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SOME COMMON FIXED POINT THEOREMS FOR SEQUENCE OF MAPPINGS IN D*- METRIC SPACE

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ABSTRACT

In this paper we establish some common Fixed Point Theorems for sequence of contraction and generalized contraction mappings in D - metric space which is introduced by Shaban Sedghi, Nabi Shobe and Haiyun Zhou [10]. In what follows (X, D*) will denote D* - metric space, N, the set of all natural number and R^+, the set of all positive real number.*

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1. INTRODUCTION

There have been an number of generalization in generalized metric space (or D-Metric space) initated by Dhage [2] in 1992. He proved the existence of unique fixed point theorems of sequence of mappings satisfying certain contractive conditions in complete and bounded D- Metric space. Dealing with D- Metric space, Ahmad etal. [1], Dhage [2, 3, 4] Rhoades [8], Singh and Sharma [9] and others made a significant contribution in fixed point theory of D- Metric space. Unfortunately almost all theorems in D-Metric space are not valid (See S.V.R Naidu and others [5-7]). Here our aim is to prove some common fixed point theorems for sequences of generalized contractive mappings in D*- Metric space as a probable modification of the definition of D-Metric spaces introduced by Dhage [2].

Definition 1.1: Let X be a non empty set. A generalized metric (or D* - metric) on X is a function $D^*: X^3 \to [0,\infty)$ that satisfies the following conditions for each x, y, z, a $\in$ X.
1. $D^*(x, y, z) \geq 0$
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$
3. $D^*(x, y, z) = D^*(\rho\{x, y, z\})$ where ρ is permutation.
4. $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D*) is called generalized metric (or D* - metric) space.

Example 1.2:

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = R^n$ then we define

$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p}$ for every $p \in R^+$

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(d) If X = R then we define

$$D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \text{Max}\{x, y, z\}, & \text{otherwise,} \end{cases}$$

Remark 1.3: In D* - metric space $D^*(x, y, y) = D^*(x, x, y)$

Definition 1.4: A open ball in a D* - metric space X with centre x and radius r is denoted by $B_{D^*}(x, r)$ and is defined by $B_{D^*}(x, r) = \{y \in X: D^*(x, y, y) < r\}$

Example 1.5: Let X=R Denote $D^*(x, y, z) = |x-y| + |y-z| + |z-x|$ for all $x, y, z \in R$.

Thus $B_{D^*}(0, 1) = \{y \in R / D^*(0, y, y) < 1\}$

$= \{y \in R / |0 - y| + |y - y| + |y - 0| < 1\}$

$= \{y \in R / |y| + |y| < 1\}$

$= \{y \in R / |y| < ½ \}$

$= \{y \in R / - ½ < y < ½\}$

$= (-½, ½)$. (Open Interval)

Definition 1.6: Let (X, D*) be a D* - metric space and $A \subseteq X$
1. If for every $x \in A$, there exist r > o such that $B_{D^*}(x, r) \subseteq A$, then subset A is called open subset of X.

2. Subset A of X is said to be D* - bounded if there exist r > o such that

$$D^*(x, y, y) < r \text{ for all } x, y \in A.$$

3. A sequence $\{x_n\}$ in X converges to x if and only if

$$D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0 \text{ as } n \to \infty.$$

That is, for each $\varepsilon > 0$ there exist $n_0 \in N$ such that for all $n \geq n_0$ implies $D^*(x, x, x_n) < \varepsilon$ This is equivalent for each $\varepsilon > 0$, there exist $n_0 \in N$ such that for all n, $\geq n_0$ implies $D^*(x, x_n, x_n) < \varepsilon$.

It is also noted that $D^*(x_n, x_n, x) = D^*(x, x, x_n) < \varepsilon$ for all $n \geq n_0$, for some $n_0 \in N$.

4. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exist $n_0 \in N$ such that $D^*(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$ The D* - metric space (X, D*) is said to complete if every Cauchy sequence is convergent.

Remark 1.7:
(1) D* is continuous function on X^3
(2) If sequence $\{x_n\}$ in X converges to x, then x is unique.
(3) Any convergent sequence in (X, D*) is a Cauchy sequence.

Definition 1.8: A point x in X is a fixed point of the map $T: X \to X$ if Tx = x.

Definition 1.9: A point x in X is a common fixed point of a sequence of maps $T_n: X \to X$ if $T_n(x) = x$ for all n..
.

Theorem 1: Let X be a D* - complete metric space and $T_n : X \to X$ be a sequence maps such that

$$D^*(T_ix, T_jy, z) \leq \alpha D^*(x, y, z) \text{ for all } i \neq j \text{ and for all } x, y, z \in X \text{ with } o \leq \alpha < ½$$

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_o \in X$ be any fixed arbitrary element Define a sequence $\{x_n\}$ in X as.

$$x_{n+1} = T_{n+1} x_n \qquad \text{for all } n = 0, 1, 2, \ldots.$$

Let $d_n = D^*(x_n, x_{n+1}, x_{n+1})$ for all n = 0, 1, 2…

Now $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+2})$

$= D^*(T_{n+1}x_n, T_{n+2}x_{n+1}, x_{n+2})$

$\leq \alpha\, D^*(x_n, x_{n+1}, x_{n+2})$

$\leq \alpha\, D^*(x_n, x_{n+1}, x_{n+1}) + \alpha\, D^*(x_{n+1}, x_{1n+2}, x_{n+2})$

$= \alpha\, d_n + \alpha\, d_{n+1}$

$(1 - \alpha)\, d_{n+1} \leq \alpha\, d_n$

$$d_{n+1} \leq \frac{\alpha}{1-\alpha}\, d_n$$

$d_{n+1} \leq k\, d_n$ for all n =0,1,2…, where $k = \dfrac{\alpha}{1-\alpha} < 1$ (Since $\alpha < ½$)

$d_n \leq k\, d_{n-1}$

.

$\leq k^n d_0 \to 0$ as $n \to \infty$

Therefore $\lim\limits_{n\to\alpha} d_n = 0$ Thus $\lim\limits_{n\to\alpha} D^*(x_n, x_{n+1}, x_{n+1}) = 0$

Now we shall prove that $\{x_n\}$ is a D* - Cauchy sequence in X.

Let m > n >, n_0 for some $n_0 \in N$.

Now $D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$$\leq \sum_{k=\propto}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0 \text{ as } m, n \to \propto$$

Thus $\mathop{Lim}\limits_{n,m\to\alpha} D^*(x_n, x_n, x_m) = 0$

Therefore $\{x_n\}$ is D* - Cauchy sequence in X.

Since X is D* - Complete $x_n \to x$ in X. we prove that x is a fixed point of T_n for all n suppose there exist an m such that $x \neq T_m x$.

Then $D^*(T_m x, x, x) = \lim\limits_{n\to\propto} D^*(T_m x, x_{n+1}, x)$

$= \lim\limits_{n\to\propto} D^*(T_m x, T_{n+1}x_n, x)$

$\leq \alpha \lim\limits_{n\to\propto} D^*(x, x_{n+1}, x)$

$= 0.$

Therefore $D^*(T_m x, x, x) = 0$, Therefore $T_n x = x$ for all n .Thus x is common fixed point of $\{T_n\}$ for all n .

UNIQUENESS

Supper x#y such that $T_n y = y$ for all n.

Then $D^*(x, y, y) = D^*(T_i x, T_j y, y)$

$\leq \alpha\, D^*(x, y, y)$

This implies $(1 - \alpha)\, D^*(x, y, y) \leq 0$

Since $x \neq y$ we have $D^*(x, y, y) > 0$ her $(1 - \alpha) < 0$.

This implies $\alpha > 1$ which contraction to $\alpha < ½$.

Thus $\{T_n\}$ have a unique common fixed point.

Theorem 2: Let X be a complete D*- metric space and $T_n: X \rightarrow X$ be a sequence of maps such that

$$D^*(T_i x, T_j y, T_k z) \leq \alpha\, D^*(x, y, z) \text{ for all } i \neq j \neq k \text{ and } x, y, z \in X \text{ with } 0 \leq \alpha < 1.$$

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any fixed arbitrary element Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}\, x_n$ for all n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$

$d_1 = D^*(x_1, x_2, x_3)$

$= D^*(T_1 x_0, T_2 x_1, T_3 x_2$

$\leq \alpha\, D^*(x_0, x_1, x_2)$

$d_1 \leq \alpha\, d_0$

$d_2 = D^*(x_2, x_3, x_4)$

$= D^*(T_2\, x_1, T_3\, x_2, T_1\, x_3)$

$\leq \alpha\, D^*(x_1, x_2, x_3)$

$\leq \alpha\, d_1$.

$\leq \alpha^2 do$

Continuing in this way we get $d_n \leq \alpha^n d_0 \rightarrow 0$ as $n \rightarrow \infty$ (since $0 \leq \alpha < 1$).

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $d^*_n = D^*(x_n, x_n, x_{n+1})$

Then $d^*_{n+1} = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d^*_n$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \rightarrow 0$ as $n \rightarrow \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d. Then $d_n^* \to d$ as $n\to\infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

$$\text{Now } d = \lim_{n_x \to \propto} d_{n+2}^*$$

$$\leq \lim_{n_x \to \propto} \{d_{n+1} + d_{n+1}^*\}$$

$$\leq \lim_{n_x \to \propto} \{\alpha\, d_n + d_{n+1}^*\}$$

$$< \lim_{n_x \to \propto} \{d_n + d_{n+1}^*\}$$

= d. which is contraction .Thus d = 0.

Hence $D^*(x_n, x_n, x_m)\to 0$ as $m, n \to \propto$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_n

To prove that $T_nx = x$ for all n.

Suppose There is an m such that $T_mx \neq x$

$$\text{Then } D^*(T_mx, x, x) = \lim_{n_x \to \propto} D^*(T_mx, x_{n+1}, x_{n+2})$$

$$= D^*(T_mx, T_{n+1}x_n, T_{n+2}x_{n+1})$$

$$\leq \alpha \lim_{n \to \propto} D^*(x, x_{n+1}, x_{n+2})$$

$$\leq \alpha\ D^*(x, x, x)$$

$$= 0$$

Thus $T_nx = x$. for all n.

Now we prove that x is a unique common fixed point of $\{T_n\}$.

Suppose $x\neq y$ and $T_ny = y$.

Then $D^*(x,y,y) = D^*(T_ix, T_jy, T_ky)$

$$\leq \alpha\, D^*(x, y, y)$$

This impulse $(1-\alpha)D^*(x, y, y) \leq 0$

Since $x\neq y$ we have $D^*(x, y, y) >0$

This $((1-\alpha) <0$

This impulse $\alpha > 1$ which in contradiction Hence $\{T_n,\}$ have a unique common fixed point

Theorem 3: Let X be a D* - complete metric space and $T_n: X \rightarrow X$ be a sequence maps such that

$$D^*(T_jT_ix, T_ix, y) \leq \alpha D^*(T_ix, x, y) \text{ for all } i \neq j \text{ and for all } x, y, z \in X \text{ with } o \leq \alpha < 1 .$$

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any fixed arbitrary element. Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for n=0; 1; 2 …

$$\begin{aligned} D^*(x_{n+1}, x_n, x_n) &= D^*(T_{n+1} T_n x_{n-1}, T_n x_{n-1}, x_n) \\ &\leq \alpha D^*(T_n x_{n-1}, x_{n-1}, x_n) \\ &= \alpha D^*(x_n, x_{n-1}, x_n) \\ &= \alpha D^*(x_n, x_{n-1}, x_{n-1}) \\ &\;\;. \\ &\;\;. \\ &\leq \alpha^n D^*(x_1, x_0, x_0) \rightarrow 0 \text{ as } n\rightarrow\infty \text{ (since } 0 < \alpha < 1). \end{aligned}$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X. Let m> n> no for some no $\in$N.

$$\begin{aligned} D^*(x_n, x_n, x_m) &\leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}, x_{k+1}) \\ &< \alpha^n / (1-\alpha) D^*(x_1, x_0, x_0) \text{ as } m\rightarrow\infty \\ &\rightarrow 0 \text{ as } n\rightarrow \infty \end{aligned}$$

$D^*(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$

Therefore $\{x_n\}$ is D* Cauchy sequence in X Since X is D* complete $x_n \rightarrow x$ in X

Now we prove that $T_n x = x$ for all n.

Suppose there is an m such that $T_m x = y$ where $y \neq x$.

$$\begin{aligned} D^*(T_m x, x, x) &= \lim_{n\rightarrow\infty} D^*(y, x_{n+2}, x_{n+1}) \\ &= \lim_{n\rightarrow\infty} D^*(y, T_{n+2} T_{n+1} x_n, T_{n+1} x_n) \\ &\leq \alpha \lim_{n\rightarrow\infty} D^*(y, T_{n+1} x_n, x_n) \\ &= \alpha \lim_{n\rightarrow\infty} D^*(y, x_{n+1}, x_n) \\ &= \alpha D^*(y, x, x) \\ &= \alpha D^*(T_m x, x, x) \\ &< D^*(T_m x, x, x) \end{aligned}$$

Therefore $T_n x = x$ for all n..

Suppose $x \neq y$ Such that that $T_n y = y$ for all n.

Then $D^*(x, x, y) = D^*(T_{n+1} T_n x, T_n x, y)$

$$\leq \alpha D^*(T_n x, x, y)$$

$$\leq \alpha D^*(x, x, y)$$

$(1-\alpha) D^*(x, x, y) \leq 0$

Thus $1 - \alpha < 0$. This implies $\alpha > 1$ which is contradiction.

Therefore $x = y$

Hence x is a unique common fixed point of the sequence of maps $\{T_n\}$. .

Theorem 4: Let X be a complete D* - metric space and T_n: X → X be a sequence of maps such that

$$D^*(T_kT_jT_ix, T_jT_ix, Tix,) \leq \alpha D^*(T_jT_ix, T_ix, x) \text{ for all } i \neq j \neq k \text{ and for all } x \in X \text{ with } 0 \leq \alpha < 1.$$

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any fixed arbitrary element Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for all n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$

$d_1 = D^*(x_1, x_2, x_3)$

$= D^*(T_1x_0, T_2 T_1x_0, T_3 T_2 T_1x_0)$

$\leq \alpha D^*(x_0, T_1x_0, T_2 T_1x_0)$

$= \alpha D^*(x_0, x_1, x_2)$

$d_1 \leq \alpha d_0$

$d_2 = D^*(x_2, x_3, x_4)$

$= D^*(T_2x_1, T_3 T_2x_1, T_4 T_3 T_2x_1)$

$\leq \alpha D^*(x_1, T_2x_1, T_3 T_2x_1)$

$= \alpha D^*(x_1, x_2, x_3)$

$= \alpha d_1.$

$\leq \alpha^2 do$

Continuing in this way we get $d_n \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$).

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $d^*_n = D^*(x_n, x_n, x_{n+1})$

Then $d^*_{n+1} = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$$

$$\leq d_n + d^*_n$$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \to d$ as $n\to\infty$.

Now we shall prove that d = 0. Suppose $d \neq 0$.

$$\text{Now } d = \lim_{n_x \to \infty} d_{n+2}^*$$

$$\leq \lim_{n_x \to \infty} \{d_{n+1} + d_{n+1}^*\}$$

$$\leq \lim_{n_x \to \infty} \{\alpha\, d_n + d_{n+1}^*\}$$

$$< \lim_{n_x \to \infty} \{d_n + d_{n+1}^*\}$$

= d. which is contraction .Thus d = 0.

Hence $D^*(x_n, x_n, x_m)\to 0$ as $m, n \to \infty$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_n

To prove that $T_n x = x$ for all n.

Suppose There is an m such that $T_m x \neq x$

$$\text{Then } D^*(T_m x, x, x) = \lim_{n_x \to \infty} D^*(T_m x, x_{n+1}, x_{n+2})$$

$$= D^*(T_m x, T_{n+1}x_n, T_{n+2}x_{n+1})$$

$$\leq \alpha \lim_{n \to \infty} D^*(x, x_{n+1}, x_{n+2})$$

$$\leq \alpha\ D^*(x, x, x)$$

$$= 0$$

Thus $T_n x = x$. for all n.

Now we prove that x is a unique common fixed point of $\{T_n\}$.

Suppose $x\neq y$ and $T_n y = y$.

Then $D^*(x,y,y) = D^*(T_i x, T_j y, T_k y)$

$$\leq \alpha\, D^*(x, y, y)$$

This implies $(1-\alpha)D^*(x, y, y) \leq 0$

Since $x\neq y$ we have $D^*(x, y, y) > 0$

This $((1- \alpha) < 0$

This implies $\alpha > 1$ which in contradiction Hence $\{T_n,\}$ have a unique common fixed point.

Theorem 5: Let X be a complete D* - metric space and $T_n : X \rightarrow X$ be a sequence of maps such that

$D^*(T_ix, T_jy, T_kz) \le a\{D^*(x, y, z) + D^*(x, T_ix, T_jy) + D^*(y, T_jy, T_kz)\}$ for all $x, y, z \in X$ with $i \ne j \ne k$ and $0 \le a < 1/3$.

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for $n = 0, 1, 2, \ldots$.

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_{n+1} x_n, T_{n+2} x_{n+1} T_{n+3} x_{n+2})$

$\le a\{D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1} T_{n+3} x_{n+2})\}$

$= a\{D^*(x_n x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$= a\{2 D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$d_{n+1} \le 2a\, d_n + a\, d_{n+1}$

$d_{n+1} \le \{2a/(1-a)\}\, d_n$

$d_{n+1} \le b\, d_n$. where $b = 2a/(1-a) < 1$.

Hence $d_n \le b^n d_0 \rightarrow 0$, as $n \rightarrow \infty$.

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\le D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\le d_n + d_n^*$

$d_{n+1}^* - d_n^* \le d_n. \le b^n d_0 \rightarrow 0$ as $n \rightarrow \infty$ (since $0 \le b < 1$)

$d_{n+1}^* \le d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d .Then $d_n^* \rightarrow d$ as $n \rightarrow \infty$.

Now we shall prove that $d = 0$ .Suppose $d \ne 0$.

Now $d = \lim_{n_x \rightarrow \infty} d_{n+2}^*$

$\le \lim_{n_x \rightarrow \infty} \{d_{n+1} + d_{n+1}^*\}$

$\le \lim_{n_x \rightarrow \infty} \{b\, d_n + d_{n+1}^*\}$

$< \lim_{n_x \rightarrow \infty} \{d_n + d_{n+1}^*\}$

$= d$. which is contraction .Thus $d = 0$.

Hence $D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \propto$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \to x$ in X

Now we prove that x is fixed point of T_n

To prove that $T_n x = x$ for all n

Suppose there is an m such that $T_m x \neq x$, Then

$$D^*(T_m x, x, x) = \lim_{n_x \to \propto} D^*(T_m x, x_{n+2}, x_{n+3})$$

$$= D^*(T_m x, T_{n+2}x_{n+1}, T_{n+3}x_{n+2})$$

$$\leq a \lim_{n \to \propto} \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x ,T_m x_, T_{n+2}\, x_{n+1}) + D^*(x_{n+1} ,T_{n+2}\, x_{n+1}\, T_{n+3}\, x_{n+2}) \}$$

$$\leq a \lim_{n \to \propto} \{D^*(x, x_{n+1}, x_{n+2}) + D^*(x, T_m x ,, x_{n+2}) + D^*(x_{n+1},x_{n+2}, x_{n+3})\}$$

$$\leq a\, D^*(x, T_m x , x)$$

$(1-a)\, D^*(x, T_m x , x) \leq 0$ Hence $(1-a) < 0$

Therefore $a>1$, which is contradiction to $a < 1$.

Thus $T_n x = x$. for all n.

Now we prove that x is a unique common fixed point of $\{T_n\}$

Suppose $x \neq y$ and $T_n y = y$ for all n .

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3\, y)$

$$\leq a \{D(x, y, y) + D^*(x ,T_1x, T_2y) + D^*(y , T_2y, T_3\, y)$$

$$= a \{D(x, y, y) + D^*(x , x, y) + D^*(y , y, y)$$

$$\leq 2a\, D^*(x, y, y)$$

$$< D^* (x, y, y)$$, which is contradiction..

Hence $\{ T_n \}$have a unique common fixed .

Theorem 6: Let X be a complete D* **-** metric space and $T_n : X \to X$ be a sequence of maps such that

$$D^* (T_ix, T_jy, T_kz) \leq a_1 D^*(x, y, z) + a_2 \{D^*(x, T_i,x, T_jy) + D^* (y , T_jy, T_kz)\}+ a_3 \{D^*(x, y, T_jy) + D^* (y , z , T_kz)\}$$

for all $x, y, z \in X$ with $i \neq j \neq k$ and $0 \leq a_1 +2a_2 +2\, a_3 < 1$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as $x_{n+1}= T_{n+1}\, x_n$ for $n = 0, 1, 2, \ldots$.

Let $d_n= D^* (x_n, x_{n+1}, x_{n+2})$. Then

$d_{n+1} = D^* (x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_{n+1} x_n , T_{n+2} x_{n+1} T_{n+3} x_{n+2})$

$\leq a_1 D^*(x_n , x_{n+1}, x_{n+2}) + a_2 (\{D^*(x_n , T_{n+1}x_n , T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\}$
$+ a_3 \{D^*(x_n , x_{n-+1}, T_2 x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})\}$

$= a_1 D^*(x_n , x_{n+1}, x_{n+2}) + a_2 \{D^*(x_n , x_{n+1}, x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$
$+ a_3 \{D^*(x_n , x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq (a_1 + a_2 + a_3) D^*(x_n , x_{n+1}, x_{n+2}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$\leq (a_1 + a_2 + a_3) d_n + (a_2 + a_3) d_{n+1}$

$(1 - a_2 - a_3) d_{n+1} \leq (a_1+a_2+a_3) d_n$

$d_{n+1} \leq \{(a_1+a_2+a_3) / (1 - a_2 - a_3)\} d_n$

$d_{n+1} \leq a\, d_n$, for all n where $a = \{ (a_1+a_2+a_3) / (1-a_2 - a_3) \} < 1$.

Hence $d_n \leq a^n d_0 \to 0$ as $n \to \propto$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X .

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d.

Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that $d = 0$. Suppose $d \neq 0$.

Now $d = \dfrac{\lim}{n_x \to \propto} d_{n+2}^*$

$\leq \dfrac{\lim}{n_x \to \propto} \{d_{n+1} + d_{n+1}^*\}$

$\leq \dfrac{\lim}{n_x \to \propto} \{b\, d_n + d_{n+1}^*\}$

$< \dfrac{\lim}{n_x \to \propto} \{d_n + d_{n+1}^*\}$

= d. which is contraction. Thus d = 0.

For m > n, we have

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m)$

$\to 0$ as $n, m \to \infty$. Hence $D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \propto$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \to x$ in X

Now we prove that x is fixed point of $\{T_n\}$

To prove that $T_n x = x$ for all n.

Suppose there is an m such that $T_m x \neq x$, Then

$$D^*(T_m x, x, x) = \lim_{n\to\infty} D^*(T_m x, x_{n+2}, x_{n+3})$$

$$= \lim_{n\to\infty} D^*(T_m x, T_{n+2}x_{n+1}, T_{n+3}x_{n+2})$$

$$\leq \lim_{n\to\infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \{D^*(x, T_m x, T_{n+2} x_{n+1}) + D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\} + a_3 \{D^*(x, x_{n-+1}, T_{n+2} x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2})\}$$

$$= \lim_{n\to\infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \{D^*(x, T_m x, x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\} + a_3 \{ D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$$

$$\leq a_2\ D^*(x, T_m x, x)$$

$< D^*(x, T_m x, x)$,which is contradiction . Thus $T_n x = x$..for all n.

Now we prove that x is a unique common fixed point of $\{T_n\}$

Suppose $x \neq y$ and $T_n y = y$ for all n.

Then $D^*(x,y,y) = D^*(T_{n+1}x, T_{n+2}y, T_{n+3}y)$

$$\leq a_1 D^*(x, y, y) + a_2 \{D^*(x, T_{n+1}x, T_{n+2}y) + D^*(y, T_{n+2}y, T_{n+3}y)\} + a_3 \{D^*(x, y, T_{n+2}y) + D^*(y, y, T_{n+3}y)\}$$

$$= a_1 D^*(x, y, y) + a_2\{D^*(x, x, y) + D^*(y, y, y)\} + a_3\{D^*(x, y, y) + D^*(y, y, y)\}$$

$$= (a_1 + a_2 + a_3) D^*(x, y, y)$$

$< D^*(x, y, y)$, which is contradiction.

Hence $\{T_n\}$ have a unique common fixed

Theorem7: Let X be a complete D* **-** metric space and $T_n : X \to X$ be a sequence of maps such that $D^*(T_i x, T_j y, T_k z) \leq a \max\{ D^*(x, y, z), \{D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z) D^*(x, y, T_j y), D^*(y, z, T_k z)\}$ for all $x, y, z \in X$, with $i \neq j \neq k$ and $0 \leq a < 1$.

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as $x_{n+1} = T_1 x_n$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

$d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_{n+1} x_n, T_{n+2} x_{n+1} T_{n+3} x_{n+2})$

$\leq a \max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_n, T_{n+1}x_n, T_{n+2} x_{n+1}), D^*(x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2}), D^*(x_n, x_{n-+1}, T_{2n+2} x_{n+1}),$

$D^*(x_{n+1}, x_{n+2}, T_{n+3}\, x_{n+2})\}$

$= a \max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3})\ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq a \max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$\leq a \max\ \{d_n, d_{n+1}\}$

$d_{n+1} \leq a\, d_n$ for all n

Hence $d_n \leq a^n\ d_0 \to 0$ as $n \to \propto$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X .

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d.

Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \dfrac{\lim}{n_x \to \propto}\ d_{n+2}^*$

$\leq \dfrac{\lim}{n_x \to \propto}\ \{d_{n+1} + d_{n+1}^*\}$

$\leq \dfrac{\lim}{n_x \to \propto}\ \{a d_n + d_{n+1}^*\}$

$< \dfrac{\lim}{n_x \to \propto}\ \{d_n + d_{n+1}^*\}$

$= d$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

For m > n we have,

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + . \ . \ . + D^*(x_{m-1}, x_{m-1}, x_m) \to 0$ as $m.\ n \to \propto$

Thus $\{x_n\}$ is a D* Cauchy sequence in X and X is D*- complete $x_n \to x$ in X .

Now we shall prove that $T_n x = x$ for all n. Suppose there is m such that $T_m x \neq x$

$D^*(T_m x, x, x) = \lim\limits_{n\to\infty} D^*(T_m x, x_{n+2}, x_{n+3})$

$$= \lim_{n\to\infty} D^*(T_m x, T_{n+2}\, x_{n+1}, T_{n+3}\, x_{n+2})$$

$$\le a \lim_{n\to\infty} \max\{D^*(x, x_{n+1}, x_{n+2}), D^*(x, T_m x, T_{n+2}\, x_{n+1}), D^*(x_{n+1}, T_2\, x_{n+1}, T_3\, x_{n+2}), D^*(x, x_{n-+1}, T_2\, x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3\, x_{n+2})\}$$

$$= a \lim_{n\to\infty} \max\{D^*(x, x_{n+1}, x_{n+2}), D^*(x, T_1 x, x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3}), D^*(x, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$

$$\le a \max\{D^*(x, T_1 x, x), 0\}$$

$$< D^*(Tx, x, x),$$

Which is a contradiction.

Thus $T_1 x = x$..

Similarly we can prove that $T_2 x = T_3 x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1 x = T_2 x = T_3 x = x$ & $T_1 y = T_2 y = T_3 y = y$

Then $D^*(x,y,y) = D^*(T_1 x, T_2 y, T_3 y)$

$$\le a \max\{D^*(x, y, y), \{D^*(x, T_1 x, T_2 y), D^*(y, T_2 y, T_3 y), D^*(x, y, T_2 y), D^*(y, y, T_3 y)\}$$

$$= a \max\{D^*(x, y, y), D^*(x, x, y), D^*(x, y, y), D^*(y, y, y)\} = a\, D^*(x, y, y)$$

$$< D^*(x, y, y),$$

which is a contradiction.

Hence T_1, T_2 &T_3 have a unique common fixed point

Theorem 8: Let X be a complete D* - metric space and $T_n : X \to X$ be a sequence of maps such that

$D^*(T_i x, T_j y, T_k z) \le a_1 D^*(x, y, z) + a_2 \max\{D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z)\}$ for all $x, y, z, \in X$ with $i \neq j \neq k$ and $0 \le a_1 + 2a_2 < 1$.

Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as $x_{n+1} = T_n\, x_n$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$$= D^*(T_{n+1}\, x_n, T_{n+2}\, x_{n+1} T_{n+3}\, x_{n+2})$$

$$\le a_1 D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x_n, T_{n+1}\, x_n, T_{n+2}\, x_{n+1}), D^*(x_{n+1}, T_{n+2}\, x_{n+1} T_{n+3}\, x_{n+2})\}$$

$$= a_1 \{D^*(x_n\, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$

$$= (a_1 + a_2) D^*(x_n, x_{n+1}, x_{n+2}) + a_2 D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$

$$d_{n+1} \le \{(a_1 + a_2)d_n + a_2\, d_{n+1}$$

$$(1 - a_2)\}d_{n+1} \le \{(a_1 + a_2)d_n$$

$d_{n+1} \leq \{(a_1 + a_2)/(1 - a_2)\}\, d_n$

$d_{n+1} \leq b\, d_n$. where $b = \{(a_1 + a_2)/(1 - a_2)\} < 1$.

Hence $d_n \leq b^n d_0 \to 0$, as $n \to \infty$.

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d .

Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \lim_{n_x \to \infty} d_{n+2}^*$

$\leq \lim_{n_x \to \infty} \{d_{n+1} + d_{n+1}^*\}$

$\leq \lim_{n_x \to \infty} \{b\, d_n + d_{n+1}^*\}$

$< \lim_{n_x \to \infty} \{d_n + d_{n+1}^*\}$

= d .which is contradiction .Thus d = 0.

Hence $D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \to x$ in X

Now we prove that x is fixed point of $\{T_n$

To prove that $T_n x = x$ for all n.

Suppose there is m such that $T_m x \neq x$ .Then

$D^*(T_m x, x, x) = \lim_{n_x \to \infty} D^*(T_m x, x_{n+2}, x_{n+3})$

$= \lim_{n \to \infty} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_n$

$\leq \lim\limits_{n\to\infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x, T_m x, T_{n+2} x_{n+1}), D^*(x_{n+1}, T_{n+2} x_{n+1} T_{n+3} x_{n+2})\}$

$\leq \lim\limits_{n\to\infty} \{a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_m x, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq a_2 D^*(x, T_m x, x)$

$< D^*(x, T_m x, x)$,which is contradiction .

Thus $T_n x = x$. for all n..

Now we prove that x is a unique common fixed point of $\{T_n\}$.

Suppose $x \neq y$ and $T_n y = y$ for all n.

Then $D^*(x,y,y) = D^*(T_{n+1}x, T_{n+2}y, T_{n+3} y)$

$\leq a_1\{D(x, y, y) + a_2 \max \{D^*(x, T_{n+1}x, T_{n+2}y), D^*(y, T_{n+2}y, T_{n+3}y)\}$

$\leq a_1 \{D(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y)\}$

$\leq (a_1 + a_2) D^*(x, y, y)$

$< D^*(x, y, y)$, which is a contradiction.

Hence $\{T_n,\}$ have a unique common fixed point

Remark 2.7: If we put $a_2 = 0$, $T_n = T$ for all n and $a_1 = a$ in the above theorem we get the following Theorem as corollary.

Corollary 2: Let (X, D^*) be a complete D^* - metric space and $T: X \to X$ be a map such that

$$D^*(Tx, Ty, Tz) \leq a D^*(x, y, z) \text{ for all } x, y, z \in X \text{ and } 0 \leq a < 1.$$

Then T has a unique fixed point.

The above Theorem is know as Banach contraction Type Theorem in D* - metric space.

Remark 2.9: If we put $a_1 = 0$, $T_n = T$ for all n and $a_2 = a$ in the above theorem 1. we get the following theorem as corollary 2.10.

Corollary 2.10: Let (X, D^*) be a complete D^* - metric space and $T: X \to X$ be a map such that

$$D^*(Tx, Ty, Tz) \leq a \max \{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\} \text{ for all } x, y, z \in X \text{ and } 0 \leq a < \frac{1}{2}.$$

Then T has a unique fixed point.

Theorem 2.11: Let X be a complete D^* - metric space and $T_n : X \to X$ be a sequence of maps such that

$D^*(T_i x, T_j y, T_k z) \leq a_1 D^*(x, y, z) + a_2 \max\{D^*(x, T_i x, T_j y), D^*(y, T_j y, T_k z)\} + a_3 \max \{D^*(x, y, T_j y), D^*(y, z, T_k z)\}$ for all $x, y, z \in X$, with $i \neq j \neq k$ and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$= D^*(T_{n+1} x_n, T_{n+2} x_{n+1} T_{n+3} x_{n+2})$

$\leq a_1 D^*(x_n , x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n , T_{n+1}x_n , T_{n+2} x_{n+1}) , D^* (x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\}$
$+ a_3 \max\{D^*(x_n , x_{n-+1}, T_{n+2} x_{n+1}) , D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2})\}$

$= a_1 D^*(x_n , x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x_n , x_{n+1}, x_{n+2}) , D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}+ a_3 \max\{ D^*(x_n , x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq (a_1 + a_2 + a_3) D^*(x_n , x_{n+1}, x_{n+2}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3})$

$\leq (a_1 + a_2 + a_3) d_n + (a_2 + a_3) d_{n+1}$

$(1 - a_2 - a_3) d_{n+1} \leq (a_1+a_2+a_3) d_n$

$d_{n+1} \leq \{(a_1+a_2+a_3) / (1 - a_2 - a_3)\} d_n$

$d_{n+1} \leq a\ d_n$, for all n where $a = \{(a_1+a_2+a_3) / (1-a_2 - a_3)\} < 1$.

Hence $d_n \leq a^n\ d_0 \to 0$ as $n \to \propto$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

Let $d_n^* = D^* (x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^* (x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^* (x_n, x_{n+1}, x_{n+2}) +D^* (x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n.\leq \alpha^n d_0 \to 0$ as $n\to\infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d.

Then $d_n^* \to d$ as $n\to\infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \dfrac{\lim}{n_x \to \propto} d_{n+2}^*$

$\leq \dfrac{\lim}{n_x \to \propto} \{d_{n+1} + d_{n+1}^*\}$

$\leq \dfrac{\lim}{n_x \to \propto} \{b\ d_n + d_{n+1}^*\}$

$< \dfrac{\lim}{n_x \to \propto} \{d_n + d_{n+1}^*\}$

= d. which is contraction .Thus d = 0.

For m > n we have

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) +. . . + D^*(x_{m-1}, x_{m-1}, x_m)$

$\rightarrow 0$ as n, m $\rightarrow\infty$.

Hence $D^*(x_n, x_n, x_m) \rightarrow 0$ as m, n $\rightarrow \propto$

Therefore $\{x_n\}$ is a D* Cauchy sequence in X.

Since X is D* complete $x_n \rightarrow x$ in X

Now we prove that x is fixed point of $\{T_n\}$

To prove that $T_n x = x$ for all n .

Suppose there is an m such that $T_m x \neq x$, Then

$$D^*(T_m x, x, x) = \frac{\lim}{n \rightarrow \propto} D^*(T_m x, x_{n+2}, x_{n+3})$$

$$= \frac{\lim}{n \rightarrow \propto} D^*(T_m x, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})$$

$$\leq \frac{\lim}{n \rightarrow \propto} \{a_1 D^*(T_m x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_1 x, T_2 x_{n+1}), D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2})\}$$
$$+ a_3 \max\{D^*(x, x_{n-+1}, T_2 x_{n+1}), D^*(x_{n+1}, x_{n+2}, T_3 x_{n+2})$$

$$= \frac{\lim}{n \rightarrow \propto} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x, T_1 x, x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$$
$$+ a_3 \max\{ D^*(x, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$$

$$\leq a_2 D^*(x, T_1 x, x)$$

$< D^*(x, T_1 x, x)$, which is contradiction . Thus $T_1 x = x$.

Similarly we can prove that $T_2 x = T_3 x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x \neq y$ and $T_1 x = T_2 x = T_3 x = x$ & $T_1 y = T_2 y = T_3 y = y$

Then $D^*(x,y,y) = D^*(T_1 x, T_2 y, T_3 y)$

$$\leq a_1 D^*(x, y, y) + a_2 \max\{D^*(x, T_1 x, T_2 y), D^*(y, T_2 y, T_3 y)\} + a_3 \max\{D^*(x, y, T_2 y), D^*(y, y, T_3 y)\}$$

$$= a_1 D^*(x, y, y) + a_2 \max\{D^*(x, x, y), D^*(y, y, y)\} + \max a_3\{D^*(x, y, y), D^*(y, y, y)\}$$

$$= (a_1 + a_2 + a_3) D^*(x, y, y)$$

$< D^*(x, y, y)$, which is contradiction.

Hence $\{T_n\}$ has a unique common fixed point.

Theorem 2.12: Let X be a complete D*- metric space and T_1, T_2, T_3: X → X be any three maps such that

$D^*(T_k x, T_j y, T_i z) \leq a \max\{D^*(x, y, z), 1/2\{D^*(x, T_i x, T_j y) + D^*(y, T_j y, T_k z)\}, 1/2\{D^*(x, y, T_j y) + D^*(y, z, T_k z)\}$ with $i \neq j \neq k$, for all x, y, z $\in$ X, and $0 \leq a < 1$.

Then $\{T_n\}$ has a unique common fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for n = 0, 1, 2, ….

Let $d_n = D^*(x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^* (x_{n+1}, x_{n+2}, x_{n+3})$

$= D^* (T_{n+1} x_n , T_{n+2} x_{n+1} T_{n+3} x_{n+2})$

$\leq a \max \{D^*(x_n , x_{n+1}, x_{n+2}), 1/2 \{D^*(x_n , T_{n+1}x_n , T_{n+2} x_{n+1}) + D^* (x_{n+1}, T_{n+2} x_{n+1}, T_{n+3} x_{n+2})\},$
$1/2 \{D^*(x_n , x_{n-+1}, T_{n+2} x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_{n+3} x_{n+2})\}\}$

$= a \max \{D^*(x_n , x_{n+1}, x_{n+2}) , 1/2 \{D^*(x_n , x_{n+1}, x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$
$1/2 \{D^*(x_n , x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \}$

$\leq a \max \{D^*(x_n , x_{n+1}, x_{n+2}) , 1/2 \{D^*(x_n , x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$\leq a \max \{d_n , 1/2 (d_n + d_{n+1}) \}$

If $\max \{d_n, 1/2 (d_n + d_{n+1})\} = ½ (d_n + d_{n+1})$ then $d_{n+1} \leq d_n$ for all n. Thus $d_{n+1} \leq d_n$ for all n.

Hence $d_n. \leq a^n d_0 \to 0$ as $n \to \propto$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

Let $d_n^* = D^* (x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^* (x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^* (x_n, x_{n+1}, x_{n+2}) + D^* (x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \to 0$ as $n \to \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d.

Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \dfrac{\lim}{n_x \to \propto} d_{n+2}^*$

$\leq \dfrac{\lim}{n_x \to \propto} \{ d_{n+1} + d_{n+1}^*\}$

$\leq \dfrac{\lim}{n_x \to \propto} \{a d_n + d_{n+1}^*\}$

$< \dfrac{\lim}{n_x \to \propto} \{d_n + d_{n+1}^*\}$

$= d$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

For m > n we have,

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \; . \; . \; . + D^*(x_{m-1}, x_{m-1}, x_m) \to 0$ as m. $n \to \propto$

Thus $\{x_n\}$ is a D* Cauchy sequence in X and X is D* - complete $x_n \to x$ in X.

Now we shall prove that $T_nx = x$ for all n. Supose there is m $T_mx \neq x$

$$D^*(T_mx, x, x) = \lim_{n\to\infty} D^*(T_mx, x_{n+2}, x_{n+3})$$

$$= \lim_{n\to\infty} D^*(T_mx, T_{n+2}\, x_{n+1}, T_{n+3}\, x_{n+2})$$

$$\leq a \lim_{n\to\infty} \max \{D^*(x , x_{n+1}, x_{n+2}), ½\{D^*(x , T_mx ,T_{n+2}\, x_{n+1}) + D^* (x_{n+1}, T_{n+2}\, x_{n+1}, T_{n+3}\, x_{n+2})\},$$
$$½\{D^*(x , x_{n-+1}, T_{n+2}\, x_{n+1}) + D^*(x_{n+1}, x_{n+2}, T_{n+3}\, x_{n+2})\}\}$$

$$= a \lim_{n\to\infty} \max D^*(x , x_{n+1}, x_{n+2}), ½\{ D^*(x , T_mx , x_{n+2}) + D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$$
$$½\{D^*(x, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$$

$$\leq a \max \{D^*(x , T_mx , x) ,0\}$$

$< D^*(T_mx , x, x)$, Which is a contradiction.

Thus $T_nx = x$. for all n.

Suppose $x\neq y$ and $T_ny = y$ for all n.

Then $D^*(x,y,y) = D^*(T_nx, T_{n+2}y, T_{n+3}\, y)$

$$\leq a \max \{ D^*(x, y, y) , ½\{D^*(x, T_{n+1}x, T_{n+2}y)+D^*(y, T_{n+2}y, T_{n+3}y)\},$$
$$1/2\{D^*(x, y, T_{n+2}y) + D^* (y, y, T_{n+3}y)\}\}$$

$$= a \max \{D^*(x, y, y) , 1/2\, D^*(x, x, y), 1/2D^*(x, y, y)\}$$

$$= a\, D^* (x, y, y)$$

$$< D^* (x, y, y),$$

which is contradiction.

Hence $\{T_n\}$ has a unique common fixed

Theorem 2.13: Let X be a complete D* - metric space and T_n: X → X be a sequence of map such that
$D^*(T_ix, T_jy, T_kz) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, T_ix , T_jy), D^*(y, T_jy, T_kz)\}+ a_3 \max\{D^*(x, y, T_jy), D^*(y, z, T_kz)\}$
for all x, y, z $\in$ X and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ a fixed arbitrary element and define a sequence $\{x_n\}$ in X as

$x_{n+1}= T_1\, x_n$

$x_{n+2}= T_2\, x_{n+1}$

$x_{n+3}= T_3\, x_{n+2}$ for n = 3k , k = 0, 1, 2, ….

Let $d_n= D^* (x_n, x_{n+1}, x_{n+2})$.

Then $d_{n+1} = D^* (x_{n+1}, x_{n+2}, x_{n+3})$

$$= D^* (T_1\, x_n ,T_2\, x_{n+1} T_3\, x_{n+2})$$

$$\leq a_1 D^*(x_n , x_{n+1}, x_{n+2}) + a_2 \max\{ D^*(x_n , T_1x_n , T_2\, x_{n+1}) , D^* (x_{n+1}, T_2\, x_{n+1}, T_3\, x_{n+2})\},$$
$$+ a_3\max \{ D^*(x_n , x_{n-+1}, T_2\, x_{n+1}) , D^*(x_{n+1}, x_{n+2}, T_3\, x_{n+2})\}\}$$

$= \; a_1 D^*(x_n, x_{n+1}, x_{n+2}) + \; a_2 \max\{ D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$
$+ a_3 \max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$

$\leq \{a_1 D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3)\max\{D^*(x_n, x_{n+1}, x_{n+2}), D^*(x_{n+1}, x_{n+2}, x_{n+3})\}$

$d_{n+1} \leq a_1 d_n + (a_2 + a_3)(d_n + d_{n+1})$

$(1 - (a_2 + a_3))\, d_{n+1} \leq (a_1 + a_2 + a_3)\, d_n$

$d_{n+1} \leq a\, d_n$ for all n ,where $a = (a_1 + a_2 + a_3) / (1 - (a_2 + a_3)) < 1$

Hence $d_n. \leq a^n d_0 \rightarrow 0$ as $n \rightarrow \propto$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

Let $d_n^* = D^*(x_n, x_n, x_{n+1})$

Then $d_{n+1}^* = D^*(x_{n+1}, x_{n+1}, x_{n+2})$

$\leq D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_n, x_{n+1}, x_{n+1})$

$\leq d_n + d_n^*$

$d_{n+1}^* - d_n^* \leq d_n. \leq \alpha^n d_0 \rightarrow 0$ as $n \rightarrow \infty$ (since $0 \leq \alpha < 1$)

$d_{n+1}^* \leq d_n^*$ for all n

Hence $\{d_n^*\}$ is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be d.

Then $d_n^* \rightarrow d$ as $n \rightarrow \infty$.

Now we shall prove that d = 0 .Suppose $d \neq 0$.

Now $d = \dfrac{\lim}{n_x \rightarrow \propto} d_{n+2}^*$

$\leq \dfrac{\lim}{n_x \rightarrow \propto} \{d_{n+1} + d_{n+1}^*\}$

$\leq \dfrac{\lim}{n_x \rightarrow \propto} \{a d_n + d_{n+1}^*\}$

$< \dfrac{\lim}{n_x \rightarrow \propto} \{d_n + d_{n+1}^*\}$

$= d$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.

For m > n we have,

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \; . \; . \; . + D^*(x_{m-1}, x_{m-1}, x_m) \rightarrow 0$ as m. $n \rightarrow \propto$

Thus $\{x_n\}$ is a D* Cauchy sequence in X and X is D* - complete $x_n \rightarrow x$ in X.

Now we shall prove that $T_1x = x$

$$D^*(T_1x, x, x) = \lim_{n\to\infty} D^*(T_1x,\ x_{n+2}, x_{n+3})$$

$$= \lim_{n\to\infty} D^*(T_1x, T_2\, x_{n+1}, T_3\, x_{n+2})$$

$$\le \lim_{n\to\infty} \{a_1\ D^*(x\,, x_{n+1}, x_{n+2}) + a_2 \max\{D^*(x\,, T_1x\ , T_2\, x_{n+1})\,, D^*(x_{n+1}, T_2\, x_{n+1,}\, T_3\, x_{n+2})\},$$
$$+ a_3 \{D^*(x\,, x_{n-+1}, T_2\, x_{n+1})\,, D^*(x_{n+1}, x_{n+2},\ T_3\, x_{n+2})\}\}$$

$$= \lim_{n\to\infty} \{a_1 D^*(x_, x_{n+1}, x_{n+2}) + a_2 \max\{ D^*(x\,, T_1x\ , x_{n+2}), D^*(x_{n-+1}, x_{n+2}, x_{n+3})\}$$
$$+ a_3 \max \{D^*(x\,, x_{n+1}, x_{n+2})\ , D^*(x_{n+1}, x_{n+2}, x_{n+3})\}\}$$

$$\le a_2\ \ D^*(x\,, T_1x\ , x)$$

$$< D^*(T_1x, x, x), \text{ Which is a contradiction.}$$

Thus $T_1x = x$..

Similarly we can prove that $T_2x = T_3x = x$.

Now we prove that x is a unique common fixed point of T_1, T_2, T_3

Suppose $x\neq y$ and $T_1x=T_2x=T_3x=x$ & $T_1y=T_2y=T_3y= y$

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3\,y)$

$$\le \{a_1 D^*(x, y, y) + a_2 \max\{D^*(x, T_1x, T_2y), D^*(y, T_2y, T_3y),+ a_3 \max\{\ D^*(x, y, T_2y), D^*(y, y,\ T_3y)\}\}$$

$$= a_1 D^*(x, y, y) + a_2\ D^*(x, x, y) + a_3 D^*(x, y, y)\}$$

$$< (a_1 + a_2 + a_3)\, D^*(x, y, y),$$

$$< D^*(x, y, y),$$

which is a contradiction.

Hence $\{T_n\}$ has a unique common fixed point .

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