)_\alpha) \in (\delta_1)_\alpha \times (\delta_2)_\alpha$

$\therefore$

$\therefore \quad (\delta_3)_\alpha \subseteq (\delta_1)_\alpha \times (\delta_2)_\alpha$ (2)

Hence from (1) and (2)

$(\delta_3)_\alpha = (\delta_1)_\alpha \times (\delta_2)_\alpha$.

Theorem: 4 Let δ be a first order fuzzy topology on I.

Let (X, $(\hat{\delta}_I)'$), (Y, $(\hat{\delta}_I)''$) and (X x Y, $(\hat{\delta}_I)'''$) be the second order fuzzy topological spaces got from (I, δ) through the association $¢_5$. Then $(\hat{\delta}_I)''' = (\hat{\delta}_I)' \times (\hat{\delta}_I)''$.

Proof: For $f \in \delta$, let $(\hat{f}_I)'$,$(\hat{f}_I)''$ and $(\hat{f}_I)'''$ denote the associated elements of $(\hat{\delta}_I)'$, $(\hat{\delta}_I)''$ and $(\hat{\delta}_I)'''$, respectively.

Here $(\hat{f}_I)'(x) = f, \ \forall \ x \in X$.

$(\hat{f}_I)''(y) = f, \ \forall \ y \in Y$.

and $(\hat{f}_I)'''(x, y) = f, \ \forall (x, y) \in X \times Y$.

Consider a basis element $(\hat{f}_I)' * (\hat{g}_I)''$ of $(\hat{\delta}_I)' \times (\hat{\delta}_I)''$

where f, g $\in \delta$.

$\therefore \quad h = f \wedge g \in \delta$ (1)

For (x, y) $\in$ X x Y, consider

$((\hat{f}_I)' * (\hat{g}_I)'')(x, y) = (\hat{f}_I)'(x) \wedge (\hat{g}_I)''(y)$

$= f \wedge g$

$= h$ ($\because$ from (1))

$= (\hat{h}_I)'''(x, y)$

$\therefore \quad (\hat{f}_I)' * (\hat{g}_I)'' = (\hat{h}_I)''' \in (\hat{\delta}_I)'''$

$\therefore (\hat{\delta}_I)' \times (\hat{\delta}_I)'' \subseteq (\hat{\delta}_I)'''$ (2)

Consider $(\hat{h}_I)''' \in (\hat{\delta}_I)'''$ where $h \in \delta$

$\therefore \quad (\hat{h}_I)' \in (\hat{\delta}_I)'$

$\therefore \quad (\hat{h}_I)' * (\hat{1}_I)'' \in (\hat{\delta}_I)' \times (\hat{\delta}_I)''$

But $(\hat{h}_I)' * (\hat{1}_I)'' = h \wedge \mathbf{1} = h = (\hat{h}_I)'''$

$\therefore \quad (\hat{h}_I)''' = (\hat{h}_I)' * (\hat{1}_I)'' \in (\hat{\delta}_I)' \times (\hat{\delta}_I)''$

$$\therefore \quad (\hat{\delta}_I)''' \subseteq (\hat{\delta}_I)' \times (\hat{\delta}_I)'' \tag{3}$$

Hence from (2) and (3) $(\hat{\delta}_I)' \times (\hat{\delta}_I)''$

The following result can be easily proved using the definition of $(\hat{\delta})_c$.

Theorem: 5 Let (X, $\hat{\delta}_1$), (Y, $\hat{\delta}_2$) be two second order fuzzy topological spaces. Let (X, $(\hat{\delta}_1)_c$) and (Y, $(\hat{\delta}_2)_c$) be the second order fuzzy topological spaces got from (X, $\hat{\delta}_1$) and (Y, $\hat{\delta}_2$), respectively through the association $¢_6$. Then

$$(\hat{\delta}_1 \times \hat{\delta}_2)_c = (\hat{\delta}_1)_c \times (\hat{\delta}_2)_c.$$

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