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The $\alpha\delta$-Kernel and $\alpha\delta$-Closure via $\alpha\delta$-Open sets in Topological Spaces

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ABSTRACT

The notion of $\alpha\delta$-closed set was introduced and studied by R. Devi, V. Kokilavani and P. Basker [2]. In this paper, we introduce the concept of weakly ultra-$\alpha\delta$-separation of two sets in a topological space using $\alpha\delta$-open sets. The $\alpha\delta$-closure and the $\alpha\delta$-kernel are defined in terms of this weakly ultra-$\alpha\delta$-separation. We also investigate some of the properties of the $\alpha\delta$-kernel and the $\alpha\delta$-closure. They also studied the relationships between these sets and several other types of open sets. It is the aim of this paper to offer some weak separation axioms by utilizing $\alpha\delta$-open sets and the $\alpha\delta$-closure operator.

Keywords: *$\alpha\delta$-open set, $\alpha\delta$-kernel, $\alpha\delta$-closure.*

MSC Classification: *54B05, 54C08.*

1. INTRODUCTION AND PRELIMINARIES

Closedness are basic concept for the study and investigation in general topological spaces. This concept has been generalized and studied by many authors from different points of views. In particular, Njastad [6] and Velicko [7] introduced α-open sets and δ-closed sets respectively. R.Devi et al.[1] introduced α-generalized closed (briefly αg-closed) sets. More recently R. Devi, V. Kokilavani and P. Basker. [2] has introduced and studied the notion of $\alpha\delta$-closed sets which is implied by that of δ-closed sets. We define that a set A is weakly ultra-$\alpha\delta$-separated from B if there exists an $\alpha\delta$-open set G containing A such that $G \cap B = \varphi$. Using this concept, we define the $\alpha\delta$-closure, $\alpha\delta$-kernel, $\alpha\delta$-derived set, $\alpha\delta$-shell of a set A of a topological space(X,τ). The aim of this paper to offer some weak separation axioms by utilizing $\alpha\delta$-open sets and the $\alpha\delta$-closure operator.

Throughout this present paper, spaces X and Y always mean topological spaces. Let X be a topological space and A, a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$, The δ-interior [7] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $Int_\delta(A)$. The subset A is called δ-open [7] if $A = Int_\delta(A)$, *i.e.*, a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set $A \subset (X,\tau)$ is called δ-closed [7] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x/x \in U \in \tau \Rightarrow int(cl(A)) \cap A \neq \varphi$. The family of all δ-open (resp. δ-closed) sets in X is denoted by $\delta O(X)$ (resp. $\delta C(X)$). A subset A of X is called α-open [6] if $A \subset int(cl(int(A)))$ and the complement of a α-open are called α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha cl(A)$, Dually, α-interior of A is defined to be the union of all α-open sets contained in A and is denoted by $\alpha int(A)$.
We recall the following definition used in sequel.

Definition 1.1. A subset A of a space X is said to be
(a) An α-generalized closed [1] (αg-closed) set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X,τ).
(b) A $\alpha\delta$-closed set [2] if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open in (X,τ).

2.$\alpha\delta$-KERNEL AND $\alpha\delta$-CLOSURE

Definition 2.1.
(a) The intersection of all $\alpha\delta$-open subsets of (X,τ) containing A is called the $\alpha\delta$-kernel of A (briefly, $\alpha\delta^{\sim Ker}(A)$) .*i.e.*,$\alpha\delta^{\sim Ker}(A) = \cap\{G \in \alpha\delta O(X,\tau): A \subseteq G\}$.
(b) Let $x \in X$. Then $\alpha\delta$-kernel of x is denoted by $\alpha\delta^{\sim Ker}(\{x\}) = \cap\{G \in \alpha\delta O(X,\tau): x \in G\}$.

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(c) Let X be a topological space and let $x \in X$. A subset N of X is said to be $\alpha\delta$-nbhd of x if there exists a $\alpha\delta$-open set G such that $x \in G \subset N$ which is denoted by $\alpha\delta$-N (x).

(d) Let A be a subset of X. A point $x \in A$ is said to be $\alpha\delta$-Interior point of A if A is a $\alpha\delta$-nbhd of x. The set of all $\alpha\delta$-Interior point of A is called the $\alpha\delta$-Interior of A and is denoted by $\alpha\delta_{Int}(A)$. [4]

(e) For a subset A of (X,τ), we define the $\alpha\delta$-closure of A as follows $\alpha\delta_{Cl}(A) = \cap \{F : F \text{ is } \alpha\delta - \text{closed in } X,\ A \subset X\}$. [4]

Theorem 2.2. Let X be a topological space. Then for any nonempty subset A of X, $\alpha\delta^{\sim Ker}(A) = \{x \in X : \alpha\delta_{Cl}(\{x\}) \cap A \neq \varphi\}$.

Proof. Let $x \in \alpha\delta^{\sim Ker}(A)$. Suppose that $\alpha\delta_{Cl}(\{x\}) \cap A = \varphi$. Then $A \subseteq X - \alpha\delta_{Cl}(\{x\})$ and $X - \alpha\delta_{Cl}(\{x\})$ is $\alpha\delta$-open set containing A but not x, which is a contradiction.

Conversely, let us assume that $x \notin \alpha\delta^{\sim Ker}(A)$ and $\alpha\delta_{Cl}(\{x\}) \cap A \neq \varphi$. Then there exist an $\alpha\delta$-open set D containing A but not x and $y \in \alpha\delta_{Cl}(\{x\}) \cap A$.

Hence an $\alpha\delta$-closed set $X - D$ contains x, and $\{x\} \subset X - D$, $y \notin X - D$. This is a contradiction to $y \in \alpha\delta_{Cl}(\{x\}) \cap A$. Therefore $x \in \alpha\delta^{\sim Ker}(A)$.

Definition 2.3. In a space X, a set A is said to be *weakly ultra-$\alpha\delta$-separated* from a set B if there exists an $\alpha\delta$-open set G such that $A \subseteq G$ and $G \cap B = \varphi$ or $A \cap \alpha\delta_{Cl}(B) = \varphi$.

By the definition 2.4 and the theorem 2.2, we have the following $x, y \in X$ of a topological space,

(a) $\alpha\delta_{Cl}(\{x\}) = \{y : \{x\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{x\}$ $\}$

(b) $\alpha\delta^{\sim Ker}(\{x\}) = \{y : \{y\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{y\}$ $\}$.

Definition 2.4. For any point x of a space X, is called

(a) $\alpha\delta$-derived (briefly, $\alpha\delta_{-D}(\{x\})$) set of x is defined to be the set.
$\alpha\delta_{-D}(\{x\}) = \alpha\delta_{Cl}(\{x\}) - \{x\} = \{ y : y \neq x$ *and* $\{y\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{x\}\}$,

(b) $\alpha\delta$-shell (briefly,$\alpha\delta_{-Shl}(\{x\})$) of a singleton set $\{x\}$ is defined to be the set.
$\alpha\delta_{-Shl}(\{x\}) = \alpha\delta^{\sim Ker}(\{x\}) - \{x\} = \{ y : y \neq x$ *and* $\{x\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{y\}$ $\}$,

Definition 2.5. Let X be a topological space. Then we define

(a) $\alpha\delta_{-N-D} = \{ x : x \in X$ and $\alpha\delta_{-D}(\{x\}) = \varphi \}$,

(b) $\alpha\delta_{-N-Shl} = \{ x : x \in X$ and $\alpha\delta_{-Shl}(\{x\}) = \varphi \}$ and

(c) $\alpha\delta$-$\langle x \rangle = \alpha\delta_{Cl}(\{x\}) \cap \alpha\delta^{\sim Ker}(\{x\})$.

Theorem 2.6. Let $x,\ y \in X$. Then the following conditions hold.

(a) $y \in \alpha\delta^{\sim Ker}(\{x\})$ if and only if $x \in \alpha\delta_{Cl}(\{y\})$,

(b) $y \in \alpha\delta_{-Shl}(\{x\})$ if and only if $x \in \alpha\delta_{-D}(\{y\})$,

(c) $y \in \alpha\delta_{Cl}(\{x\})$ implies $\alpha\delta_{Cl}(\{y\}) \subseteq \alpha\delta_{Cl}(\{x\})$ and

(d) $y \in \alpha\delta^{\sim Ker}(\{x\})$ implies $\alpha\delta^{\sim Ker}(\{y\}) \subseteq \alpha\delta^{\sim Ker}(\{x\})$

Proof. The proof of (a) and (b) are obvious.

(c) Let $z \in \alpha\delta_{Cl}(\{y\})$. Then $\{z\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{y\}$. So there exists an $\alpha\delta$-open set Gcontaining z such that $G \cap \{y\} \neq \varphi$. Hence $y \in G$ and by assumption $G \cap \{x\} \neq \varphi$. Hence $\{z\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{x\}$. So $z \in \alpha\delta_{Cl}(\{x\})$. Therefore $\alpha\delta_{Cl}(\{y\}) \subseteq \alpha\delta_{Cl}(\{x\})$.

(d) Let $z \in \alpha\delta^{\sim Ker}(\{y\})$. Then $\{y\}$ *is not weakly ultra-$\alpha\delta$-separated from* $\{z\}$. So $y \in \alpha\delta_{Cl}(\{z\})$. Hence $\alpha\delta_{Cl}(\{y\}) \subseteq \alpha\delta_{Cl}(\{z\})$. By assumption $y \in \alpha\delta^{\sim Ker}(\{x\})$ and then $x \in \alpha\delta_{Cl}(\{y\})$. So $\alpha\delta_{Cl}(\{x\}) \subseteq \alpha\delta_{Cl}(\{y\})$. Ultimately $\alpha\delta_{Cl}(\{x\}) \subseteq \alpha\delta_{Cl}(\{z\})$. Hence $x \in \alpha\delta_{Cl}(\{z\})$, that is $z \in \alpha\delta^{\sim Ker}(\{x\})$. Therefore $\alpha\delta^{\sim Ker}(\{y\}) \subseteq \alpha\delta^{\sim Ker}(\{x\})$.

Let us recall that a subset A of X is called a degenerate set if A is either a null set or a singleton set.

Theorem 2.7. Let $x,\ y \in X$. Then,

(a) for every $x \in X$, $\alpha\delta_{-Shl}(\{x\})$ is degenerate if and only if for all $x,\ y \in X, x \neq y$, $\alpha\delta_{-D}(\{x\}) \cap \alpha\delta_{-D}(\{y\}) = \varphi$,

(b) for every $x \in X$, $\alpha\delta_{-D}(\{x\})$ is degenerate if and only if for every $x,\ y \in X, x \neq y$, $\alpha\delta_{-Shl}(\{x\}) \cap \alpha\delta_{-Shl}(\{y\}) = \varphi$.

Proof. (a) Let $\alpha\delta_{-D}(\{x\}) \cap \alpha\delta_{-D}(\{y\}) \neq \varphi$. Then there exists a $z \in X$ such that $z \in \alpha\delta_{-D}(\{x\})$ and $z \in \alpha\delta_{-D}(\{y\})$.

Then $z \neq y \neq x$ and $z \in \alpha\delta_{Cl}(\{x\})$ and $z \in \alpha\delta_{Cl}(\{y\})$, that is $x,\ y \in \alpha\delta^{\sim Ker}(\{z\})$. Hence $\alpha\delta^{\sim Ker}(\{z\})$ and so $\alpha\delta_{-Shl}(\{z\})$ is not a degenerate set.

Conversely, let $x,\ y\ \in\ \alpha\delta_{-Shl}(\{z\})$. Then we get $x \neq z$, $x \in \alpha\delta^{\sim Ker}(\{z\})$ and $y \neq z$, $y \in \alpha\delta^{\sim Ker}(\{z\})$ and hence z is an element of both $\alpha\delta_{Cl}(\{x\})$ and $\alpha\delta_{Cl}(\{y\})$, which is a contradiction.
(b) Obvious.

Theorem 2.8. If $y \in \alpha\delta$-$\langle x\rangle$, then $\alpha\delta$-$\langle x\rangle = \alpha\delta$-$\langle y\rangle$.

Proof. If $y \in \alpha\delta$-$\langle x\rangle$, then $y\ \in\ \alpha\delta_{Cl}(\{x\}) \cap \alpha\delta^{\sim Ker}(\{x\})$. Hence $y \in \alpha\delta_{Cl}(\{x\})$ and $y\ \in \alpha\delta^{\sim Ker}(\{x\})$ and so we have $\alpha\delta_{Cl}(\{y\}) \subseteq \alpha\delta_{Cl}(\{x\})$ and $\alpha\delta^{\sim Ker}(\{y\}) \subseteq \alpha\delta^{\sim Ker}(\{x\})$. Then $\alpha\delta_{Cl}(\{y\}) \cap \alpha\delta^{\sim Ker}(\{y\}) \subseteq \alpha\delta_{Cl}(\{x\}) \cap \alpha\delta^{\sim Ker}(\{x\})$. Hence $\alpha\delta$-$\langle y\rangle \subseteq \alpha\delta$-$\langle x\rangle$. The fact that $y \in \alpha\delta_{Cl}(\{x\})$ implies $x \in \alpha\delta^{\sim Ker}(\{y\})$ and $y \in \alpha\delta^{\sim Ker}(\{x\})$ implies $x \in \alpha\delta_{Cl}(\{y\})$. Then we have that $\alpha\delta$-$\langle x\rangle \subseteq \alpha\delta$-$\langle y\rangle$. So $\alpha\delta$-$\langle x\rangle = \alpha\delta$-$\langle y\rangle$.

Theorem 2.9. For all $x,\ y\ \in\ X$, either $\alpha\delta$-$\langle x\rangle \cap \alpha\delta$-$\langle y\rangle = \varphi$ or $\alpha\delta$-$\langle x\rangle = \alpha\delta$-$\langle x\rangle$.

Proof. $\alpha\delta$-$\langle x\rangle \cap \alpha\delta$-$\langle y\rangle \neq \varphi$, then there exists $z\ \in X$ such that $z \in \alpha\delta$-$\langle x\rangle$ and $z \in \alpha\delta$-$\langle y\rangle$. So by *Theorem 2.8*, $\alpha\delta$-$\langle z\rangle = \alpha\delta$-$\langle x\rangle = \alpha\delta$-$\langle y\rangle$. Hence the result.

Theorem 2.10. For any two points $x,\ y\ \in\ X$, the following statements are equivalent.
(a) $\alpha\delta^{\sim Ker}(\{x\}) \neq\ \alpha\delta^{\sim Ker}(\{y\})$ and
(b) $\alpha\delta_{Cl}(\{x\}) \neq\ \alpha\delta_{Cl}(\{y\})$.

Proof. $(a) \Rightarrow (b)$ Let us assume $\alpha\delta^{\sim Ker}(\{x\}) \neq \alpha\delta^{\sim Ker}(\{y\})$. Then there exists a $z \in \alpha\delta^{\sim Ker}(\{x\})$ but $z \notin \alpha\delta^{\sim Ker}(\{y\})$. As $z \in \alpha\delta^{\sim Ker}(\{x\})$, $x \in \alpha\delta_{Cl}(\{z\})$ and $\alpha\delta_{Cl}(\{x\}) \subseteq \alpha\delta_{Cl}(\{z\})$. Also we have taken $z \notin \alpha\delta^{\sim Ker}(\{y\})$, by Theorem 2.2, $\alpha\delta_{Cl}(\{z\}) \cap \{y\} = \varphi$, so $\alpha\delta_{Cl}(\{x\}) \cap \{y\} = \varphi$ and so $\{y\}$ is weakly ultra-$\alpha\delta$-separated from $\{x\}$ and hence we get that $y \notin \alpha\delta_{Cl}(\{x\})$. Hence $\alpha\delta_{Cl}(\{x\})\ \neq\ \alpha\delta_{Cl}(\{y\})$.
$(b) \Rightarrow (a)$ Suppose $\alpha\delta_{Cl}(\{x\}) \neq\ \alpha\delta_{Cl}(\{y\})$. Then there exists a point $z \in \alpha\delta_{Cl}(\{x\})$ but $z \notin \alpha\delta_{Cl}(\{y\})$.

So, we get an $\alpha\delta$-open set containing z and x but not y. That is $y \notin \alpha\delta^{\sim Ker}(\{x\})$. Hence $\alpha\delta^{\sim Ker}(\{x\}) \neq\ \alpha\delta^{\sim Ker}(\{y\})$.

Theorem 2.12 Let X be a topological space and each $x \in X$, Let $\alpha\delta$-$N\ (X,\tau)$ be the collection of all $\alpha\delta$-nbhd of x. Then we have the following results.
(a) $\forall x\ \in\ X,\ \alpha\delta$-$N\ (x) \neq\ \varphi$
(b) $N\ \in \alpha\delta$-$N\ (x)\ \Rightarrow\ x\ \in\ N$.
(c) $N\ \in \alpha\delta$-$N\ (x), M\ \supset\ N\ \Rightarrow\ M\ \in \alpha\delta$-$N\ (x)$
(d) $N\ \in \alpha\delta$-$N\ (x)\ \Rightarrow$ there exists $M\ \in \alpha\delta$-$N\ (x)$ such $M\ \subset\ N$ and $M\ \in\ \alpha\delta$-$N\ (y)$ for every $y\ \in\ M$.

Proof.
(a) Since X is $\alpha\delta$-open set, it is a $\alpha\delta$-nbhd of every $x\ \in\ X$. Hence there exists at least one $\alpha\delta$-nbhd (namely-X) for each $x\ \in\ X$. Hence $\alpha\delta$-$N\ (x) \neq \varphi$ for every $x\ \in\ X$.

(b) If $N\ \in \alpha\delta$-$N\ (x)$, then N is a $\alpha\delta$-nbhd of x. So by definition of $\alpha\delta$-nbhd, $x\ \in\ N$.

(c) Let $N\ \in \alpha\delta$-$N\ (x)$ and $M\ \supset\ N$. Then there is a $\alpha\delta$-open set U such that $x\ \in\ U\ \subset\ N$. Since $N\ \subset\ M, x\ \in\ U\ \subset\ M$ and M is $\alpha\delta$-nbhd of x. Hence $M\ \in \alpha\delta$-$N\ (x)$.

(d) If $N\ \in \alpha\delta$-$N\ (x)$, then there exists a $\alpha\delta$-open set M such that $x\ \in\ M\ \subset\ N$. Since M is a $\alpha\delta$-open set, it is $\alpha\delta$-nbhd of each of its points. Therefore $M\ \in \alpha\delta$-$N\ (y)$ for every $y\ \in\ M$.

Theorem 2.13. Let X be a nonempty set, for each $x\ \in\ X$, let $\alpha\delta$-$N\ (x)$ be nonempty
collection of subsets of X satisfying following conditions.
(a) $N\ \in \alpha\delta$-$N\ (X,\ \tau)\ \Rightarrow\ x\ \in\ N$.
(b) Let τ consists of the empty set and all those non-empty subsets of U of X having the property that $x\ \in\ U$ implies that there exists an $N\ \in \alpha\delta$-$N\ (x)$ such that $x \in N \subset U$, Then τ is a topology for X.

Proof.
(i) $\varphi\ \in\ \tau$ by definition. We now show that $x\ \in\ \tau$. Let x be any arbitrary element of X. Since $\alpha\delta$-$N\ (x)$ is nonempty, there is an $N\ \in \alpha\delta$-$N\ (x)$ and so $x \in N$. Since N is a subset of X, we have $x\ \in\ N\ \subset\ X$. Hence $X\ \in\ \tau$.

(ii) Let $U_\lambda\ \in\ \tau$ for every $\lambda\ \in\ \Lambda$. If $x\ \in \cup\ \{U_\lambda\ :\ \lambda\ \in\ \Lambda\}$, then $x\ \in\ U_{\lambda_x}$ for some $\lambda_x\ \in \Lambda$.

Since $U_{\lambda_x}\ \in\ \tau$, there exists an $N\ \in \alpha\delta$-$N\ (x)$ such that $x\ \in\ N\ \subset\ U_{\lambda_x}$ and consequently $x\ \in\ N\ \subset \cup \{U_\lambda\ :\ \lambda\ \in\ \Lambda\}$.

Hence $\cup\ \{\ U_\lambda\ :\ \lambda\ \in\ \Lambda\} \in\ \tau$. It follows that τ is topology for X.

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