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# 2 - EQUITABLE DOMINATION IN GRAPHS

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***ABSTRACT***

*Let $G = (V, E)$ be a graph. A subset $D$ of $V(G)$ is called an equitable dominating set of a graph $G$ if for every $v \in (V - D)$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. An equitable dominating set $D$ is said to be a connected equitable dominating set if the subgraph $\langle D \rangle$ induced by $D$ is connected. In this paper we introduce the 2- equitable domination and 2-connected equitable domination in a graph, bounds and exact values for some standard graphs are found.*

***Keywords**:equitable domination number, 2-equitable dominating set, 2-connected equitable dominating.*

***Mathematics Subject Classification:** 05C69.*

## 1. INTRODUCTION

Introduction: By a graph $G = (V, E)$ we mean a finite, undirected with neither loops nor multiple edges the order and size of $G$ are denoted by $p$ and $q$ respectively for graph theoretic terminology we refer to Chartrand and Lesnaik [1] A subset $S$ of $V$ is called a dominating set if $N[S] = V$ the minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)$, $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes etal [4] A survey of several advanced topics in domination is given in the book edited by Haynes et al. [5]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [4]. Sampathkumar and Walikar [7] introduced the concept of connected domination in graphs. Let $G = (V, E)$ be a graph and let $v \in V$ the open neighborhood and the closed neighborhood of $v$ are denoted by $N(v)$ and $N[v] = N(v) \cup v$ respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of $u$ with respect to $S$ is defined by $Pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

A dominating set $S$ of $G$ is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected the minimum cardinality of a connected dominating set of $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$. A dominating set $S$ of a connected graph $G$ is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a ncd-set of $G$ is called the neighborhood connected domination number of $G$ and is denoted by $\gamma_{nc}(G)$. A ncd-set $S$ is said to be minimal if no proper subset of $S$ is a ncd-set. A coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that no two adjacent vertices receive the same color. The minimum integer $K$ for which a graph $G$ is $k-$colorable is called the chromatic number of $G$ and is denoted by $\chi(G)$.

A subset $S$ of $V$ is called an equitable dominating set if for every $v \in V - S$ there exist a vertex $u \in S$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such an equitable dominating set is denoted by $\gamma_e$ and is called the equitable domination number of $G$. A vertex $u \in V$ is said to be degree equitable with a vertex

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$v \in V$ if $|d(u)-d(v)| \le 1$. If $S$ is an equitable dominating set then any super set of $S$ is an equitable dominating set. An equitable set $S$ is said to be a minimal equitable dominating set if no proper subset of $S$ is an equitable dominating set. The minimal upper equitable dominating number is $\Gamma_e$ the upper equitable dominating set of G. If $u \in V$ such that $|d(u)-d(v)| \ge 2$ for every $v \in N(u)$ then $u$ is in every equitable dominating set such points are called an equitable isolated. $I_e$ denotes the set of all equitable isolates. An equitable dominating $S$ of connected graph $G$ is called an equitable connected dominating set (ecd-set) if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a ecd-set of $G$ is called the equitable connected domination number of $G$ and is denoted by $\gamma_{ec}(G)$. Let $G = (V, E)$ be a graph and let $u \in V$ the equitable neighborhood of $u$ denoted by $N_e(u)$ is defined as $N_e(u) = \{v \in V : |v \in N(u), |d(u)-d(v)| \le 1$ . The maximum and minimum equitable degree of a point in $G$ are denoted by $\Delta_e(G)$ and $\delta_e(G)$ that is $\Delta_e(G) = max_{u \in V(G)} |N_e(u)|$ and $\delta_e(G) = min_{u \in V(G)} |N_e(u)|$. The open equitable neighbourhood and closed equitable neighbourhood of $v$ are denoted by $N_e(v)$ and $N_e[v] = N_e(v) \cup \{v\}$ respectively. If $S \subseteq V$ then $N_e(S) = \cup_{v \in S} N_e(v)$ and $N[S] = N_e(S) \cup S$ . If $S \subseteq V$ and $u \in S$ then the private equitable neighbor set of $u$ with respect to $S$ is defined by $pne[u, S] = N_e[u] - N_e[S - \{u\}]$.

If $G$ is connected graph, then a vertex cut of $G$ is a subset $R$ of $V(G)$ with the property that the subgraph of $G$ induced by $V(G) - R$ is disconnected. If $G$ is not a complete Graph, then the vertex connectivity number $k(G)$ is the minimum cardinality of a vertex cut. If $G$ is complete graph $K_p$ it is known that $k(G) = p-1$

## 2. 2-EQUITABLE DOMINATION IN A GRAPH

**Definition.** Let $G = (V, E)$ be a graph. An equitable dominating set $S$ of a graph $G$ is called $2$-equitable dominating set ($2$-ed-set) if for any vertex $v$ in $G$ either $v \in S$ or $v$ is equitable dominated by at least $2$ vertices in $S$ . The minimum cardinality of a an $2$-equitable dominating set of $G$ is called the $2$-equitable domination number of $G$ and is denoted by $\gamma_{\times 2e}(G)$ .

**Proposition 2.1.** *The 2-equitable domination number of some standard graphs are*

(1) $\gamma_{\times 2e}(K_p) = 2$

(2) $\gamma_{\times 2e}(P_p) = \lfloor \frac{p+2}{2} \rfloor$

(3) $\gamma_{\times 2e}(C_p) = \lfloor \frac{p+1}{2} \rfloor$

(4) $\gamma_{\times 2e}(W_p) = \begin{cases} 1 + \lfloor \frac{p}{2} \rfloor, & if \quad p \ge 5; \\ 2, & otherwise. \end{cases}$

(5) $\gamma_{\times 2e}(K_{r,t}) = \begin{cases} r+t, & if \ |r-t| \ge 2; \\ 4, & otherwise. \end{cases}$

**Proposition 2.2.** *For any graph* $G$ ,

(1) $\gamma(G) \le \gamma_e(G) \le \gamma_{\times 2e}(G)$.

 

(2) $\gamma_{\times 2}(G) \le \gamma_{\times 2e}(G)$.

**Proof.** From the definition of the 2-equitable dominating set of a graph $G$, it is clearly that for any graph $G$ any 2-equitable dominating set $D$ is also an equitable dominating set and every equitable dominating set is also dominating set. Hence $\gamma(G) \le \gamma_e(G) \le \gamma_{\times 2e}(G)$. Similarly (2) since every 2-equitable dominating set is 2-dominating set for any graph $G$. Hence $\gamma_{\times 2}(G) \le \gamma_{\times 2e}(G)$.

**Observation 2.3.** *In any graph $G$, if $v$ be any vertex of equitable degree less than two, then $v$ must be in every 2-equitable dominating set.*

A 2-equitable dominating set $S$ is said to be minimal if no proper subset of $S$ is 2-equitable dominating set.

**Theorem 2.4.** *Let $G = (V, E)$ be a graph. An 2-equitable dominating set $S$ of $G$ is minimal if and only if for every vertex $v \in S$, either*

(i) $|N_e(v) \cap S| < 2$, or

(ii) There exists a vertex $u \in V - S$ such that $|N_e(v) \cap S| = 2$ and $u \in N_e(v)$.

**Proof.** Let $S$ be a minimal 2-equitable dominating set of $G$. Suppose that the two condition (i) and (ii) are not hold. That is there exist a vertex $v$ in $S$ such that $|N_e(v) \cap S| \ge 2$ and for every vertex $u \in V - S$ either $|N_e(v) \cap S| > 2$ or $u \notin N_e(v)$ and consider $S' = S - \{v\}$ and since $v$ equitable adjacent to at least 2 vertices of $S'$. Therefore $S'$ is an 2-equitable dominating set, a contradiction with minimality of $S$.

Conversely, let $S$ be an 2-equitable dominating set of $G$ satisfying the conditions (i)and (ii). Consider $S' = S - \{v\}$ for any vertex $v \in S$. Now if (i) holds then $S'$ is not 2-equitable dominating set, and if (ii) holds then there exist a vertex $u \in V - S$ such that $|N_e(v) \cap S| = 2$ and $u \in N_e(v)$ and in this case $S'$ is not an 2-equitable dominating set(because $S'$ not 2-equitable dominate $u$). Therefore in the two cases $S'$ is not an 2-equitable dominating set. Hence $S$ is $S'$ a minimal 2-equitable dominating set of $G$.

**Proposition 2.5.** *For any graph $G$ with $p$ vertices,*

(1) $2 \le \gamma_{\times e}(G) \le p$

(2) $\gamma_{\times e}(G) = p$ if and only if $\Delta_e(G) < 2$.

$\gamma_{\times e}(G) = 2$ if there exist at least two vertices $v, u \in G$, such that $deg_e(v) = deg_e(u) = p - 1$ or $deg_e(v) = deg_e(u) = p - 2$

**Theorem 2.6.** *Let $G$ be a graph with $\delta_e(G) \ge 2$. If $S$ is a minimal 2-equitable dominating set, then $V - S$ contains a minimal equitable dominating set.*

**Proof.** Let $S$ be a minimal 2-equitable dominating set of $G$. Suppose that $V - S$ is not an equitable dominating set, then there exist at least one vertex $v \in S$ which is not equitable adjacent to any vertex in $V - S$. Therefore is equitable adjacent to at least two vertices in $S$. Then $S - \{v\}$ is an 2-equitable dominating set a contradiction. Hence every vertex in $S$ must be equitable adjacent to at least one vertex in $V - S$. Hence $V - S$ is an equitable dominating set which contains minimal equitable dominating set.

**Observation 2.7.** *Every 2-equitable dominating set of a graph $G$ contains the leaves and support vertices of $G$.*

**Proposition 2.8.** *Let $G$ be a connected graph has no non-equitable edge and $H$ is spanning subgraph of $G$. Then $\gamma_{\times 2e}(G) \le \gamma_{\times 2e}(H)$.*

 

**Proof.** Let $G$ be a connected Graph and let $H$ is the spanning subgraph of $H$. Suppose That $D$ is the minimum 2-equitable dominating set of $G$. Then $D$ also an 2-equitable dominate all the vertices in $V(H)-D$ that is $D$ is an 2-equitable dominating set in $H$. Hence $\gamma_{\times 2e}(G) \le \gamma_{\times 2e}(H)$.

**Theorem 2.9.** *For any graph* $G$, $\gamma_e(G)+1 \le \gamma_{\times 2e}(G)$.

**Proof.** Let $S$ be $\gamma_e$ -set. Then for any vertex $v \in S$, $S-\{v\}$ is equitable dominating set. Hence $\gamma_e(G)+1 \le \gamma_{\times 2e}(G)$. Further if $G = C_4$, the equality hold.

## 3. CONNECTED 2-EQUITABLE DOMINATION IN A GRAPH

**Definition.** Let $G=(V,E)$ be a graph. An 2-equitable dominating set $D \subseteq V(G)$ if the subgraph of $G$ induced by $D$ is connected. The connected 2-equitable domination of $G$ is the size of its smallest connected 2-equitable dominating set, and is denoted by $\gamma_{\times 2ce}$.

**Proposition 3.1.** *The connected 2-equitable domination number of some standard graphs are*

(1) $\gamma_{\times 2ce}(K_p) = 2$

(2) $\gamma_{\times 2ce}(P_p) =$

(3) $\gamma_{\times 2ce}(C_p) = p-1$

(4) $\gamma_{\times 2ce}(W_p) = \begin{cases} p-1, & if \quad p \ge 5; \\ 2, & otherwise. \end{cases}$

(5) $\gamma_{\times 2ce}(K_{r,t}) = \begin{cases} r+t, & if \ |r-t| \ge 2; \\ 4, & otherwise. \end{cases}$

**Observation 3.2.** *For any tree* $T$ *with* $p$ *vertices,*

$$\gamma_{\times 2ce}(T) = p.$$

It is clear for any graph $G$, any connected 2-equitable dominating set is 2-equitable dominating set and any 2-equitable dominating set is equitable dominating set, and also any equitable dominating set is dominating set, then the following proposition is straightforward.

**Proposition 3.3.** *For any Graph,* $G$

$$\gamma(G) \le \gamma_e(G) \le \gamma_{\times 2e}(G) \le \gamma_{\times 2ce}(G).$$

**Theorem 3.4.** *For any connected graph* $G$ *with diameter equal to* $k$.

$$k-1 \le \gamma_{\times 2ce}(G).$$

**Proof.** Let $u$ and $v$ be any two vertices such that the distance between them $d(u,v)=k$, if $u,v \in D$, then $D$ has at least $k+1$ vertices, if $u \in D$ but $v \notin D$, then since $v$ must be adjacent to at least two vertices in $D$. hence $|D| \ge k$, and if $u$ and $v$ both not in $D$ clearly $|D| \ge k-1$.

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