

SOME COMMON FIXED POINT THEOREMS  
FOR SEQUENCE OF MAPPINGS IN  $D^*$ - METRIC SPACE

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ABSTRACT

In this paper we establish some common Fixed Point Theorems for sequence of contraction and generalized contraction mappings in  $D^*$  - metric space which is introduced by Shaban Sedghi, Nabi Shobe and Haiyun Zhou [10]. In what follows  $(X, D^*)$  will denote  $D^*$  - metric space,  $N$ , the set of all natural number and  $R^+$ , the set of all positive real number.

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1. INTRODUCTION

There have been an number of generalization in generalized metric space (or D-Metric space) initiated by Dhage [2] in 1992. He proved the existence of unique fixed point theorems of sequence of mappings satisfying certain contractive conditions in complete and bounded D- Metric space. Dealing with D- Metric space, Ahmad etal. [1], Dhage [2, 3, 4] Rhoades [8], Singh and Sharma [9] and others made a significant contribution in fixed point theory of D- Metric space. Unfortunately almost all theorems in D-Metric space are not valid (See S.V.R Naidu and others [5-7]). Here our aim is to prove some common fixed point theorems for sequences of generalized contractive mappings in  $D^*$ - Metric space as a probable modification of the definition of D-Metric spaces introduced by Dhage [2].

**Definition 1.1:** Let  $X$  be a non empty set. A generalized metric (or  $D^*$  - metric) on  $X$  is a function  $D^*: X^3 \rightarrow [0, \infty)$  that satisfies the following conditions for each  $x, y, z, a \in X$ .

1.  $D^*(x, y, z) \geq 0$
2.  $D^*(x, y, z) = 0$  if and only if  $x = y = z$
3.  $D^*(x, y, z) = D^*(\rho\{x, y, z\})$  where  $\rho$  is permutation.
4.  $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$ .

The pair  $(X, D^*)$  is called generalized metric (or  $D^*$  - metric) space.

**Example 1.2:**

(a)  $D^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$ ,

(b)  $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ .

Here,  $d$  is the ordinary metric on  $X$ .

(c) If  $X = R^n$  then we define

$$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p} \text{ for every } p \in R^+$$

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(d) If  $X = \mathbb{R}$  then we define

$$D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \text{Max } \{x, y, z\}, & \text{otherwise,} \end{cases}$$

**Remark 1.3:** In  $D^*$  - metric space  $D^*(x, y, y) = D^*(x, x, y)$

**Definition 1.4:** A open ball in a  $D^*$  - metric space  $X$  with centre  $x$  and radius  $r$  is denoted by  $B_{D^*}(x, r)$  and is defined by  $B_{D^*}(x, r) = \{y \in X: D^*(x, y, y) < r\}$

**Example 1.5:** Let  $X = \mathbb{R}$  Denote  $D^*(x, y, z) = |x-y| + |y-z| + |z-x|$  for all  $x, y, z \in \mathbb{R}$ .

Thus  $B_{D^*}(0, 1) = \{y \in \mathbb{R} / D^*(0, y, y) < 1\}$

$$= \{y \in \mathbb{R} / |0 - y| + |y - y| + |y - 0| < 1\}$$

$$= \{y \in \mathbb{R} / |y| + |y| < 1\}$$

$$= \{y \in \mathbb{R} / |y| < 1/2\}$$

$$= \{y \in \mathbb{R} / -1/2 < y < 1/2\}$$

$$= (-1/2, 1/2). \text{ (Open Interval)}$$

**Definition 1.6:** Let  $(X, D^*)$  be a  $D^*$  - metric space and  $A \subseteq X$

1. If for every  $x \in A$ , there exist  $r > 0$  such that  $B_{D^*}(x, r) \subseteq A$ , then subset  $A$  is called open subset of  $X$ .

2. Subset  $A$  of  $X$  is said to be  $D^*$  - bounded if there exist  $r > 0$  such that

$$D^*(x, y, y) < r \text{ for all } x, y \in A.$$

3. A sequence  $\{x_n\}$  in  $X$  converges to  $x$  if and only if

$$D^*(x_n, x_n, x) = D^*(x, x, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

That is, for each  $\varepsilon > 0$  there exist  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  implies  $D^*(x, x, x_n) < \varepsilon$  This is equivalent for each  $\varepsilon > 0$ , there exist  $n_0 \in \mathbb{N}$  such that for all  $n, \geq n_0$  implies  $D^*(x, x_n, x_n) < \varepsilon$ .

It is also noted that  $D^*(x_n, x_n, x) = D^*(x, x, x_n) < \varepsilon$  for all  $n \geq n_0$ , for some  $n_0 \in \mathbb{N}$ .

4. A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if for each  $\varepsilon > 0$ , there exist  $n_0 \in \mathbb{N}$  such that  $D^*(x_n, x_n, x_m) < \varepsilon$  for each  $n, m \geq n_0$  The  $D^*$  - metric space  $(X, D^*)$  is said to complete if every Cauchy sequence is convergent.

**Remark 1.7:**

(1)  $D^*$  is continuous function on  $X^3$

(2) If sequence  $\{x_n\}$  in  $X$  converges to  $x$ , then  $x$  is unique.

(3) Any convergent sequence in  $(X, D^*)$  is a Cauchy sequence.

**Definition 1.8:** A point  $x$  in  $X$  is a fixed point of the map  $T: X \rightarrow X$  if  $Tx = x$ .

**Definition 1.9:** A point  $x$  in  $X$  is a common fixed point of a sequence of maps  $T_n: X \rightarrow X$  if  $T_n(x) = x$  for all  $n$ .

**Theorem 1:** Let  $X$  be a  $D^*$  - complete metric space and  $T_n: X \rightarrow X$  be a sequence maps such that

$$D^*(T_i x, T_j y, z) \leq \alpha D^*(x, y, z) \text{ for all } i \neq j \text{ and for all } x, y, z \in X \text{ with } 0 \leq \alpha < 1/2$$

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  be any fixed arbitrary element Define a sequence  $\{x_n\}$  in  $X$  as.

$$x_{n+1} = T_{n+1} x_n \quad \text{for all } n = 0, 1, 2, \dots$$

Let  $d_n = D^*(x_n, x_{n+1}, x_{n+1})$  for all  $n = 0, 1, 2, \dots$

Now  $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+2})$

$$\begin{aligned} &= D^*(T_{n+1}x_n, T_{n+2}x_{n+1}, x_{n+2}) \\ &\leq \alpha D^*(x_n, x_{n+1}, x_{n+2}) \\ &\leq \alpha D^*(x_n, x_{n+1}, x_{n+1}) + \alpha D^*(x_{n+1}, x_{n+2}, x_{n+2}) \\ &= \alpha d_n + \alpha d_{n+1} \end{aligned}$$

$$(1 - \alpha) d_{n+1} \leq \alpha d_n$$

$$d_{n+1} \leq \frac{\alpha}{1 - \alpha} d_n$$

$$d_{n+1} \leq k d_n \quad \text{for all } n = 0, 1, 2, \dots, \text{ where } k = \frac{\alpha}{1 - \alpha} < 1 \text{ (Since } \alpha < 1/2)$$

$$d_n \leq k d_{n-1}$$

$$\leq k^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore  $\lim_{n \rightarrow \infty} d_n = 0$  Thus  $\lim_{n \rightarrow \infty} D^*(x_n, x_{n+1}, x_{n+1}) = 0$

Now we shall prove that  $\{x_n\}$  is a  $D^*$  - Cauchy sequence in  $X$ .

Let  $m > n > n_0$  for some  $n_0 \in \mathbb{N}$ .

Now  $D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$$\leq \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Thus  $\lim_{n, m \rightarrow \infty} D^*(x_n, x_n, x_m) = 0$

Therefore  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

Since  $X$  is  $D^*$  - Complete  $x_n \rightarrow x$  in  $X$ . we prove that  $x$  is a fixed point of  $T_n$  for all  $n$  suppose there exist an  $m$  such that  $x \neq T_m x$ .

Then  $D^*(T_m x, x, x) = \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+1}, x)$

$$= \lim_{n \rightarrow \infty} D^*(T_m x, T_{n+1} x_n, x)$$

$$\leq \alpha \lim_{n \rightarrow \infty} D^*(x, x_{n+1}, x)$$

$$= 0.$$

Therefore  $D^*(T_m x, x, x) = 0$ , Therefore  $T_n x = x$  for all  $n$ . Thus  $x$  is common fixed point of  $\{T_n\}$  for all  $n$ .

UNIQUENESS

Supper  $x \neq y$  such that  $T_n y = y$  for all n.

Then  $D^*(x, y, y) = D^*(T_i x, T_j y, y)$

$$\leq \alpha D^*(x, y, y)$$

This implies  $(1 - \alpha) D^*(x, y, y) \leq 0$

Since  $x \neq y$  we have  $D^*(x, y, y) > 0$  her  $(1 - \alpha) < 0$ .

This implies  $\alpha > 1$  which contraction to  $\alpha < 1/2$ .

Thus  $\{T_n\}$  have a unique common fixed point.

**Theorem 2:** Let X be a complete D\*- metric space and  $T_n: X \rightarrow X$  be a sequence of maps such that

$$D^*(T_i x, T_j y, T_k z) \leq \alpha D^*(x, y, z) \text{ for all } i \neq j \neq k \text{ and } x, y, z \in X \text{ with } 0 \leq \alpha < 1.$$

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  be any fixed arbitrary element Define a sequence  $\{x_n\}$  in X as  $x_{n+1} = T_{n+1} x_n$  for all n = 0, 1, 2, ...

Let  $d_n = D^*(x_n, x_{n+1}, x_{n+2})$

$$\begin{aligned} d_1 &= D^*(x_1, x_2, x_3) \\ &= D^*(T_1 x_0, T_2 x_1, T_3 x_2) \\ &\leq \alpha D^*(x_0, x_1, x_2) \end{aligned}$$

$$d_1 \leq \alpha d_0$$

$$\begin{aligned} d_2 &= D^*(x_2, x_3, x_4) \\ &= D^*(T_2 x_1, T_3 x_2, T_1 x_3) \\ &\leq \alpha D^*(x_1, x_2, x_3) \\ &\leq \alpha d_1 \\ &\leq \alpha^2 d_0 \end{aligned}$$

Continuing in this way we get  $d_n \leq \alpha^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$  (since  $0 \leq \alpha < 1$ ).

Now we shall prove that  $\{x_n\}$  is a Cauchy sequence in X.

Let  $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then  $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\begin{aligned} &\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \\ &\leq d_n + d_n^* \end{aligned}$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all n}$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d. Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\begin{aligned} \text{Now } d &= \lim_{n \rightarrow \infty} d_{n+2}^* \\ &\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\} \\ &\leq \lim_{n \rightarrow \infty} \{\alpha d_n + d_{n+1}^*\} \\ &< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\} \\ &= d. \text{ which is contraction. Thus } d = 0. \end{aligned}$$

Hence  $D^*(x_n, x_n, x_m) \rightarrow 0$  as  $m, n \rightarrow \infty$

Therefore  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$ .

Since  $X$  is  $D^*$  complete  $x_n \rightarrow x$  in  $X$

Now we prove that  $x$  is fixed point of  $T_n$

To prove that  $T_n x = x$  for all  $n$ .

Suppose There is an  $m$  such that  $T_m x \neq x$

$$\begin{aligned} \text{Then } D^*(T_m x, x, x) &= \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+1}, x_{n+2}) \\ &= D^*(T_m x, T_{n+1} x_n, T_{n+2} x_{n+1}) \\ &\leq \alpha \lim_{n \rightarrow \infty} D^*(x, x_{n+1}, x_{n+2}) \\ &\leq \alpha D^*(x, x, x) \\ &= 0 \end{aligned}$$

Thus  $T_n x = x$  for all  $n$ .

Now we prove that  $x$  is a unique common fixed point of  $\{T_n\}$ .

Suppose  $x \neq y$  and  $T_n y = y$ .

$$\begin{aligned} \text{Then } D^*(x, y, y) &= D^*(T_i x, T_j y, T_k y) \\ &\leq \alpha D^*(x, y, y) \end{aligned}$$

This implies  $(1-\alpha)D^*(x, y, y) \leq 0$

Since  $x \neq y$  we have  $D^*(x, y, y) > 0$

This  $((1-\alpha) < 0$

This impulse  $\alpha > 1$  which in contradiction Hence  $\{T_n\}$  have a unique common fixed point

**Theorem 3:** Let X be a D\* - complete metric space and  $T_n: X \rightarrow X$  be a sequence maps such that

$$D^*(T_i T_j x, T_i x, y) \leq \alpha D^*(T_i x, x, y) \text{ for all } i \neq j \text{ and for all } x, y, z \in X \text{ with } 0 \leq \alpha < 1.$$

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  be any fixed arbitrary element. Define a sequence  $\{x_n\}$  in X as  $x_{n+1} = T_{n+1} x_n$  for  $n=0; 1; 2 \dots$

$$\begin{aligned} D^*(x_{n+1}, x_n, x_n) &= D^*(T_{n+1} T_n x_{n-1}, T_n x_{n-1}, x_n) \\ &\leq \alpha D^*(T_n x_{n-1}, x_{n-1}, x_n) \\ &= \alpha D^*(x_n, x_{n-1}, x_n) \\ &= \alpha D^*(x_n, x_{n-1}, x_{n-1}) \\ &\vdots \\ &\leq \alpha^n D^*(x_1, x_0, x_0) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 < \alpha < 1). \end{aligned}$$

Now we shall prove that  $\{x_n\}$  is a Cauchy sequence in X. Let  $m > n > n_0$  for some  $n_0 \in \mathbb{N}$ .

$$\begin{aligned} D^*(x_n, x_m, x_m) &\leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}, x_{k+1}) \\ &< \alpha^n / (1-\alpha) D^*(x_1, x_0, x_0) \text{ as } m \rightarrow \infty \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

$$D^*(x_n, x_m, x_m) \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

Therefore  $\{x_n\}$  is D\* Cauchy sequence in X Since X is D\* complete  $x_n \rightarrow x$  in X

Now we prove that  $T_n x = x$  for all n.

Suppose there is an m such that  $T_m x = y$  where  $y \neq x$ .

$$\begin{aligned} D^*(T_m x, x, x) &= \lim_{n \rightarrow \infty} D^*(y, x_{n+2}, x_{n+1}) \\ &= \lim_{n \rightarrow \infty} D^*(y, T_{n+2} T_{n+1} x_n, T_{n+1} x_n) \\ &\leq \alpha \lim_{n \rightarrow \infty} D^*(y, T_{n+1} x_n, x_n) \\ &= \alpha \lim_{n \rightarrow \infty} D^*(y, x_{n+1}, x_n) \\ &= \alpha D^*(y, x, x) \\ &= \alpha D^*(T_m x, x, x) \\ &< D^*(T_m x, x, x) \end{aligned}$$

Therefore  $T_n x = x$  for all n.

Suppose  $x \neq y$  Such that that  $T_n y = y$  for all n.

$$\begin{aligned} \text{Then } D^*(x, x, y) &= D^*(T_{n+1} T_n x, T_n x, y) \\ &\leq \alpha D^*(T_n x, x, y) \\ &\leq \alpha D^*(x, x, y) \end{aligned}$$

$$(1-\alpha) D^*(x, x, y) \leq 0$$

Thus  $1 - \alpha < 0$ . This implies  $\alpha > 1$  which is contradiction.

Therefore  $x = y$

Hence  $x$  is a unique common fixed point of the sequence of maps  $\{T_n\}$ .

**Theorem 4:** Let  $X$  be a complete  $D^*$  - metric space and  $T_n: X \rightarrow X$  be a sequence of maps such that

$$D^*(T_k T_j T_i x, T_j T_i x, T_i x) \leq \alpha D^*(T_j T_i x, T_i x, x) \text{ for all } i \neq j \neq k \text{ and for all } x \in X \text{ with } 0 \leq \alpha < 1.$$

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  be any fixed arbitrary element Define a sequence  $\{x_n\}$  in  $X$  as  $x_{n+1} = T_{n+1} x_n$  for all  $n = 0, 1, 2, \dots$

$$\text{Let } d_n = D^*(x_n, x_{n+1}, x_{n+2})$$

$$\begin{aligned} d_1 &= D^*(x_1, x_2, x_3) \\ &= D^*(T_1 x_0, T_2 T_1 x_0, T_3 T_2 T_1 x_0) \\ &\leq \alpha D^*(x_0, T_1 x_0, T_2 T_1 x_0) \\ &= \alpha D^*(x_0, x_1, x_2) \end{aligned}$$

$$d_1 \leq \alpha d_0$$

$$\begin{aligned} d_2 &= D^*(x_2, x_3, x_4) \\ &= D^*(T_2 x_1, T_3 T_2 x_1, T_4 T_3 T_2 x_1) \\ &\leq \alpha D^*(x_1, T_2 x_1, T_3 T_2 x_1) \\ &= \alpha D^*(x_1, x_2, x_3) \\ &= \alpha d_1. \end{aligned}$$

$$\leq \alpha^2 d_0$$

Continuing in this way we get  $d_n \leq \alpha^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$  (since  $0 \leq \alpha < 1$ ).

Now we shall prove that  $\{x_n\}$  is a Cauchy sequence in  $X$ .

$$\text{Let } d_n^* = D^*(x_n, x_n, x_{n+1})$$

$$\begin{aligned} \text{Then } d_{n+1}^* &= D^*(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \\ &\leq d_n + d_n^* \end{aligned}$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be  $d$ . Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\begin{aligned} \text{Now } d &= \lim_{n_x \rightarrow \infty} d_{n+2}^* \\ &\leq \frac{\lim_{n_x \rightarrow \infty}}{n_x} \{d_{n+1} + d_{n+1}^*\} \\ &\leq \frac{\lim_{n_x \rightarrow \infty}}{n_x} \{\alpha d_n + d_{n+1}^*\} \\ &< \frac{\lim_{n_x \rightarrow \infty}}{n_x} \{d_n + d_{n+1}^*\} \\ &= d. \text{ which is contraction. Thus } d = 0. \end{aligned}$$

Hence  $D^*(x_n, x_n, x_m) \rightarrow 0$  as  $m, n \rightarrow \infty$

Therefore  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$ .

Since  $X$  is  $D^*$  complete  $x_n \rightarrow x$  in  $X$

Now we prove that  $x$  is fixed point of  $T_n$

To prove that  $T_n x = x$  for all  $n$ .

Suppose There is an  $m$  such that  $T_m x \neq x$

$$\begin{aligned} \text{Then } D^*(T_m x, x, x) &= \frac{\lim_{n_x \rightarrow \infty}}{n_x} D^*(T_m x, x_{n+1}, x_{n+2}) \\ &= D^*(T_m x, T_{n+1} x_n, T_{n+2} x_{n+1}) \\ &\leq \alpha \frac{\lim_{n \rightarrow \infty}}{n} D^*(x, x_{n+1}, x_{n+2}) \\ &\leq \alpha D^*(x, x, x) \\ &= 0 \end{aligned}$$

Thus  $T_n x = x$  for all  $n$ .

Now we prove that  $x$  is a unique common fixed point of  $\{T_n\}$ .

Suppose  $x \neq y$  and  $T_n y = y$ .

$$\begin{aligned} \text{Then } D^*(x, y, y) &= D^*(T_i x, T_j y, T_k y) \\ &\leq \alpha D^*(x, y, y) \end{aligned}$$

This implies  $(1-\alpha)D^*(x, y, y) \leq 0$

Since  $x \neq y$  we have  $D^*(x, y, y) > 0$

This  $((1-\alpha) < 0$



This implies  $\alpha > 1$  which in contradiction Hence  $\{T_n\}$  have a unique common fixed point.

**Theorem 5:** Let X be a complete D\* - metric space and  $T_n : X \rightarrow X$  be a sequence of maps such that

$$D^*(T_i x, T_j y, T_k z) \leq a\{D^*(x, y, z) + D^*(x, T_i x, T_j y) + D^*(y, T_j y, T_k z)\} \text{ for all } x, y, z \in X \text{ with } i \neq j \neq k \text{ and } 0 \leq a < 1/3.$$

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in X as  $x_{n+1} = T_{n+1} x_n$  for  $n = 0, 1, 2, \dots$

$$\text{Let } d_n = D^*(x_n, x_{n+1}, x_{n+2}).$$

$$\text{Then } d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$$

$$\begin{aligned} &= D^*(T_{n+1} x_n, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\ &\leq a\{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\} \\ &= a\{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\ &= a\{2 D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \end{aligned}$$

$$d_{n+1} \leq 2a d_n + a d_{n+1}$$

$$d_{n+1} \leq \{2a/(1-a)\} d_n$$

$$d_{n+1} \leq b d_n, \text{ where } b = 2a/(1-a) < 1.$$

Hence  $d_n \leq b^n d_0 \rightarrow 0$ , as  $n \rightarrow \infty$ .

Now we shall prove that  $\{x_n\}$  is a Cauchy sequence in X.

$$\text{Let } d_n^* = D^*(x_n, x_n, x_{n+1})$$

$$\text{Then } d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\begin{aligned} &\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \\ &\leq d_n + d_n^* \end{aligned}$$

$$d_{n+1}^* - d_n^* \leq d_n \leq b^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq b < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d. Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\begin{aligned} \text{Now } d &= \lim_{n \rightarrow \infty} d_{n+2}^* \\ &\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\} \\ &\leq \lim_{n \rightarrow \infty} \{b d_n + d_{n+1}^*\} \\ &< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\} \end{aligned}$$

= d. which is contraction .Thus d = 0.

Hence  $D^*(x_n, x_n, x_m) \rightarrow 0$  as  $m, n \rightarrow \infty$

Therefore  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$ .

Since  $X$  is  $D^*$  complete  $x_n \rightarrow x$  in  $X$

Now we prove that  $x$  is fixed point of  $T_n$

To prove that  $T_n x = x$  for all  $n$

Suppose there is an  $m$  such that  $T_m x \neq x$ , Then

$$\begin{aligned} D^*(T_m x, x, x) &= \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+2}, x_{n+3}) \\ &= D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\ &\leq a \lim_{n \rightarrow \infty} \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x, T_m x, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\} \\ &\leq a \lim_{n \rightarrow \infty} \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x, T_m x, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\ &\leq a D^*(x, T_m x, x) \end{aligned}$$

$$(1-a) D^*(x, T_m x, x) \leq 0 \text{ Hence } (1-a) < 0$$

Therefore  $a > 1$ , which is contradiction to  $a < 1$ .

Thus  $T_n x = x$ . for all  $n$ .

Now we prove that  $x$  is a unique common fixed point of  $\{T_n\}$

Suppose  $x \neq y$  and  $T_n y = y$  for all  $n$ .

Then  $D^*(x, y, y) = D^*(T_1 x, T_2 y, T_3 y)$

$$\begin{aligned} &\leq a \{D(x, y, y) + D^*(x, T_1 x, T_2 y) + D^*(y, T_2 y, T_3 y)\} \\ &= a \{D(x, y, y) + D^*(x, x, y) + D^*(y, y, y)\} \\ &\leq 2a D^*(x, y, y) \\ &< D^*(x, y, y), \text{ which is contradiction..} \end{aligned}$$

Hence  $\{T_n\}$  have a unique common fixed .

**Theorem 6:** Let  $X$  be a complete  $D^*$  - metric space and  $T_n : X \rightarrow X$  be a sequence of maps such that

$$D^*(T_i x, T_j y, T_k z) \leq a_1 D^*(x, y, z) + a_2 \{D^*(x, T_i x, T_j y) + D^*(y, T_j y, T_k z)\} + a_3 \{D^*(x, y, T_j y) + D^*(y, z, T_k z)\}$$

for all  $x, y, z \in X$  with  $i \neq j \neq k$  and  $0 \leq a_1 + 2a_2 + 2a_3 < 1$ . Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in  $X$  as  $x_{n+1} = T_{n+1} x_n$  for  $n = 0, 1, 2, \dots$

Let  $d_n = D^*(x_n, x_{n+1}, x_{n+2})$ . Then

$$d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$$

$$\begin{aligned}
 &= D^*(T_{n+1} x_n, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\
 &\leq a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \{D^*(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\} \\
 &\quad + a_3 \{D^*(x_n, x_{n+1}, T_2 x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\} \\
 &= a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 &\quad + a_3 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 &\leq (a_1 + a_2 + a_3) D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3}) \\
 &\leq (a_1 + a_2 + a_3) d_n + (a_2 + a_3) d_{n+1}
 \end{aligned}$$

$$(1 - a_2 - a_3) d_{n+1} \leq (a_1 + a_2 + a_3) d_n$$

$$d_{n+1} \leq \{(a_1 + a_2 + a_3) / (1 - a_2 - a_3)\} d_n$$

$$d_{n+1} \leq a d_n, \text{ for all } n \text{ where } a = \{(a_1 + a_2 + a_3) / (1 - a_2 - a_3)\} < 1.$$

Hence  $d_n \leq a^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

Let  $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then  $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\begin{aligned}
 &\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \\
 &\leq d_n + d_n^*
 \end{aligned}$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be  $d$ .

Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\begin{aligned}
 \text{Now } d &= \lim_{n \rightarrow \infty} d_{n+2}^* \\
 &\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\} \\
 &\leq \lim_{n \rightarrow \infty} \{b d_n + d_{n+1}^*\} \\
 &< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\}
 \end{aligned}$$

=  $d$ . which is contraction. Thus  $d = 0$ .

For  $m > n$ , we have

$$\begin{aligned}
 D^*(x_n, x_n, x_m) &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m) \\
 &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m)
 \end{aligned}$$

$\rightarrow 0$  as  $n, m \rightarrow \infty$ . Hence  $D^*(x_n, x_n, x_m) \rightarrow 0$  as  $m, n \rightarrow \infty$

Therefore  $\{x_n\}$  is a D\* Cauchy sequence in X.

Since X is D\* complete  $x_n \rightarrow x$  in X

Now we prove that x is fixed point of  $\{T_n\}$

To prove that  $T_n x = x$  for all n.

Suppose there is an m such that  $T_m x \neq x$ , Then

$$\begin{aligned} D^*(T_m x, x, x) &= \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+2}, x_{n+3}) \\ &= \lim_{n \rightarrow \infty} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\ &\leq \lim_{n \rightarrow \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \{ D^*(x, T_m x, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \} \\ &\quad + a_3 \{ D^*(x, x_{n+1}, T_{n+2} x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2}) \} \} \\ &= \lim_{n \rightarrow \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \{ D^*(x, T_m x, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \\ &\quad + a_3 \{ D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \} \\ &\leq a_2 D^*(x, T_m x, x) \\ &< D^*(x, T_m x, x) \text{ , which is contradiction . Thus } T_n x = x \text{ for all n.} \end{aligned}$$

Now we prove that x is a unique common fixed point of  $\{T_n\}$

Suppose  $x \neq y$  and  $T_n y = y$  for all n.

$$\begin{aligned} \text{Then } D^*(x, y, y) &= D^*(T_{n+1} x, T_{n+2} y, T_{n+3} y) \\ &\leq a_1 D^*(x, y, y) + a_2 \{ D^*(x, T_{n+1} x, T_{n+2} y) + D^*(y, T_{n+2} y, T_{n+3} y) \} + a_3 \{ D^*(x, y, T_{n+2} y) + D^*(y, y, T_{n+3} y) \} \\ &= a_1 D^*(x, y, y) + a_2 \{ D^*(x, x, y) + D^*(y, y, y) \} + a_3 \{ D^*(x, y, y) + D^*(y, y, y) \} \\ &= (a_1 + a_2 + a_3) D^*(x, y, y) \\ &< D^*(x, y, y), \text{ which is contradiction.} \end{aligned}$$

Hence  $\{T_n\}$  have a unique common fixed

**Theorem7:** Let X be a complete D\* - metric space and  $T_n : X \rightarrow X$  be a sequence of maps such that  $D^*(T_i x, T_j y, T_k z) \leq a \max \{ D^*(x, y, z), \{ D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z), D^*(x, y, T_j y), D^*(y, z, T_k z) \} \}$  for all  $x, y, z \in X$ , with  $i \neq j \neq k$  and  $0 \leq a < 1$ .

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in X as  $x_{n+1} = T_1 x_n$  for  $n = 0, 1, 2, \dots$

$$\text{Let } d_n = D^*(x_n, x_{n+1}, x_{n+2}).$$

$$\begin{aligned} d_{n+1} &= D^*(x_{n+1}, x_{n+2}, x_{n+3}) \\ &= D^*(T_{n+1} x_n, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\ &\leq a \max \{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}), D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}), D^*(x_n, x_{n+1}, T_{2n+2} x_{n+1}), \end{aligned}$$

$$\begin{aligned}
 & D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2}) \} \\
 & = a \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3}), D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 & \leq a \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 & \leq a \max \{d_n, d_{n+1}\}
 \end{aligned}$$

$$d_{n+1} \leq a d_n \text{ for all } n$$

Hence  $d_n \leq a^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

$$\text{Let } d_n^* = D^*(x_n, x_n, x_{n+1})$$

$$\begin{aligned}
 \text{Then } d_{n+1}^* &= D^*(x_{n+1}, x_{n+1}, x_{n+2}) \\
 &\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \\
 &\leq d_n + d_n^*
 \end{aligned}$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be  $d$ .

Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\begin{aligned}
 \text{Now } d &= \lim_{n \rightarrow \infty} d_{n+2}^* \\
 &\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\} \\
 &\leq \lim_{n \rightarrow \infty} \{ad_n + d_{n+1}^*\} \\
 &< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\} \\
 &= d
 \end{aligned}$$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

For  $m > n$  we have,

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m) \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Thus  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$  and  $X$  is  $D^*$ - complete  $x_n \rightarrow x$  in  $X$ .

Now we shall prove that  $T_n x = x$  for all  $n$ . Suppose there is  $m$  such that  $T_m x \neq x$

$$D^*(T_m x, x, x) = \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+2}, x_{n+3})$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\
 &\leq a \lim_{n \rightarrow \infty} \max \{D^*(x, x_{n+1}, x_{n+2}), D^*(x, T_m x, T_{n+2} x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}), D^*(x, x_{n+1}, T_2 x_{n+1}), \\
 &\quad D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\} \\
 &= a \lim_{n \rightarrow \infty} \max \{D^*(x, x_{n+1}, x_{n+2}), D^*(x, T_1 x, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3}), D^*(x, x_{n+1}, x_{n+2}), \\
 &\quad D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 &\leq a \max \{D^*(x, T_1 x, x), 0\} \\
 &< D^*(T x, x, x),
 \end{aligned}$$

Which is a contradiction.

Thus  $T_1 x = x$ .

Similarly we can prove that  $T_2 x = T_3 x = x$ .

Now we prove that  $x$  is a unique common fixed point of  $T_1, T_2, T_3$

Suppose  $x \neq y$  and  $T_1 x = T_2 x = T_3 x = x$  &  $T_1 y = T_2 y = T_3 y = y$

Then  $D^*(x, y) = D^*(T_1 x, T_2 y, T_3 y)$

$$\begin{aligned}
 &\leq a \max \{D^*(x, y, y), \{D^*(x, T_1 x, T_2 y), D^*(y, T_2 y, T_3 y), D^*(x, y, T_2 y), D^*(y, y, T_3 y)\}\} \\
 &= a \max \{D^*(x, y, y), D^*(x, x, y), D^*(x, y, y), D^*(y, y, y)\} = a D^*(x, y, y) \\
 &< D^*(x, y, y),
 \end{aligned}$$

which is a contradiction.

Hence  $T_1, T_2$  &  $T_3$  have a unique common fixed point

**Theorem 8:** Let  $X$  be a complete  $D^*$  - metric space and  $T_n : X \rightarrow X$  be a sequence of maps such that

$$D^*(T_i x, T_j y, T_k z) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z)\} \text{ for all } x, y, z, \in X \text{ with } i \neq j \neq k \text{ and } 0 \leq a_1 + 2a_2 < 1.$$

Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in  $X$  as  $x_{n+1} = T_n x_n$  for  $n = 0, 1, 2, \dots$

Let  $d_n = D^*(x_n, x_{n+1}, x_{n+2})$ .

Then  $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$$\begin{aligned}
 &= D^*(T_{n+1} x_n, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\
 &\leq a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}), D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\} \\
 &= a_1 \{D^*(x_n, x_{n+1}, x_{n+2})\} + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 &= (a_1 + a_2) D^*(x_n, x_{n+1}, x_{n+2}) + a_2 D^*(x_{n+1}, x_{n+2}, x_{n+3})
 \end{aligned}$$

$$d_{n+1} \leq (a_1 + a_2) d_n + a_2 d_{n+1}$$

$$(1 - a_2) d_{n+1} \leq (a_1 + a_2) d_n$$

$$d_{n+1} \leq \{(a_1 + a_2)/(1 - a_2)\} d_n$$

$$d_{n+1} \leq b d_n \text{ where } b = \{(a_1 + a_2)/(1 - a_2)\} < 1.$$

Hence  $d_n \leq b^n d_0 \rightarrow 0$ , as  $n \rightarrow \infty$ .

Now we shall prove that  $\{x_n\}$  is a Cauchy sequence in X.

$$\text{Let } d_n^* = D^*(x_n, x_n, x_{n+1})$$

$$\text{Then } d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$$

$$\leq d_n + d_n^*$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d .

Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$  .

Now we shall prove that  $d = 0$  .Suppose  $d \neq 0$  .

$$\text{Now } d = \lim_{n \rightarrow \infty} d_{n+2}^*$$

$$\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\}$$

$$\leq \lim_{n \rightarrow \infty} \{b d_n + d_{n+1}^*\}$$

$$< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\}$$

$$= d \text{ .which is contradiction .Thus } d = 0.$$

Hence  $D^*(x_n, x_n, x_m) \rightarrow 0$  as  $m, n \rightarrow \infty$

Therefore  $\{x_n\}$  is a  $D^*$  Cauchy sequence in X.

Since X is  $D^*$  complete  $x_n \rightarrow x$  in X

Now we prove that x is fixed point of  $\{T_n\}$

To prove that  $T_n x = x$  for all n.

Suppose there is m such that  $T_m x \neq x$  .Then

$$D^*(T_m x, x, x) = \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+2}, x_{n+3})$$

$$= \lim_{n \rightarrow \infty} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_n)$$

$$\begin{aligned} &\leq \lim_{n \rightarrow \infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x, T_m x, T_{n+2} x_{n+1}), D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\}\} \\ &\leq \lim_{n \rightarrow \infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x, T_m x, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\} \\ &\leq a_2 D^*(x, T_m x, x) \\ &< D^*(x, T_m x, x), \text{ which is contradiction.} \end{aligned}$$

Thus  $T_n x = x$  for all  $n$ .

Now we prove that  $x$  is a unique common fixed point of  $\{T_n\}$ .

Suppose  $x \neq y$  and  $T_n y = y$  for all  $n$ .

$$\begin{aligned} \text{Then } D^*(x, y, y) &= D^*(T_{n+1} x, T_{n+2} y, T_{n+3} y) \\ &\leq a_1 \{D(x, y, y) + a_2 \max \{D^*(x, T_{n+1} x, T_{n+2} y), D^*(y, T_{n+2} y, T_{n+3} y)\}\} \\ &\leq a_1 \{D(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y)\}\} \\ &\leq (a_1 + a_2) D^*(x, y, y) \\ &< D^*(x, y, y), \text{ which is a contradiction.} \end{aligned}$$

Hence  $\{T_n\}$  have a unique common fixed point

**Remark 2.7:** If we put  $a_2 = 0$ ,  $T_n = T$  for all  $n$  and  $a_1 = a$  in the above theorem we get the following Theorem as corollary.

**Corollary 2:** Let  $(X, D^*)$  be a complete  $D^*$ - metric space and  $T: X \rightarrow X$  be a map such that

$$D^*(Tx, Ty, Tz) \leq a D^*(x, y, z) \text{ for all } x, y, z \in X \text{ and } 0 \leq a < 1.$$

Then  $T$  has a unique fixed point.

The above Theorem is know as Banach contraction Type Theorem in  $D^*$ - metric space.

**Remark 2.9:** If we put  $a_1 = 0$ ,  $T_n = T$  for all  $n$  and  $a_2 = a$  in the above theorem 1. we get the following theorem as corollary 2.10.

**Corollary 2.10:** Let  $(X, D^*)$  be a complete  $D^*$ - metric space and  $T: X \rightarrow X$  be a map such that

$$D^*(Tx, Ty, Tz) \leq a \max \{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\} \text{ for all } x, y, z \in X \text{ and } 0 \leq a < \frac{1}{2}.$$

Then  $T$  has a unique fixed point.

**Theorem 2.11:** Let  $X$  be a complete  $D^*$ - metric space and  $T_n : X \rightarrow X$  be a sequence of maps such that

$$D^*(T_i x, T_j y, T_k z) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z)\} + a_3 \max \{D^*(x, y, T_j y), D^*(y, z, T_k z)\}$$

for all  $x, y, z \in X$ , with  $i \neq j \neq k$  and  $0 \leq a_1 + 2a_2 + 2a_3 < 1$ . Then  $\{T_n\}$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in  $X$  as  $x_{n+1} = T_{n+1} x_n$  for  $n = 0, 1, 2, \dots$

$$\text{Let } d_n = D^*(x_n, x_{n+1}, x_{n+2}).$$

$$\text{Then } d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$$

$$= D^*(T_{n+1} x_n, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})$$



$$\begin{aligned} &\leq a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, T_{n+1}x_n, T_{n+2}x_{n+1}), D^*(x_{n+1}, T_{n+2}x_{n+1}, T_{n+3}x_{n+2})\} \\ &\quad + a_3 \max \{D^*(x_n, x_{n+1}, T_{n+2}x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_{n+3}x_{n+2})\} \\ &= a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} + a_3 \max \{D^*(x_n, x_{n+1}, x_{n+2}), \\ &\quad D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\ &\leq (a_1 + a_2 + a_3) D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3}) \\ &\leq (a_1 + a_2 + a_3) d_n + (a_2 + a_3) d_{n+1} \end{aligned}$$

$$(1 - a_2 - a_3) d_{n+1} \leq (a_1 + a_2 + a_3) d_n$$

$$d_{n+1} \leq \{(a_1 + a_2 + a_3) / (1 - a_2 - a_3)\} d_n$$

$$d_{n+1} \leq a d_n, \text{ for all } n \text{ where } a = \{(a_1 + a_2 + a_3) / (1 - a_2 - a_3)\} < 1.$$

Hence  $d_n \leq a^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

Let  $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then  $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$$

$$\leq d_n + d_n^*$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be  $d$ .

Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\text{Now } d = \lim_{n \rightarrow \infty} d_{n+2}^*$$

$$\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\}$$

$$\leq \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\}$$

$$< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\}$$

$$= d. \text{ which is contraction. Thus } d = 0.$$

For  $m > n$  we have

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$$

$$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m)$$

$\rightarrow 0$  as  $n, m \rightarrow \infty$ .

Hence  $D^*(x_n, x_n, x_m) \rightarrow 0$  as  $m, n \rightarrow \infty$

Therefore  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$ .

Since  $X$  is  $D^*$  complete  $x_n \rightarrow x$  in  $X$

Now we prove that  $x$  is fixed point of  $\{T_n\}$

To prove that  $T_n x = x$  for all  $n$ .

Suppose there is an  $m$  such that  $T_m x \neq x$ , Then

$$\begin{aligned} D^*(T_m x, x, x) &= \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+2}, x_{n+3}) \\ &= \lim_{n \rightarrow \infty} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\ &\leq \lim_{n \rightarrow \infty} \{a_1 D^*(T_m x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_1 x, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\} \\ &\quad + a_3 \max\{D^*(x, x_{n+1}, T_2 x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\} \\ &= \lim_{n \rightarrow \infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_1 x, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\ &\quad + a_3 \max\{D^*(x, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\ &\leq a_2 D^*(x, T_1 x, x) \\ &< D^*(x, T_1 x, x), \text{ which is contradiction. Thus } T_1 x = x. \end{aligned}$$

Similarly we can prove that  $T_2 x = T_3 x = x$ .

Now we prove that  $x$  is a unique common fixed point of  $T_1, T_2, T_3$

Suppose  $x \neq y$  and  $T_1 x = T_2 x = T_3 x = x$  &  $T_1 y = T_2 y = T_3 y = y$

Then  $D^*(x, y, y) = D^*(T_1 x, T_2 y, T_3 y)$

$$\begin{aligned} &\leq a_1 D^*(x, y, y) + a_2 \max\{D^*(x, T_1 x, T_2 y), D^*(y, T_2 y, T_3 y)\} + a_3 \max\{D^*(x, y, T_2 y), D^*(y, y, T_3 y)\} \\ &= a_1 D^*(x, y, y) + a_2 \max\{D^*(x, x, y), D^*(y, y, y)\} + \max a_3 \{D^*(x, y, y), D^*(y, y, y)\} \\ &= (a_1 + a_2 + a_3) D^*(x, y, y) \\ &< D^*(x, y, y), \text{ which is contradiction.} \end{aligned}$$

Hence  $\{T_n\}$  has a unique common fixed point.

**Theorem 2.12:** Let  $X$  be a complete  $D^*$ - metric space and  $T_1, T_2, T_3: X \rightarrow X$  be any three maps such that

$D^*(T_k x, T_j y, T_i z) \leq a \max\{D^*(x, y, z), 1/2\{D^*(x, T_i x, T_j y) + D^*(y, T_j y, T_k z)\}, 1/2\{D^*(x, y, T_j y) + D^*(y, z, T_k z)\}$  with  $i \neq j \neq k$ , for all  $x, y, z \in X$ , and  $0 \leq a < 1$ .

Then  $\{T_n\}$  has a unique common fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in  $X$  as  $x_{n+1} = T_{n+1} x_n$  for  $n = 0, 1, 2, \dots$

Let  $d_n = D^*(x_n, x_{n+1}, x_{n+2})$ .

$$\begin{aligned}
 \text{Then } d_{n+1} &= D^*(x_{n+1}, x_{n+2}, x_{n+3}) \\
 &= D^*(T_{n+1} x_n, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\
 &\leq a \max \{D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{D^*(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\}, \\
 &\quad 1/2 \{D^*(x_n, x_{n+1}, T_{n+2} x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2})\}\} \\
 &= a \max \{D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 &\quad 1/2 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\} \\
 &\leq a \max \{D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\} \\
 &\leq a \max \{d_n, 1/2 (d_n + d_{n+1})\}
 \end{aligned}$$

If  $\max \{d_n, 1/2 (d_n + d_{n+1})\} = 1/2 (d_n + d_{n+1})$  then  $d_{n+1} \leq d_n$  for all n. Thus  $d_{n+1} \leq d_n$  for all n.

Hence  $d_n \leq a^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in X.

Let  $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then  $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\begin{aligned}
 &\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1}) \\
 &\leq d_n + d_n^*
 \end{aligned}$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all n}$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d.

Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\begin{aligned}
 \text{Now } d &= \lim_{n \rightarrow \infty} d_{n+2}^* \\
 &\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\} \\
 &\leq \lim_{n \rightarrow \infty} \{ad_n + d_{n+1}^*\} \\
 &< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\} \\
 &= d
 \end{aligned}$$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in X.

For  $m > n$  we have,

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m) \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Thus  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$  and  $X$  is  $D^*$  - complete  $x_n \rightarrow x$  in  $X$ .

Now we shall prove that  $T_n x = x$  for all  $n$ . Suppose there is  $m$   $T_m x \neq x$

$$\begin{aligned} D^*(T_m x, x, x) &= \lim_{n \rightarrow \infty} D^*(T_m x, x_{n+2}, x_{n+3}) \\ &= \lim_{n \rightarrow \infty} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}) \\ &\leq a \lim_{n \rightarrow \infty} \max \{D^*(x, x_{n+1}, x_{n+2}), \frac{1}{2}\{D^*(x, T_m x, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\}, \\ &\quad \frac{1}{2}\{D^*(x, x_{n+1}, T_{n+2} x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2})\}\} \\ &= a \lim_{n \rightarrow \infty} \max \{D^*(x, x_{n+1}, x_{n+2}), \frac{1}{2}\{D^*(x, T_m x, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\ &\quad \frac{1}{2}\{D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\} \\ &\leq a \max \{D^*(x, T_m x, x), 0\} \\ &< D^*(T_m x, x, x), \text{ Which is a contradiction.} \end{aligned}$$

Thus  $T_n x = x$ . for all  $n$ .

Suppose  $x \neq y$  and  $T_n y = y$  for all  $n$ .

Then  $D^*(x, y, y) = D^*(T_n x, T_{n+2} y, T_{n+3} y)$

$$\begin{aligned} &\leq a \max \{D^*(x, y, y), \frac{1}{2}\{D^*(x, T_{n+1} x, T_{n+2} y) + D^*(y, T_{n+2} y, T_{n+3} y)\}, \\ &\quad \frac{1}{2}\{D^*(x, y, T_{n+2} y) + D^*(y, y, T_{n+3} y)\}\} \\ &= a \max \{D^*(x, y, y), \frac{1}{2} D^*(x, x, y), \frac{1}{2} D^*(x, y, y)\} \\ &= a D^*(x, y, y) \\ &< D^*(x, y, y), \end{aligned}$$

which is contradiction.

Hence  $\{T_n\}$  has a unique common fixed

**Theorem 2.13:** Let  $X$  be a complete  $D^*$  - metric space and  $T_n: X \rightarrow X$  be a sequence of map such that  $D^*(T_i x, T_j y, T_k z) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z)\} + a_3 \max \{D^*(x, y, T_j y), D^*(y, z, T_k z)\}$  for all  $x, y, z \in X$  and  $0 \leq a_1 + 2a_2 + 2a_3 < 1$ . Then  $T$  has a unique fixed point.

**Proof:** Let  $x_0 \in X$  a fixed arbitrary element and define a sequence  $\{x_n\}$  in  $X$  as

$$\begin{aligned} x_{n+1} &= T_1 x_n \\ x_{n+2} &= T_2 x_{n+1} \\ x_{n+3} &= T_3 x_{n+2} \text{ for } n = 3k, k = 0, 1, 2, \dots \end{aligned}$$

Let  $d_n = D^*(x_n, x_{n+1}, x_{n+2})$ .

Then  $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$$\begin{aligned} &= D^*(T_1 x_n, T_2 x_{n+1}, T_3 x_{n+2}) \\ &\leq a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, T_1 x_n, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}, \\ &\quad + a_3 \max \{D^*(x_n, x_{n+1}, T_2 x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\} \end{aligned}$$

$$= a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \\ + a_3 \max \{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \\ \leq \{ a_1 D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) \max \{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \}$$

$$d_{n+1} \leq a_1 d_n + (a_2 + a_3) (d_n + d_{n+1})$$

$$(1 - (a_2 + a_3)) d_{n+1} \leq (a_1 + a_2 + a_3) d_n$$

$$d_{n+1} \leq a d_n \text{ for all } n, \text{ where } a = (a_1 + a_2 + a_3) / (1 - (a_2 + a_3)) < 1$$

Hence  $d_n \leq a^n d_0 \rightarrow 0$  as  $n \rightarrow \infty$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

Let  $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then  $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$$

$$\leq d_n + d_n^*$$

$$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (since } 0 \leq \alpha < 1)$$

$$d_{n+1}^* \leq d_n^* \text{ for all } n$$

Hence  $\{d_n^*\}$  is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be  $d$ .

Then  $d_n^* \rightarrow d$  as  $n \rightarrow \infty$ .

Now we shall prove that  $d = 0$ . Suppose  $d \neq 0$ .

$$\text{Now } d = \lim_{n \rightarrow \infty} d_{n+2}^*$$

$$\leq \lim_{n \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\}$$

$$\leq \lim_{n \rightarrow \infty} \{ad_n + d_{n+1}^*\}$$

$$< \lim_{n \rightarrow \infty} \{d_n + d_{n+1}^*\}$$

$$= d$$

Now we prove that  $\{x_n\}$  is  $D^*$  - Cauchy sequence in  $X$ .

For  $m > n$  we have,

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m) \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Thus  $\{x_n\}$  is a  $D^*$  Cauchy sequence in  $X$  and  $X$  is  $D^*$  - complete  $x_n \rightarrow x$  in  $X$ .

Now we shall prove that  $T_1x = x$

$$\begin{aligned}
 D^*(T_1x, x, x) &= \lim_{n \rightarrow \infty} D^*(T_1x, x_{n+2}, x_{n+3}) \\
 &= \lim_{n \rightarrow \infty} D^*(T_1x, T_2 x_{n+1}, T_3 x_{n+2}) \\
 &\leq \lim_{n \rightarrow \infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_1x, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}, \\
 &\quad + a_3 \{D^*(x, x_{n+1}, T_2 x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}\} \\
 &= \lim_{n \rightarrow \infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_1x, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \\
 &\quad + a_3 \max\{D^*(x, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\} \\
 &\leq a_2 D^*(x, T_1x, x) \\
 &< D^*(T_1x, x, x), \text{ Which is a contradiction.}
 \end{aligned}$$

Thus  $T_1x = x$ .

Similarly we can prove that  $T_2x = T_3x = x$ .

Now we prove that  $x$  is a unique common fixed point of  $T_1, T_2, T_3$

Suppose  $x \neq y$  and  $T_1x = T_2x = T_3x = x$  &  $T_1y = T_2y = T_3y = y$

$$\begin{aligned}
 \text{Then } D^*(x, y, y) &= D^*(T_1x, T_2y, T_3y) \\
 &\leq \{a_1 D^*(x, y, y) + a_2 \max\{D^*(x, T_1x, T_2y), D^*(y, T_2y, T_3y)\} + a_3 \max\{D^*(x, y, T_2y), D^*(y, y, T_3y)\}\} \\
 &= a_1 D^*(x, y, y) + a_2 D^*(x, x, y) + a_3 D^*(x, y, y) \\
 &< (a_1 + a_2 + a_3) D^*(x, y, y), \\
 &< D^*(x, y, y),
 \end{aligned}$$

which is a contradiction.

Hence  $\{T_n\}$  has a unique common fixed point .

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