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THE DETERMINATION OF THE NUMBER OF DISTINCTFUZZY SUBGROUPS OF GROUP $Z_{p_1\times p_2\times\ldots\times p_n}$ AND THEDIHEDRAL GROUP $D_{2\times p_1\times p_2\times\ldots\times p_n}$

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ABSTRACT

In this paper, we use the natural equivalence of fuzzy subgroups studied by Iranmanesh and Naraghi [3] to determine the number of distinct fuzzy subgroups of some finite groups. We focus on the determination of the number of distinct fuzzy subgroups of group $Z_{p_1\times p_2\times\ldots\times p_n}$ and the dihedral group $D_{2\times p_1\times p_2\times\ldots\times p_n}$ using this equivalence relation.

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1. INTRODUCTION

Zadeh introduced the notion of fuzzy sets and fuzzy set operations, in his classic paper [15] of 1965. In an analogous application with groups, Rosenfeld [13] formulated the elements of a theory of fuzzy groups. One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid evolutionin the last years. Many papers have treated the particular case of finite cyclic groups. Thus, in [8] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [11, 12, 14] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). In the present paper we establish the recurrence relation verified by the number of distinct fuzzy subgroups of group $Z_{p_1\times p_2\times\ldots\times p_n}$ and the dihedral group $D_{2\times p_1\times p_2\times\ldots\times p_n}$ such that $p_1,p_2,\ldots,p_n$ are distinct primes.

2. PRELIMINARIES

First of all, we present some basic notions and results of fuzzy sub group theory (for more details, see [4, 7, and 3]).

The dihedral group of order 2n, for $n\geq 2$, denoted by D_{2n}. A fuzzy sub set of a set X is a mapping $\mu : X \to [0,1]$. Fuzzy subset μ of a group G is called a fuzzy subgroup of G if:

(G_1) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all x; y $\in$ G;
(G_2) $\mu(x^{-1}) \geq \mu(x)$ for all x $\in$ G

The set of all fuzzy subgroup of a group G is denoted by F (G).

Definition 2.1: Let G be a group and $\mu \in F(G)$. The set of $\{x \in G \,|\, \mu(x) > 0\}$ is called the support of μ and denoted by supp μ.

Let G be a group and $\mu \in F(G)$. We shall write $\operatorname{Im}\mu$ for the image set of μ and F μ for the family $\{\mu_t \,|\, t \in \operatorname{Im}\mu\}$.

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Theorem 2.2: Let G be a fuzzy group. If μ is a fuzzy subset of G, then $\mu \in F(G)$ if and only if for all $\mu_t \in F_\mu$, μ_t is a subgroup of G.

Let $F_1(G)$ be the set of all fuzzy subgroups μ of G such that $\mu(e)=1$, and let $\sim_R$ be an equivalence relation on $F_1(G)$. We denote the set $\{v \in F_1(G) \mid v \sim_R \mu\}$ by $\frac{\mu}{\sim_R}$ and the set $\left\{\frac{\mu}{\sim_R} \mid \mu \in F_1(G)\right\}$ by $\frac{F_1(G)}{\sim_R}$.

Definition 2.3: Let G be a group, and $\mu, v \in F_1(G)$. μ is equivalent to v, written as $\mu \sim v$ if
(1) $\mu(x) > \mu(y) \Leftrightarrow v(x) > v(y)$ for all $x, y \in G$.
(2) $\mu(x) = 0 \Leftrightarrow v(x) = 0$ for all $x \in G$.

The number of the equivalence classes ~ on $F_1(G)$ is denoted by $s(G)$. We mean the number of distinct fuzzy subgroups of G is s (G).

Theorem 2.4: Let G be a finite group. The number of distinct fuzzy subgroups of G such that their support is exactly equal to G is $\frac{s(G)+1}{2}$.

Let G be a finite group. The number of distinct fuzzy subgroups of G such that their support is exactly equal to G is denoted by $s^*(G)$.

Theorem 2.5: [3] Let G be a finite group. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of G is $\frac{s(G)-1}{2}$.

Theorem 2.6: [3] Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly equal to H is $\frac{s(H)+1}{2}$.

Corollary 2.7: [3] Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G such that their support is exactly a subgroup of H is $\frac{s(H)-1}{2}$.

Proposition 2.8: [6] Let $n \in N$. Then there are $2^{n+1}-1$ distinct equivalence classes of fuzzy subgroups of Z_{p^n}.

3. THE NUMBER OF THE DISTINCT FUZZY SUBGROUPS OF THE ABELIANGROUP $Z_{p_1 \times p_2 \times \ldots \times p_n}$

In this section, we characterize fuzzy subgroups of the abelian group $Z_{p_1 \times p_2 \times \ldots p_n}$ such that $p_1, p_2, \ldots, p_n$ are distinct primes numbers (n > 1).

Proposition 3.1: Suppose that p and q are distinct primes. Then there are11 distinct equivalence classes of fuzzy subgroups of Z_{pq}.

Proof: See Theorem 8.2.4 of [6].

Proposition 3.2: Suppose that p, q and r are distinct primes. Then there are 51 distinct equivalence classes of fuzzy subgroups of Z_{pqr}.

Proof: We know that Z_{pqr} has the following maximal chains:

$$Z_{pqr} \supset Z_{pq} \supset Z_p \supset \{0\}, Z_{pqr} \supset Z_{pq} \supset Z_q \supset \{0\}, Z_{pqr} \supset Z_{pr} \supset Z_p \supset \{0\},$$
$$Z_{pqr} \supset Z_{pr} \supset Z_r \supset \{0\}, Z_{pqr} \supset Z_{qr} \supset Z_q \supset \{0\} \text{ and } Z_{pqr} \supset Z_{qr} \supset Z_r \supset \{0\}.$$

All of subgroups of the group Z_{pqr} are $Z_{pq}, Z_{pr}, Z_{qr}, Z_p, Z_q, Z_r$ and $\{0\}$ .Thus

$$\frac{s(G)-1}{2} = s^*(\{0\}) + s^*(Z_{pq}) + s^*(Z_{pr}) + s^*(Z_{qr}) + s^*(Z_p) + s^*(Z_q) + s^*(Z_r),$$

therefore

$$\frac{s(G)-1}{2} = 1 + 3s^*(Z_{pq}) + 3s^*(Z_p) = 1 + 3(6) + 3(2) = 2 \quad .$$

Hence

$$s(G) = 51.$$

Theorem 3.3: Suppose that $p_1, p_2, ..., p_n$ are distinct primes. If $G = Z_{p_1 \times p_2 \times ... \times p_n}$ and n>1, then $s(G) = \sum_{i=1}^{n-1} \binom{n}{i} s(Z_{\prod_{j=1}^{i} p_j}) + 2^n + 1$.

Proof: Denote $\prod_k = \{p_{i1} \times ... \times p_{ik} \mid i_1, ..., i_k \in \{1, ..., n\}, i_1 < ... < i_k\}$, $k = 1, 2, ..., n$. We know that $G = Z_{p_1 \times p_2 \times ... \times p_n}$ has the following maximal chains each of which can be identified with the chain $Z_{\pi_n} \supset Z_{\pi_{n-1}} \supset .... \supset Z_{\pi_1} \supset \{0\}$ such that $\pi_i \in \Pi_i$, for all $i \in \{1, ..., n\}$. For all $i = 1, .., n$, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the number of subgroups of the group G as Z_{π_i} . Therefore by theorem 2.5, $\frac{s(G)-1}{2} = s^*(\{0\}) + \sum_{i=1}^{n-1} \binom{n}{i} s^*(Z_{\pi_i})$ and hence $s(G) = 2\sum_{i=1}^{n-1} \binom{n}{i} s^*(Z_{\pi_i}) + 3$ .By theorem 2.4, $s(G) = \sum_{i=1}^{n-1} \binom{n}{i} s(Z_{\pi_i}) + 2^n + 1$.

4. THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF THE DIHEDRALGROUP $D_{2 \times p_1 \times p_2 \times ... \times p_n}$

In this section, we determine the number of distinct fuzzy subgroups ofthe dihedral group $D_{2 \times p_1 \times p_2 \times ... \times p_n}$ such that $p_1, p_2, ..., p_n$ are odd distinct primes.

Theorem 4.1: Suppose that p is a prime and $p \geq 3$. If G is the dihedral group of order 2p, then s(G) = 4p + 7.

Proof: We know that D_{2p} has the following maximal chains:
$D_{2p} \supset Z_p \supset \{0\}$ and $D_{2p} \supset Z_2 \supset \{0\}$ whose the number is p. Now 2 is the number of distinct fuzzy subgroups whose support is Z_p, $2^1 p$ is the number of distinct fuzzy subgroups whose support is Z_2, and 2^0 is the number of fuzzy subgroups whose support is $\{0\}$. Thus $\frac{s(G)-1}{2} = 2p + 2 + 1$, therefore $s(G) = 4p + 7$.

Theorem 4.2: Suppose that p and q are odd distinct primes. If G is the dihedral group of order 2pq, then $s(G) = 12pq + 8(p+q) + 23$.

Proof: We know that D_{2pq} has the following maximal chains:
$D_{2pq} \supset D_{2p} \supset Z_2 \supset \{0\}$, $D_{2pq} \supset D_{2p} \supset Z_p \supset \{0\}$, $D_{2pq} \supset D_{2q} \supset Z_2 \supset \{0\}$, $D_{2pq} \supset D_{2q} \supset Z_q \supset \{0\}$,
$D_{2pq} \supset D_{pq} \supset Z_p \supset \{0\}$, $D_{2pq} \supset D_{pq} \supset Z_q \supset \{0\}$. Clearly, pq is the number of the subgroups Z_2 of the dihedral group D_{2pq} , q is the number of the subgroups D_{2p} and p is the number of the subgroups D_{2q} of the dihedral group D_{2pq} and the dihedral group D_{2pq} has just one subgroup as Z_{qp}, Z_q, Z_p . So that

$$\frac{s(G)-1}{2} = s^*(\{0\}) + pqs^*(Z_2) + qs^*(D_{2p}) + ps^*(D_{2q}) + s^*(Z_{pq}) + s^*(Z_p) + s^*(Z_q).$$

Thus

$$\frac{s(G)-1}{2}=1+2pq+q(2p+4)+p(2q+4)+6+2+2,$$

therefore

$$s(G)=12pq+8(p+q)+23.$$

Table 1: The number of distinct fuzzy subgroups of dihedral group D_{2pq} for some selected primes.

$G=D_{2pq}$	s(G)
$p=3,\ q=5$	267
$p=3,\ q=7$	355
$p=3,\ q=11$	531
$p=3,\ q=13$	619
$p=13,\ q=17$	2915

Theorem 4.3: Suppose that p, q and r are odd distinct primes. If G is the dihedral group of order 2pqr, then $s(G)=52pqr+24(pq+pr+qr)+24(p+q+r)+103$.

Proof: We have

$$D_{2n}=<x,y\mid x^n=y^2=1, yxy=x^{-1}>.$$

It is well Known that for every divisor r of n, D_{2n} possesses a subgroup isomorphic to Z_r, namely $H_0^r=<x^{\frac{n}{r}}>$ and $\frac{n}{r}$ subgroups isomorphic to D_{2r}, namely $H_i^r=<x^{\frac{n}{r}},x^{i-1}y>$, $i=1,2,\dots,\frac{n}{r}$. We know that D_{2pqr} has the following maximal chains each of which can be identified with the Chain,

$$D_{2pqr}\supset D_{2pq}\supset D_{2p}\supset Z_p\supset\{0\}, D_{2pqr}\supset D_{2pq}\supset D_{2q}\supset Z_q\supset\{0\}, D_{2pqr}\supset D_{2pr}\supset D_{2p}\supset Z_p\supset\{0\},$$
$$D_{2pqr}\supset D_{2pr}\supset D_{2r}\supset Z_r\supset\{0\}, D_{2pqr}\supset D_{2qr}\supset D_{2q}\supset Z_q\supset\{0\}, D_{2pqr}\supset D_{2qr}\supset D_{2r}\supset Z_r\supset\{0\},$$
$$D_{2pqr}\supset D_{2pq}\supset D_{2p}\supset Z_2\supset\{0\}, D_{2pqr}\supset D_{2pq}\supset D_{2q}\supset Z_2\supset\{0\}, D_{2pqr}\supset D_{2pr}\supset D_{2p}\supset Z_2\supset\{0\},$$
$$D_{2pqr}\supset D_{2pr}\supset D_{2r}\supset Z_2\supset\{0\}, D_{2pqr}\supset D_{2qr}\supset D_{2q}\supset Z_2\supset\{0\}, D_{2pqr}\supset D_{2qr}\supset D_{2r}\supset Z_2\supset\{0\},$$
$$D_{2pqr}\supset Z_{pqr}\supset Z_{pq}\supset Z_p\supset\{0\}, D_{2pqr}\supset Z_{pqr}\supset Z_{pq}\supset Z_q\supset\{0\}, D_{2pqr}\supset Z_{pqr}\supset Z_{pr}\supset Z_p\supset\{0\},$$
$$D_{2pqr}\supset Z_{pqr}\supset Z_{pr}\supset Z_r\supset\{0\}, D_{2pqr}\supset Z_{pqr}\supset Z_{rq}\supset Z_q\supset\{0\} \text{ and } D_{2pqr}\supset Z_{pqr}\supset Z_{rq}\supset Z_r\supset\{0\}.$$

Clearly, pqr is the number of subgroups Z_2 of the dihedral group D_{2pqr}, qr is the number of the subgroups D_{2p}, pr is the number of the subgroups D_{2q}, pq is the number of the subgroups D_{2r} and p is the number of the subgroups D_{2qr}, q is the number of the subgroups D_{2pr} and r is the number of the subgroups D_{2pq} of the dihedral group D_{2pqr} and the dihedral group D_{2pqr} has just one subgroup as Z_{pqr}, Z_{pq}, Z_{pr}, Z_{qr}, Z_q, Z_p, Z_r.So that

$$\frac{s(G)-1}{2}=s^*(\{0\})+pqrs^*(Z_2)+s^*(Z_p)+s^*(Z_q)+s^*(Z_r)+s^*(Z_{pq})+s^*(Z_{pr})+s^*(Z_{qr})+s^*(Z_{pqr})$$
$$+qrs^*(D_{2p})+prs^*(D_{2q})+pqs^*(D_{2r})+ps^*(D_{2qr})+qs^*(D_{2pr})+rs^*(D_{2pq}),$$

Therefore

$$\frac{s(G)-1}{2}=1+2pqr+3(2Z_p)+3(6)+26+qr(2p+4)+pr(2q+4)+pq(2r+4)$$
$$+p\frac{(12qr+8(q+r)+24)}{2}+q\frac{(12pr+8(p+r)+24)}{2}+r\frac{(12pq+8(p+q)+24)}{2}.$$

Thus

$$s(G)=52pqr+24(pq+pr+qr)+24(p+q+r)+103.$$

Table 2: The number of distinct fuzzy subgroups the dihedral group D_{2pqr} for some selected primes.

$G = D_{2pqr}$	s(G)
$p=3, q=5, r=7$	7627
$p=3, q=7, r=11$	15763
$p=5, q=7, r=13$	28947
$p=7, q=13, r=17$	91779
$p=13, q=17, r=19$	238611

Theorem 4.4: Suppose that $p_1, p_2, ..., p_n$ are odd distinct primes and $P = 2\times p_1 \times p_2 \times ... \times p_n$. If $G = D_p$ and $n>1$, then

$$s(G) = 2P + \sum_{i=1}^{n}\binom{n}{i} s(Z_{\prod_{j=1}^{i} p_j}) + \frac{P}{2}\sum_{t|P, 2<t<P} s(D_{2t}) + \frac{P}{2}(2^{n+1}-3) + 2^n + 2 .$$

Proof: We have

$$D_{2n} = < x, y \mid x^n = y^2 = 1, yxy = x^{-1} > .$$

It is well Known that for every divisor r of n, D_{2n} possesses a sub group isomorphic to Z_r, namely $H_0^r = < x^{\frac{n}{2}} >$ and $\frac{n}{r}$ subgroups isomorphic to D_{2r}, namely $H_i^r = < x^{\frac{n}{r}}, x^{i-1}y >, i = 1, 2, ..., \frac{n}{r}$. Let

$$\Pi_k = \left\{ p_{i_1} \times .... \times p_{i_k} \mid i_1, ..., i_k \in \{1, ..., n\}, i_1 < ... < i_k \right\}, k = 1, 2, ..., n .$$

We know that $G = D_P$ has the following maximal chains each be identified with the chain

$$D_{2\pi_n} \supset D_{2\pi_{n-1}} \supset ..... \supset D_{2\pi_1} \supset Z_{\pi_1} \supset \{0\},$$
$$D_{2\pi_n} \supset Z_{\pi_n} \supset Z_{\pi_{n-1}} \supset .... \supset Z_{\pi_1} \supset \{0\},$$
$$D_{2\pi_n} \supset D_{2\pi_{n-1}} \supset ..... \supset D_{2\pi_1} \supset Z_2 \supset \{0\},$$

such that $\pi_i \in \Pi_i$ for all $i \in \{1, ..., n\}$. Now $\frac{P}{2}$ is the number of subgroups of the group G as Z_2, and for all $i = 1, ..., n$, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the number of subgroups of the group G as Z_{π_i}. Also $\frac{P}{2t}$ is the number of subgroups of the group G as D_{2t}, for every divisor t of $\frac{P}{2}$. Therefore by theorem 2.5,

$$\frac{s(G)-1}{2} = s^*(\{0\}) + \frac{P}{2}s^*(Z_2) + \sum_{i=1}^{n}\binom{n}{i} s^*(Z_{\pi_i}) + \sum_{t|P, 2<t<P}\frac{P}{2t}s^*(D_{2t}),$$

Then

$$s(G) = 2P + 2\sum_{i=1}^{n}\binom{n}{i} s^*(Z_{\pi_i}) + P\sum_{t|P, 2<t<P} s(D_{2t}) + \frac{P}{t}s^*(D_{2t}) + 3,$$

Thus

$$s(G) = 2P + \sum_{i=1}^{n}\binom{n}{i} s(Z_{\pi_i}) + \frac{P}{2}\sum_{t|P, 2<t<P}\frac{s(D_{2t})+1}{t} + 2^n + 2 .$$

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