International Journal of Mathematical Archive-3(4), 2012, Page: 1712-1717
Available online through www.ijma.info ISSN 2229 – 5046

# THE DETERMINATION OF THE NUMBER OF DISTINCTFUZZY SUBGROUPS OF GROUP $Z_{p_1\times p_2\times\ldots\times p_n}$ AND THEDIHEDRAL GROUP $D_{2\times p_1\times p_2\times\ldots\times p_n}$

HASSAN NARAGHI[1*] & HOSEIN NARAGHI[2]

[1]*Department of Mathematics, Islamic Azad University, Ashtian Branch, P.O. Box 39618-13347, Ashtian, Iran*
[2] *Department of Mathematics, Payame Noor University, Ray Branch, Iran*

*E-mail: [1]naraghi@mail.aiau.ac.ir, [2]h.naraghi56@yahoo.com,*

*(Received on: 03-04-12; Accepted on: 27-04-12)*

---

*ABSTRACT*

*In this paper, we use the natural equivalence of fuzzy subgroups studied by Iranmanesh and Naraghi [3] to determine the number of distinct fuzzy subgroups of some finite groups. We focus on the determination of the number of distinct fuzzy subgroups of group $Z_{p_1\times p_2\times\ldots\times p_n}$ and the dihedral group $D_{2\times p_1\times p_2\times\ldots\times p_n}$ using this equivalence relation.*

***2000 Mathematics Subject Classification:*** *20N25.*

***Key words and phrases:*** *Dihedral group, Equivalence relation, Fuzzy subgroups.*

---

## 1. INTRODUCTION

Zadeh introduced the notion of fuzzy sets and fuzzy set operations, in his classic paper [15] of 1965. In an analogous application with groups, Rosenfeld [13] formulated the elements of a theory of fuzzy groups. One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid evolutionin the last years. Many papers have treated the particular case of finite cyclic groups. Thus, in [8] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [11, 12, 14] deal with this number for cyclic groups of order $p^nq^m$ (p, q primes). In the present paper we establish the recurrence relation verified by the number of distinct fuzzy subgroups of group $Z_{p_1\times p_2\times\ldots\times p_n}$ and the dihedral group $D_{2\times p_1\times p_2\times\ldots\times p_n}$ such that $p_1,p_2,\ldots,p_n$ are distinct primes.

## 2. PRELIMINARIES

First of all, we present some basic notions and results of fuzzy sub group theory (for more details, see [4, 7, and 3]).

The dihedral group of order 2n, for $n\geq 2$, denoted by $D_{2n}$. A fuzzy sub set of a set X is a mapping $\mu : X \to [0,1]$. Fuzzy subset $\mu$ of a group G is called a fuzzy subgroup of G if:

($G_1$) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all x; y $\in$ G;
($G_2$) $\mu(x^{-1}) \geq \mu(x)$ for all x $\in$ G

The set of all fuzzy subgroup of a group G is denoted by F (G).

**Definition 2.1:** Let G be a group and $\mu \in F(G)$. The set of $\{x \in G \mid \mu(x) > 0\}$ is called the support of $\mu$ and denoted by supp $\mu$.

Let G be a group and $\mu \in F(G)$. We shall write $\operatorname{Im}\mu$ for the image set of $\mu$ and F $\mu$ for the family $\{\mu_t \mid t \in \operatorname{Im}\mu\}$.

---

*Corresponding author: [1]HASSAN NARAGHI, *E-mail: naraghi@mail.aiau.ac.ir*

**Theorem 2.2:** Let G be a fuzzy group. If $\mu$ is a fuzzy subset of G, then $\mu \in F(G)$ if and only if for all $\mu_t \in F_\mu$, $\mu_t$ is a subgroup of G.

Let $F_1(G)$ be the set of all fuzzy subgroups $\mu$ of G such that $\mu(e)=1$, and let $\sim_R$ be an equivalence relation on $F_1(G)$. We denote the set $\{v \in F_1(G) | v \sim_R \mu\}$ by $\frac{\mu}{\sim_R}$ and the set $\left\{\frac{\mu}{\sim_R} | \mu \in F_1(G)\right\}$ by $\frac{F_1(G)}{\sim_R}$.

**Definition 2.3:** Let G be a group, and $\mu, v \in F_1(G)$. $\mu$ is equivalent to $v$, written as $\mu \sim v$ if

(1) $\mu(x) > \mu(y) \Leftrightarrow v(x) > v(y)$ for all $x, y \in G$.

(2) $\mu(x) = 0 \Leftrightarrow v(x) = 0$ for all $x \in G$.

The number of the equivalence classes $\sim$ on $F_1(G)$ is denoted by $s(G)$. We mean the number of distinct fuzzy subgroups of G is s (G).

**Theorem 2.4:** Let G be a finite group. The number of distinct fuzzy subgroups of G such that their support is exactly equal to G is $\frac{s(G)+1}{2}$.

Let G be a finite group. The number of distinct fuzzy subgroups of G such that their support is exactly equal to G is denoted by $s^*(G)$.

**Theorem 2.5:** [3] Let G be a finite group. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of G is $\frac{s(G)-1}{2}$.

**Theorem 2.6:** [3] Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly equal to H is $\frac{s(H)+1}{2}$.

**Corollary 2.7:** [3] Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of H is $\frac{s(H)-1}{2}$.

**Proposition 2.8:** [6] Let $n \in N$. Then there are $2^{n+1}-1$ distinct equivalence classes of fuzzy subgroups of $Z_{p^n}$.

## 3. THE NUMBER OF THE DISTINCT FUZZY SUBGROUPS OF THE ABELIANGROUP $Z_{p_1 \times p_2 \times \ldots \times p_n}$

In this section, we characterize fuzzy subgroups of the abelian group $Z_{p_1 \times p_2 \times \ldots p_n}$ such that $p_1, p_2, \ldots, p_n$ are distinct primes numbers (n > 1).

**Proposition 3.1:** Suppose that p and q are distinct primes. Then there are11 distinct equivalence classes of fuzzy subgroups of $Z_{pq}$.

**Proof:** See Theorem 8.2.4 of [6].

**Proposition 3.2:** Suppose that p, q and r are distinct primes. Then there are 51 distinct equivalence classes of fuzzy subgroups of $Z_{pqr}$.

**Proof:** We know that $Z_{pqr}$ has the following maximal chains:

$$Z_{pqr} \supset Z_{pq} \supset Z_p \supset \{0\}, Z_{pqr} \supset Z_{pq} \supset Z_q \supset \{0\}, Z_{pqr} \supset Z_{pr} \supset Z_p \supset \{0\},$$
$$Z_{pqr} \supset Z_{pr} \supset Z_r \supset \{0\}, Z_{pqr} \supset Z_{qr} \supset Z_q \supset \{0\} \text{ and } Z_{pqr} \supset Z_{qr} \supset Z_r \supset \{0\}.$$

All of subgroups of the group $Z_{pqr}$ are $Z_{pq}, Z_{pr}, Z_{qr}, Z_p, Z_q, Z_r$ and $\{0\}$ .Thus

$$\frac{s(G)-1}{2} = s^*(\{0\}) + s^*(Z_{pq}) + s^*(Z_{pr}) + s^*(Z_{qr}) + s^*(Z_p) + s^*(Z_q) + s^*(Z_r),$$

therefore

$$\frac{s(G)-1}{2} = 1 + 3s^*(Z_{pq}) + 3s^*(Z_p) = 1 + 3(6) + 3(2) = 2 \quad .$$

Hence

$$s(G) = 51.$$

**Theorem 3.3:** Suppose that $p_1, p_2, ..., p_n$ are distinct primes. If $G = Z_{p_1 \times p_2 \times ... \times p_n}$ and n>1, then $s(G) = \sum_{i=1}^{n-1} \binom{n}{i} s(Z_{\prod_{j=1}^{i} p_j}) + 2^n + 1$.

**Proof:** Denote $\prod_k = \{p_{i1} \times ... \times p_{ik} \mid i_1, ..., i_k \in \{1, ..., n\}, i_1 < ... < i_k\}$, $k = 1, 2, ..., n$. We know that $G = Z_{p_1 \times p_2 \times ... \times p_n}$ has the following maximal chains each of which can be identified with the chain $Z_{\pi_n} \supset Z_{\pi_{n-1}} \supset .... \supset Z_{\pi_1} \supset \{0\}$ such that $\pi_i \in \Pi_i$ , for all $i \in \{1, ..., n\}$. For all $i = 1, .., n$, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the number of subgroups of the group G as $Z_{\pi_i}$ . Therefore by theorem 2.5, $\frac{s(G)-1}{2} = s^*(\{0\}) + \sum_{i=1}^{n-1} \binom{n}{i} s^*(Z_{\pi_i})$ and hence $s(G) = 2\sum_{i=1}^{n-1} \binom{n}{i} s^*(Z_{\pi_i}) + 3$.By theorem 2.4, $s(G) = \sum_{i=1}^{n-1} \binom{n}{i} s(Z_{\pi_i}) + 2^n + 1$.

## 4. THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF THE DIHEDRALGROUP $D_{2 \times p_1 \times p_2 \times ... \times p_n}$

In this section, we determine the number of distinct fuzzy subgroups ofthe dihedral group $D_{2 \times p_1 \times p_2 \times ... \times p_n}$ such that $p_1, p_2, ..., p_n$ are odd distinct primes.

**Theorem 4.1:** Suppose that p is a prime and $p \geq 3$. If G is the dihedral group of order 2p, then s(G) = 4p + 7.

**Proof:** We know that $D_{2p}$ has the following maximal chains:
$D_{2p} \supset Z_p \supset \{0\}$ and $D_{2p} \supset Z_2 \supset \{0\}$ whose the number is p. Now 2 is the number of distinct fuzzy subgroups whose support is $Z_p$, $2^1 p$ is the number of distinct fuzzy subgroups whose support is $Z_2$, and $2^0$ is the number of fuzzy subgroups whose support is $\{0\}$ . Thus $\frac{s(G)-1}{2} = 2p + 2 + 1$, therefore $s(G) = 4p + 7$.

**Theorem 4.2:** Suppose that p and q are odd distinct primes. If G is the dihedral group of order 2pq, then $s(G) = 12pq + 8(p+q) + 23$.

**Proof:** We know that $D_{2pq}$ has the following maximal chains:
$D_{2pq} \supset D_{2p} \supset Z_2 \supset \{0\}$, $D_{2pq} \supset D_{2p} \supset Z_p \supset \{0\}$, $D_{2pq} \supset D_{2q} \supset Z_2 \supset \{0\}$, $D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}$,
$D_{2pq} \supset D_{pq} \supset Z_p \supset \{0\}$, $D_{2pq} \supset D_{pq} \supset Z_q \supset \{0\}$. Clearly, pq is the number of the subgroups $Z_2$ of the dihedral group $D_{2pq}$ , q is the number of the subgroups $D_{2p}$ and p is the number of the subgroups $D_{2q}$ of the dihedral group $D_{2pq}$ and the dihedral group $D_{2pq}$ has just one subgroup as $Z_{qp}, Z_q, Z_p$ . So that

$$\frac{s(G)-1}{2} = s^*(\{0\}) + pqs^*(Z_2) + qs^*(D_{2p}) + ps^*(D_{2q}) + s^*(Z_{pq}) + s^*(Z_p) + s^*(Z_q).$$

Thus

$$\frac{s(G)-1}{2}=1+2pq+q(2p+4)+p(2q+4)+6+2+2,$$

therefore

$$s(G)=12pq+8(p+q)+23.$$

**Table 1:** The number of distinct fuzzy subgroups of dihedral group $D_{2pq}$ for some selected primes.

| $G=D_{2pq}$ | s(G) |
|---|---|
| $p=3,\ q=5$ | 267 |
| $p=3,\ q=7$ | 355 |
| $p=3,\ q=11$ | 531 |
| $p=3,\ q=13$ | 619 |
| $p=13,\ q=17$ | 2915 |

**Theorem 4.3:** Suppose that p, q and r are odd distinct primes. If G is the dihedral group of order 2pqr, then $s(G)=52pqr+24(pq+pr+qr)+24(p+q+r)+103$.

**Proof:** We have

$$D_{2n}=<x,y\mid x^n=y^2=1, yxy=x^{-1}>.$$

It is well Known that for every divisor r of n, $D_{2n}$ possesses a subgroup isomorphic to $Z_r$, namely $H_0^r=<x^{\frac{n}{r}}>$ and $\frac{n}{r}$ subgroups isomorphic to $D_{2r}$, namely $H_i^r=<x^{\frac{n}{r}},x^{i-1}y>,\ i=1,2,...,\frac{n}{r}$. We know that $D_{2pqr}$ has the following maximal chains each of which can be identified with the Chain,

$$\begin{aligned}
&D_{2pqr}\supset D_{2pq}\supset D_{2p}\supset Z_p\supset\{0\},\ D_{2pqr}\supset D_{2pq}\supset D_{2q}\supset Z_q\supset\{0\},\ D_{2pqr}\supset D_{2pr}\supset D_{2p}\supset Z_p\supset\{0\},\\
&D_{2pqr}\supset D_{2pr}\supset D_{2r}\supset Z_r\supset\{0\},\ D_{2pqr}\supset D_{2qr}\supset D_{2q}\supset Z_q\supset\{0\},\ D_{2pqr}\supset D_{2qr}\supset D_{2r}\supset Z_r\supset\{0\},\\
&D_{2pqr}\supset D_{2pq}\supset D_{2p}\supset Z_2\supset\{0\},\ D_{2pqr}\supset D_{2pq}\supset D_{2q}\supset Z_2\supset\{0\},\ D_{2pqr}\supset D_{2pr}\supset D_{2p}\supset Z_2\supset\{0\},\\
&D_{2pqr}\supset D_{2pr}\supset D_{2r}\supset Z_2\supset\{0\},\ D_{2pqr}\supset D_{2qr}\supset D_{2q}\supset Z_2\supset\{0\},\ D_{2pqr}\supset D_{2qr}\supset D_{2r}\supset Z_2\supset\{0\},\\
&D_{2pqr}\supset Z_{pqr}\supset Z_{pq}\supset Z_p\supset\{0\},\ D_{2pqr}\supset Z_{pqr}\supset Z_{pq}\supset Z_q\supset\{0\},\ D_{2pqr}\supset Z_{pqr}\supset Z_{pr}\supset Z_p\supset\{0\},\\
&D_{2pqr}\supset Z_{pqr}\supset Z_{pr}\supset Z_r\supset\{0\},\ D_{2pqr}\supset Z_{pqr}\supset Z_{rq}\supset Z_q\supset\{0\}\text{ and }D_{2pqr}\supset Z_{pqr}\supset Z_{rq}\supset Z_r\supset\{0\}.
\end{aligned}$$

Clearly, pqr is the number of subgroups $Z_2$ of the dihedral group $D_{2pqr}$, qr is the number of the subgroups $D_{2p}$, pr is the number of the subgroups $D_{2q}$, pq is the number of the subgroups $D_{2r}$ and p is the number of the subgroups $D_{2qr}$, q is the number of the subgroups $D_{2pr}$ and r is the number of the subgroups $D_{2pq}$ of the dihedral group $D_{2pqr}$ and the dihedral group $D_{2pqr}$ has just one subgroup as $Z_{pqr}$, $Z_{pq}$, $Z_{pr}$, $Z_{qr}$, $Z_q$, $Z_p$, $Z_r$.So that

$$\begin{aligned}\frac{s(G)-1}{2}=&\,s^*(\{0\})+p\,qs^*(Z_2)+s^*(Z_p)+s^*(Z_q)+s^*(Z_r)+s^*(Z_{pq})+s^*(Z_{pr})+s^*(Z_{qr})+s^*(Z_{pqr})\\
&+qrs^*(D_{2p})+prs^*(D_{2q})+pqs^*(D_{2r})+ps^*(D_{2qr})+qs^*(D_{2pr})+rs^*(D_{2pq}),\end{aligned}$$

Therefore

$$\begin{aligned}\frac{s(G)-1}{2}=&\,1+2pqr+3(2Z_p)+3(6)+26+qr(2p+4)+pr(2q+4)+pq(2r+4)\\
&+p\frac{(12qr+8(q+r)+24)}{2}+q\frac{(12pr+8(p+r)+24)}{2}+r\frac{(12pq+8(p+q)+24)}{2}.\end{aligned}$$

Thus

$$s(G)=52pqr+24(pq+pr+qr)+24(p+q+r)+103.$$

**Table 2:** The number of distinct fuzzy subgroups the dihedral group $D_{2pqr}$ for some selected primes.

| $G = D_{2pqr}$ | s(G) |
|---|---|
| $p=3,\ q=5,\ r=7$ | 7627 |
| $p=3,\ q=7,\ r=11$ | 15763 |
| $p=5,\ q=7,\ r=13$ | 28947 |
| $p=7,\ q=13,\ r=17$ | 91779 |
| $p=13,\ q=17,\ r=19$ | 238611 |

**Theorem 4.4**: Suppose that $p_1, p_2, ..., p_n$ are odd distinct primes and $P = 2\times p_1 \times p_2 \times ... \times p_n$ . If $G = D_p$ and $n > 1$, then

$$s(G) = 2P + \sum_{i=1}^{n}\binom{n}{i} s(Z_{\prod_{j=1}^{i} p_j}) + \frac{P}{2}\sum_{t|P,2<t<P} s(D_{2t}) + \frac{P}{2}(2^{n+1}-3) + 2^n + 2 .$$

**Proof:** We have

$$D_{2n} =< x, y \mid x^n = y^2 = 1, yxy = x^{-1} > .$$

It is well Known that for every divisor r of n, $D_{2n}$ possesses a sub group isomorphic to $Z_r$ , namely $H_0^r =< x^{\frac{n}{2}} >$ and $\frac{n}{r}$ subgroups isomorphic to $D_{2r}$ , namely $H_i^r =< x^{\frac{n}{r}}, x^{i-1}y >, i = 1,2,...,\frac{n}{r}$. Let

$$\Pi_k = \{ p_{i_1} \times .... \times p_{i_k} \mid i_1, ...., i_k \in \{1,....,n\}, i_1 < ... < i_k \}, k = 1,2,...,n .$$

We know that $G = D_P$ has the following maximal chains each be identified with the chain

$$D_{2\pi_n} \supset D_{2\pi_{n-1}} \supset ..... \supset D_{2\pi_1} \supset Z_{\pi_1} \supset \{0\},$$
$$D_{2\pi_n} \supset Z_{\pi_n} \supset Z_{\pi_{n-1}} \supset .... \supset Z_{\pi_1} \supset \{0\},$$
$$D_{2\pi_n} \supset D_{2\pi_{n-1}} \supset ..... \supset D_{2\pi_1} \supset Z_2 \supset \{0\},$$

such that $\pi_i \in \Pi_i$ for all $i \in \{1,...,n\}$. Now $\frac{P}{2}$ is the number of subgroups of the group G as $Z_2$, and for all $i = 1,...,n$ , $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the number of subgroups of the group G as $Z_{\pi_i}$ . Also $\frac{P}{2t}$ is the number of subgroups of the group G as $D_{2t}$ ,for every divisor t of $\frac{P}{2}$. Therefore by theorem 2.5,

$$\frac{s(G)-1}{2} = s^*(\{0\}) + \frac{P}{2}s^*(Z_2) + \sum_{i=1}^{n}\binom{n}{i}s^*(Z_{\pi_i}) + \sum_{t|P,2<t<P}\frac{P}{2t}s^*(D_{2t}),$$

Then

$$s(G) = 2P + 2\sum_{i=1}^{n}\binom{n}{i}s^*(Z_{\pi_i}) + P\sum_{t|P,2<t<P} s(D_{2t}) + \frac{P}{t}s^*(D_{2t}) + 3,$$

Thus

$$s(G) = 2P + \sum_{i=1}^{n}\binom{n}{i}s(Z_{\pi_i}) + \frac{P}{2}\sum_{t|P,2<t<P}\frac{s(D_{2t})+1}{t} + 2^n + 2 .$$

## REFERENCES

[1] P. S. Das, Fuzzy groups and level subgroups, Math. Appl.8 (1981), 264-269.

[2] C. Degang and J. Jiashang, Some notes on equivalence fuzzy sets and fuzzy sub-groups, Fuzzy sets and systems.152 (2005) 403-409.

[3] A. Iranmanesh and H. Naraghi, THE CONNECTION BETWEEN SOME EQUIVALENCE RELATIONS ON FUZZY SUBGROUPS, Iranian Journal of Fuzzy Systems, **8**(5)(2011)69-80.

[4] R. Kumar, Fuzzy Algebra, vol. I, University of Delhi, Publication Division, (1993).

[5] M. Mashinchi and M. Mukaidonon, On fuzzy subgroups classification, Research Report of Meiji University, Japan.9 (65) (1993), 31-36.

[6] John N. Mordeson, Kiran R. Bhutani and Azriel Rosenfeld, Fuzzy Group Theory, Springer-Verlag Berlin Heidelberg, (2005).

[7] J.N. Mordeson, N. Kuroki, D.S. Malik, Fuzzy Semigroups, Springer, Berlin, (2003).

[8] V. Murali and B.B. Makamba, On an equivalence of fuzzy subgroups I, Fuzzy sets andsystems.**123** (2001)259-264.

[9] V. Murali and B.B. Makamba, On an equivalence of fuzzy subgroups II, Fuzzy sets andsystems.**136** (1) (2003)93-104.

[10] V. Murali and B. B. Makamba, On an equivalence of fuzzy subgroups III, Internat. J. Math. Sci.**36** (2003)2303-2313.

[11] V. Murali and B.B. Makamba, Counting the number of fuzzy subgroups of an abelian group of order pnqm, Fuzzy sets and systems.**144** (2004) 459{470.

[12] V. Murali and B. B. Makamba, Fuzzy subgroups of infinite abelian groups, FJMS.14 (1) (2004) 113-125.

[13] A. Rosenfeld, Fuzzy groups, J.Math.Anal.Appl.**35** (1971)512-517.

[14] M. Tarnauceanu and L. Bentea, On the number of fuzzy subgroup S of finite abelian groups, Fuzzy Sets and systems, 159(2008) 1084-1096.

[15] L. A. Zadeh, Fuzzysets, information and control.8 (1965) 338-353.

[16] Y. Zhang and K. Zou, A not on an equivalence relation on fuzzy subgroups, Fuzzy Sets and Systems, **95**(1998)243-247.

**Source of support: Nil, Conflict of interest: None Declared**