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SOME OTHER COMMUTATIVE - TRANSITIVE FINITE RINGS

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ABSTRACT

The ring R is said to be commutative- transitive if for each $a, b, c \in R\backslash Z(R)$, $ab = ba$ and $bc = cb$ imply $ac = ca$. In this paper, we present other examples of commutative- transitive rings.

We show that a ring R is commutative- transitive iff commutative graph R is a union of complete graphs. So we show non-commutative rings of order p^4 are commutative- transitive.

***Keywords:** transitive rings, commutative- transitive, centralizer,*

INTRODUCTION

In reference [1], the structure of commutative- transitive finite rings has been described, it is shown in that paper that simple and commutative- transitive finite rings are fields or 2×2 matrices rings on fields or $\frac{R}{J(R)} = F_1 \times F_2$ for two fields and . Also, the structure of irreducible commutative- transitive ring were specified.

COMMUTATIVE - TRANSITIVE FINITE RINGS

Definition 1: Ring *R* is said to be commutative-transitive if for each *a,b,c* ∈*R*, *ab=ba* and *bc=cb*, imply *ac=ca* [1].

Theorem 1: The following conditions are equivalent for ring *R* :
A) *R* is commutative- transitive.
B) For each *x,y* ∈*R/Z(R)* , if *xy= yx* , then *c(x)= c(y).*
C) The centralizers of all non central elements of *R* are commutative.

Definition 2: Let *R* is a ring.Commutative graph *μ(R)* as set of vertices *μ(R)* is all elements non central of *R* and distinct vertices *a,b* in *μ(R)* adjacent iff *ab=ba*.

Result 1: The ring *R* is commutative-transitive iff commutative graph *μ(R)* is union of complete graphs. [ ] { } | F

Example 1: For an arbitrary field , the ring *R*= $\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in F\right\}$ is commutative -transitive. Let *A*= $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a non-central elements of *R*. We determine the centralizer of *A*.

If *B* = $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in C(A)$, Then *AB=BA*. So *(a−c)y =(x−z) b* we consider the following cases:

Case 1: if *a=c* , then *C(A)*= $\left\{\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \mid x, y \in F\right\}$.

Case 2: If *a≠c*, then *C(A)*= $\left\{\begin{bmatrix} x & (x-z)b(a-c)^{-1} \\ 0 & z \end{bmatrix} \mid x, z \in F\right\}$.

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In each case $C(A)$ is commutative. So R is commutative- transitive. The commutative graph of R is the union of $|F|+1$ complete graphs which an of them has $|F|^2 - |F|$ vertices.

GROUP RINGS

Definition 3: Let G is a group and R is a ring. Then RG is defined as $RG = \{\sum_{g\in G} r_g g \mid r_g \in R\}$ in which $r_g = 0$, except for some finite numbers. In RG, addition and multiplication are defined naturally and distributedly, respectively. RG is called a group ring on R [3].

Example 2: Let $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$ is an eight-element quaternion group. We show that the group ring of Z_2Q_8 is commutative- transitive. For simplicity, we put $1' = -1$, $i' = -I$, $j' = -j$, $k' = -k$. It is shown that: $Z(Z_2Q_8) = \{a_1 1 + a_2 1' + a_3(i+i') + a_4(j+j') + a_5(k+k') \mid a_1 \ldots a_5 \in z_2\}$. Regarding the symmetry of Z_2Q_8 elements. We must find $c(i)$, $c(i+j)$ and $c(i+j+k)$ to prove that centralizer is commutative for each non-central element. we can get with a little calculation

$c(i) = \{a_1 1 + a_2 1' + a_3 i + a_4 i' + a_5(j+j') + a_6(k+k') \mid a_1 \ldots a_6 \in z\}$

$c(i+j) = \{a_1 1 + a_2 1' + a_3 i + a_4 j + (a_5 - a_4) i' + (a_5 - a_3) j' + a_6(k+k') \mid a_1 \ldots a_6 \in z_2\}$.

and $C(i+j+k) = \{a_1 1 + a_2 1' + a_3 i + a_4 i' + a_5 - a_4)j + (a_5 - a_3) + (a_6 - a_4)k + (a_6 - a_3)k' \mid a_1 \ldots a_6 \in z_2$.

As all above three centralizer are 6- dimensional and each contains a 5- dimensional subspace $Z(Z_2Q_8)$, they are commutative. Consequently group ring of Z_2Q_8 is commutative-transitive.

Theorem 2: If R is a non-commutative ring with identity of order p^4, in which p is prime number, then R is commutative-transitive.

Proof: As $|R| = p^4$, char R is a power of P. Therefore, elements of $0, 1, \ldots, p-1$ are distinct. For each $a \in R$, we have $|c(a)| > p$. so for a $a \notin Z(R)$, $|c(a)| = p^2$ or p^3. We show that $c(a)$ is commutative.

Case 1: If $|c(a)| = p^2$, we know each ring with identity of order p^2 is commutative.

Case 2: If $|c(a)| = p^3$, we put $S = C(a)$ and as element $a, 0, \ldots, p-1$ are all in the center of S, we get $|Z(s)| > p$ Therefore, $|Z(S)| = p^2$ or p^3. If $|Z(S)| = p^3$, then $Z(S) = S$. If $|Z(S)| = p^2$ for each $b \notin Z(S)$. We have $Z(S) \subsetneq C(b) \subsetneq S$ which is impossible, because $|S| = p^3$ and $|Z(S)| = p^2$. The proof is complete.

Example 3: Let the following rings have 16 elements.

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a^2 \end{bmatrix} \mid a, b \in F_4 \right\}, S = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} \mid a, b, c, d \in F_2 \right\}.$$

Therefore , based on the above theorem , they are commutative-transitive and the commutative of R is the union of one K_6 graph and four K_2 graphs and the commutative graph of S is the union of three K_4 graphs.

Note 1: We have a single non commutative ring of $p3$ order with identity the following ring:

$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in Z_p \right\}$ That is also commutative-transitive [2].

That is also commutative-transitive [2].

SKEW POLYNOMIAL RING

Definition 4: If R is a ring and $\sigma : R \to R$ is a endomorphism, let $R[x; \sigma]$ denote the ring of polynomials over R, that is , all formal polynomials in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ [4].

Theorem 3: Let F is a field and σ is an endomorphism of F. Then $R = \frac{F[X;\sigma]}{<x^2>}$ is commutative-transitive.

Proof: Let $Z(R) = Fix(\sigma) = K$. Now show centralizer of each non-central elements is commutative. Let $a+bx$ is a non-central element of R and $\alpha+\mu x \in c(a+bx)$

Case 1: If $\sigma(a) \neq a$ then $C(a+bx) = \left\{\alpha + \frac{b(\sigma(\alpha)-\alpha)}{\sigma(\alpha)-\alpha} x \mid \alpha \in F\right\}$ in this case centralizer is commutative.

Case 2: If $\sigma(a)=a$ in this case $\sigma(\alpha)=\alpha$ and $b\neq s$ so $C(a+bx)=\{\alpha + \mu x \mid \mu \in F, \alpha \in k\}$.

In this case centralizer is commutative. Therefore ring R is commutative-transitive. commutative graph of R is the union of $|F|$ complete graphs which an of them has $|F|-|k|$ vertices and and one complete graph with $|k| \mid F| |k|$ vertices.In the following, we preset a simple and short proof of a Corollary (21) of [1].

Theorem 4: Let R is a local ring and $R/J(R) \cong F_p^{\ r}$, in which r is prime. If $J(R)$ is commutative, then R is commutative-ransitive.

Proof: Let a is a non-centeral element of R. show $C(a)$ is commutative. Since $J(C(a)) = C(a) \cap J(R)$, *so*

$$Z_p \subseteq \frac{C(a)}{C(a)\cap J(R)} \cong \frac{C(a)\cap J(R)}{J(R)} \subseteq \frac{R}{J(R)} \cong F_{p^r} .$$

Since r is prime so there is not any field between Z_p and $F_p^{\ r}$. On the other hand $\frac{C(a)}{C(a)\cap J(R)}$ is a field since $C(a)$ is a local ring with maximal ideal of $C(a)\cap J(R)$

Therefore , $\frac{C(a)}{C(a)\cap J(R)} \cong Z_p$ or $\frac{C(a)}{C(a)\cap J(R)} \cong F_{p^r}$.

Case 1: If $\frac{C(a)}{C(a)\cap J(R)} \cong Z_p$ then $C(a)=Z_p+(C(a) \cap J(R))$ so $C(a)$ is commutative.

Case 2: If $\frac{C(a)}{C(a)\cap J(R)} \cong F_{p^r}$, in this case $a \notin Z_p + J(R)$, since if $a \in Z_p + J(R)$ then Since $J(R)$ is commutative therefore $J(R) \subseteq C(a)$ Since $\frac{C(a)}{J(R)} \cong F_{p^r}$ and $\frac{R}{J(R)} \cong F_{p^r}$ so $C(a)=R$ this meaning a is centeral which is a contradiction. As $a \in Z(C(a))$ so $\frac{Z(C(a))}{Z(C(a))\cap J(R)} \ncong Z_p$.

On the other hand $Z_p \subseteq \frac{Z(C(a))}{Z(C(a))\cap J(R)} \cong \frac{Z(C(a))+J(R)}{J(R)} \subseteq \frac{R}{J(R)} \cong F_{p^r}$.

Therefore $\frac{Z(C(a))+J(R)}{J(R)} \cong F_{p^r}$, since $\frac{C(a)+J(R)}{J(R)} \cong F_{p^r}$, So $Z(C(a))+J(R)=C(a)+J(R)$ then $C(a)=Z(C(a))+(C(a)\cap J(R))$.

And as $J(a)$ is commutative so $C(a)$ is commutative, Therefore R is commutative- transitive.

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