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ADMITTING A CONFORMAL TRANSFORMATION GROUP ON KAEHLERIAN RECURRENT SPACES

U. S. Negi* & Kailash Gairola

Department of Mathematics, H. N. B. Garhwal University,
Campus Badshahi Thaul, Tehri Garhwal – 249 199, Uttarakhand, India

E-mail: usnegi7@gmail.com

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ABSTRACT

Yano and Sawaki (1968) have studied Riemannian manifolds admitting a conformal transformation group. Yano (1969) has studied on Riemannian manifolds with constant scalar curvature admitting a conformal transformation group. In this paper, we have studied admitting a conformal transformation group on Kaehlerian recurrent spaces and several theorems have been obtained.

Key Words & Phrases: *Kaehlerian, Conformal, Recurrent, Symmetric, transformation group, Space.*

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1. INTRODUCTION.

Let K_n be a connected (C^∞ -) differentiable Kaehlerian spaces of dimension **n** and g_{ji}, ∇_j, $R^h{}_{kji}$, R_{ji} and R, respectively the components of metric tensor field, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor field, the Ricci tensor field and the scalar curvature field. The indices a,b,c, …,i,j,k,…….run over the range 1,2,3,…..,n. We shall denote $g^{ja}\nabla_a$ by ∇^j and the Laplace-Beltrami operator by Δ. In this paper, we assume that Kaehlerian spaces are connected and differentiable and functions are also differentiable.

An infinitesimal transformation v^h on K_n is said to be conformal, if it satisfies

(1.1) $$£_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji},$$

for some function ρ on K_n, where denote the operator of Lie-derivation with respect to v^h and $v_j = g_{ja} v^a$. The ρ satisfies $\rho = \nabla_a v^a/n$ .If ρ in (1.1) is a constant, the transformation is said to be homothetic anf if $\rho = 0$, the transformation is called to be isometric. Hereafter, we shall denote the gradient of ρ by $\rho_j = \nabla_j \rho$ We, now, put

$$G_{ji} = R_{ji} - R g_{ji}/n,$$

$$Z_{Kj\,ih} = R_{kji\,h} - R(g_{Kh} g_{ji} - g_{jh} g_{ki}\,/\,n(n-1).$$

We then have

(1. 2) $G_{ji}\, g^{ji} = 0$, $Z^a{}_{aji} - G_{ji}$.

Here, Yano and Sawaki (1968) introduced the covariant tensor field

(1.3) $W_{kjih} = aZ_{kjih} + b\,(g_{Kh}\, G_{ji} - g_{jh}\, G_{ki} + G_{Kh}\, g_{ji} - G_{jh}\, g_{ki}) / (n\text{-}2)$,

Where a and b being constants, not both zero. It is easily seen that

$$W_{kjih}\, g^{kh} = (a+b)\, G_{ji} .$$

***Corresponding author: U. S. Negi*, *E-mail: usnegi7@gmail.com**

Hereafter, we shall use the following notations:

$$f = G_{ji}\, G^{ji},\ \ z = Z_{kjih}\, Z^{kjih},\ \ w = W_{kjih}\, W^{kjih}\ .$$

Also, Yano and Sawaki (1968) give the following properties:

Definition (1.1): Suppose that a compact orientable Riemannian space M_n with constant scalar curvature field R and of dimension >2 satiesfies

$$\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z = constant,$$

where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constant, not all zero, such that if n > 6.

(1.4) $$8\, R(n-1)^{-1}\alpha_1 \geq (n-6)\alpha_0 \geq 0,$$

$$8\, R(n-1)^{-1}\beta_1 \geq (n-6)\beta_0 \geq 0.$$

If M_n admits and infinitesimal non-isometric conformal transformation $v^h\!: \pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, then M_n is isometric to a sphere.

Definition (1.2): If a compact orientable Riemannian space M_n with constant curvature field R and of dimension >2 admits an infinitesimal non-isometric conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, such that

$$\pounds_v \pounds_v (\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z) \leq 0,$$

Where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constants, not all zero, such that

(1.5) $$4(n-1)R^{-1}\alpha_0 \geq (n+6)\alpha_1 \geq 0,$$

$$4R(n-1)R^{-1}\beta_0 \geq (n+6)\beta_1 \geq 0.$$

then M_n is isometric to a sphere.

Definition (1.3): Suppose that a compact orientable Riemannian space M_n with constant scalar curvature field R and of dimention >2 admits an infinitesimal non-isometric conformal transformation

$$v^h : \pounds_v g_{ji} = 2\rho g_{ji}\ ,\ \rho \neq 0$$

If $\pounds_v \pounds_v\, w = 0$, a and b being constant such that a+b $\neq 0$, then M_n is isometric to a sphere.

2. ADMITTING A CONFORMAL TRANSFORMATION GROUP ON KAEHLERIAN RECURRENT SPACES

In a Riemannian space M_n, for an infinitesimal conformal transformation $v^h\!: \pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$.we have Yano (1957)

$$\pounds_v R^h{}_{kji} = -\delta^h_k \nabla_j \rho_i + \delta^h_j \nabla_k \rho_i - (\nabla_k \rho^h) g_{ji} + (\nabla_k \rho^h) g_{ki},$$

$$\pounds_v R_{ji} = -(n-2)\nabla_j \rho_i - \Delta\rho\, g_{ji}$$

$$\pounds_v R = -2(n-1)\Delta\rho - 2\rho R$$

Thus, for K_n with constant scalar curvature field R,

$$\Delta\rho = -R\rho/(n-1)\ \ and$$

(2.1) $$\nabla_j G_{ji} = 1/2\ (n-1)n^{-1}\, \nabla_i R = 0$$

We have, from (1.2),

(2.2) $$\pounds_v G_{ji} = -(n-2)(\nabla_i \rho_i - \Delta\rho\, g_{ji}/n)$$ and

$$£_v Z_{kjih} = -g_{kh}\nabla_i\rho_i - g_{jh}\nabla_k\rho_i - (\nabla_k\rho_h)g_{ji} + (\nabla_j\rho_h)g_{ki} + 2\nabla\rho(g_{kh}g_{ji} - g_{jh}g_{ki})/n + 2\rho Z_{kjih}.$$

By straightforward calculations, we have, in view of (1.3) and (2.2)

$$(£_v W_{kjih})W^{kjih} = -4(a+b)^2(\nabla^j\rho^i)g_{ji} + 2\rho W_{kjih}W^{kjih} \quad .$$

On the other hand, we get

$$(£_v W^{kjih})W_{kjih} = (£_v W_{kjih})W^{kjih} - 8\rho W_{kjih}W^{kjih} \quad .$$

Thus, we have

(2.3) $$£_v w = -8(a+b)^2(\nabla^j\rho^i)G_{ji} - 4\rho w.$$

Now, we have the following Lemmas:

Lemma (2.1): If a compact orientable Kaehlerian space K_n of dimension n admits an infinitesimal conformal transformation $v^h: £_v g_{ji} = 2\rho g_{ji}$ then foe any function F on K_n,

(2.4) $$\int_{Kn} \rho F dv = -1/n\int_{Kn} £_v F\, dv\,.$$

Proof: Since $\rho = \nabla_a v^a/n$, we have by using Green's Theorem

$$\int_{Kn} \nabla_a(Fv^a)dv = \int_{Kn} £_v F dv + \int_{Kn} \rho F dv = 0\,,$$

Which proves (2.4).

Lemma (2.2): If a compact Orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension n admits an infinitesimal conformal transformation on $v^h: £_v g_{ji} = 2\rho g_{ji}$, then $\int_{Kn} \rho(\nabla^j\rho^i)G_{ji}\,dv = -\int_{Kn} G_{ji}\rho^j\rho^i dv\,.$

Proof: This follows from (2.1) and

$$\int_{Kn} \nabla_j(G_{ji}\rho^j\rho)dv = \int_{Kn} (\nabla^j G_{ji})\rho^i\rho\, dv + \int_{Kn} G_{ji}(\nabla^j\rho^i)\rho\, dv + \int_{Kn} G_{ji}\rho^j\rho^i\, dv = 0.$$

Lemma (2.3): Hiramatu gives, If a compact orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension n admits an infinitesimal conformal transformation $v^h: £_v g_{ji} = 2\rho g_{ji}$, then

(2.5) $$\int_{Kn} £_v £_v f dv = -2n(n-2)\int_{Kn} G_{ji}\rho^j\rho^i dv + 4n\int_{Kn} \rho^2 f dv\,,$$

$$\int_{Kn} £_v £_v z dv = -8n\int_{Kn} G_{ji}\rho^j\rho^i dv + 4n\int_{Kn} \rho^2 z dv\,,$$

Lemma (2.4): If a compact orientable Kaehlerian space K_n with
constant scalar curvature field R and of dimension n admits an infinitesimal conformal transformation $v^h: £_v g_{ji} = 2\rho g_{ji}$, then

(2.6) $$\int_{Kn} £_v £_v w dv = -8n(a+b)^2\int_{Kn} G_{ji}\rho^j\rho^i dv + 4n\int_{Kn} \rho^2 w dv.$$

Proof: Making use of (2.3) and Lemmas (2.1) and (2.2), we have

$$\begin{aligned}\int_{Kn} £_v £_v w dv &= -n\int_{Kn} \rho £_v w dv \\ &= 8n(a+b)^2\int_{Kn}\rho(\nabla^j\rho^i)G_{ji}\,dv + 4n\int_{Kn} \rho^2 w dv\end{aligned}$$

Which shows (2.6).

Lemma (2.5): Hiramatu gives, If a compact Orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension $n \geq 2$ admits an infinitesimal conformal transformation $v^h: £_v g_{ji} = 2\rho g_{ji}$, then for any function F on K_n ,

$$\int_{Kn} £_v £_v \nabla F dv = -\frac{R}{n-1}\int_{Kn} £_v £_v F dv + \frac{n(n+2)}{2}\int_{Kn} (\rho^2\Delta F)dv$$

Lemma (2.6): Yano (1966) gives, If a compact Oreintable Kaehlerian space K_n with constant scalar curvature field R and of dimension >2 admits on infinitesimal non-isometric conformal transformation $v^h : £_v g_{ji} = 2\rho g_{ji}, \rho \neq 0$, then

$$\int_{Kn} G_{ji} \rho^j \rho^i dv \leq 0 \,,$$

The equality holds if and only if K_n is isometric to a sphere.

We have the following

Theorem (2.1): If a compact orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension >2 admits an infinitesimal non - isometric conformal transformation

$$v^h : £_v g_{ji} = 2\rho g_{ji} \,,\ \rho \neq 0 \text{ then}$$

(2.7) $$\int £_{Kn} £_v £_v (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv \geq \frac{n(n+2)}{2} \int_{Kn} \rho^2 (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv$$

Holds, where dv denotes the volume element of K_n and $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constants, not all zero, such that if n>6, (1.4) holds, the equality in (2.7) holds if and only if K_n is isometric to a sphere.

Proof: Making use of (2.5) in Lemma (2.3) and (2.5) we get

$$\int_{Kn} £_v £_v (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv - \frac{n(n+2)}{2} \int_{Kn} \rho^2 (\alpha_0 f + \ \Delta f - \beta_1 \Delta z) dv$$

$$= \int_{Kn} £_v £_v (\alpha_0 f + \beta_0 z) dv - \frac{R}{n-1} \int_{Kn} £_v £_v (-\alpha_1 f - \beta_1 z) dv + \frac{n(n+2)}{2} \{ \int_{Kn} \rho^2 (-\alpha_1 \Delta f - \beta_1 \Delta z) dv - \int_{Kn} \rho^2 (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv \}$$

$$= \left(\alpha_0 + \frac{R}{n-1}\alpha_1\right) \int_{Kn} £_v £_v f dv + \left(\beta_0 + \frac{R}{n-1}\beta_1\right) \int_{Kn} £_v £_v z dv - \frac{n(n+2)}{2} \int_{Kn} \rho^2 (\alpha_0 f - \beta_0 z) dv$$

$$= -[2n(n-2) \left(\alpha_0 + \frac{R}{n-1}\alpha_1\right) + 8n\left(\beta_0 + \frac{R}{n-1}\beta_1\right)] \int_{Kn} G_{ji} \rho^j \rho^i dv + n\left(\frac{4R}{n-1}\alpha_1 - \frac{n-6}{2}\alpha_0\right) \int_{Kn} \rho^2 dv + n\left(\frac{4R}{n-1}\beta_1 - \frac{n-6}{2}\beta_0\right) \int_{Kn} \rho^2 z dv$$

From Lemma (2.6) and our assumption, we can see that the right hands side of the above relation in non-negative and consequently (2.7) holds. If the equality in (2.7) holds, then, from our assumption, we have

(2.8) $$\int_{Kn} G_{ji} \rho^j \rho^i dv = 0,$$

And K_n is isometric to a sphere, by virtue of Lemma (2.6).

Conversely, If K_n is isometric to a sphere, we get G_{ji}=0 and $Z_{kji\,h}$=0 and the equality in (2.7) holds.

Remark (2.1): If we assueme that $\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z = c$ (constant), from Theorem (2.1), we have $c \leq 0$. On the other hand $c \geq 0$ holds, because

$$c \int_{Kn} dv = \int_{Kn} c dv = \int_{Kn} (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv$$

$$= \alpha_0 \int_{Kn} f dv + \beta_0 \int_{Kn} z dv \geq 0 \,.$$

Thus, if $\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z$ is a constant, then the constant is equal to zero and consequently the equality in (2.7) holds, and K_n is isometric to a sphere.

Theorem (2.2): If a compact orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension >2 admits an infinitesimal non-isometric conformal transformation

$$v^h : £_v g_{ji} = 2\rho g_{ji} \,,\ \rho \neq 0 \text{ then}$$

(2.9) $$\int_{Kn} £_v £_v (\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z) dv \geq 0 \,,$$

Holds, where $\alpha_0, \alpha_1, \beta_0$ and β_1, are non-negative constant, not all zero, such that (1.5) holds, the equality in (2.9) holds if and only if K_n is isometric to a sphere.

Proof: Making use of (2.5) in Lemmas (2.3), (2.5) and

(2.10) $\quad 1/2\nabla\rho^2 = \rho_i\rho^i - R\rho^2/(n-1),$

We have

$$\int_{Kn} £_v £_v(\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z)dv$$
$$= (\alpha_0 - \frac{R}{n-1}\alpha_1)\int_{Kn} £_v £_v f dv + (\beta_0 - \frac{R}{n-1}\beta_1)\int_{Kn} £_v £_v z dv + \frac{n(n+2)}{2}\alpha_1 \int_{Kn}(\Delta\rho^2) f dv + \frac{n(n+2)}{2}\beta_1 \int_{Kn}(\Delta\rho^2) z dv$$

$$= (\alpha_0 - \frac{R}{n-1}\alpha_1)\int_{Kn} £_v £_v f dv + (\beta_0 - \frac{R}{n-1}\beta_1)\int_{Kn} £_v £_v z dv + \quad n(n+2)\int_{Kn}\rho_i\rho^i(\alpha_1 f + +\beta_1 z)dv + n(n+2)\frac{R}{n-1}\int_{Kn}\rho^2(\alpha_1 f + \beta_1 z)dv$$
$$= -[2n(n-2)\left(\alpha_0 - \frac{R}{n-1}\alpha_1\right) + 8n(\beta_0 - \frac{R}{n-1}\beta_1)\int_{Kn} G_{ji}\rho^j\rho^i dv + n(n-2)\int_{Kn}\rho_i\rho^i(\alpha_1 f + +\beta_1 z)dv + n[4\alpha_0 - \frac{(n+6)R}{n-1}\alpha_1]\int_{Kn}\rho^2 f dv + n[4\beta_0 - \frac{(n+6)R}{n-1}\beta_1]\int_{Kn}\rho^2 z dv.$$

From Lemma (2.6) and our assumption, we can see that the right hand side of the above equation is non-negative and consequently (2.9) holds, because it follows from our assumption that

$$\alpha_0\, R(n-1)^{-1}\alpha_1 and\, \beta_0\, R(n-1)^{-1}\beta_1$$

are non-negative and not both zero. If the equality in (2.9) holds, then we have (2.8) and K_n is isometric to a sphere by virtue of Lemma (2.6).

Conversely, If K_n is isometric to a sphere, we get $G_{ji} = 0$ and $Z_{kji\,h} = 0$ and the equality in (2.9) holds.

Theorem (2.3): If a compact orientable space K_n with constant scalar curvature field R and of dimension >2 admits an infinitesimal non-isometric conformal transformation on $v^h: £_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$ then

(2.11) $\quad \int_{Kn} £_v £_v(\alpha_0\omega - \alpha_1\Delta\omega)dv \geq \frac{n(n+2)}{2}\int_{Kn}\rho^2(\alpha_0\omega - \alpha_1\Delta\omega)dv$, holds,

where a and b are constants such that a+b≠0, α_0 and α_1 , are non-negative constants such that if n > 6 the first inequality in (1.4) holds, the equality in (2.11) holds if and only if K_n is isometric to a sphere.

Proof: Similarly, as in the proof of the Theorem (2.1), by using (2.6) in Lemma (2.4) and Lemma (2.5) and (2.6), we get

$$\int_{Kn} £_v £_v(\alpha_0 w - \alpha_1\Delta w)dv - \frac{n(n+2)}{2}, \int_{Kn}\rho^2(\alpha_0 w - \alpha_1\Delta w)dv, = \left(\alpha_0 + \frac{R}{n-1}\alpha_1\right)\int_{Kn} £_v £_v w dv - \frac{n(n+2)}{2}\alpha_0)\int_{Kn}\rho^2 w dv$$
$$= -8n(a+b)^2\left(\alpha_0 + \frac{R}{n-1}\alpha_1\right)\int_{Kn} G_{ji}\rho^j\rho^i dv + n(\frac{4R}{n-1}\alpha_1 - \frac{n-6}{2}\alpha_0)\int_{Kn}\rho^2 w dv \geq 0\,,$$

Which proves (2.11). It is easily proved from Lemma (2.6) and our assumption that the equality in (2.11) holds if only if K_n is isometric to a sphere.

Theorem (2.4): If a compact orientable space K_n with constant scalar curvature field R and of dimension > 2 admits an infinitesimal non- isometric conformal transformation $v^h: £_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$. then

(2.12) $\quad \int_{Kn} £_v £_v(\alpha_0\omega - \alpha_1\Delta\omega)dv \geq 0$

holds, where a and b are constant such that a+b ≠0 and α_0 and α_1 are non-negative constant, not both zero, such that the first inequality in (1.5) holds, the equality in (2.12) holds if K_n is isometric to a sphere.

Proof: Similarly, as in the proof of Theorem (2.2), by using (2.6) in Lemma (2.4), (2.5) and (2.6) and (2.10), we have

$$\int_{Kn} £_v £_v(\alpha_0 w + \alpha_1 \Delta w)dv$$
$$= -8n(a+b)^2 \left(\alpha_0 - \frac{R}{n-1}\alpha_1 \right) \int_{Kn} g_{ji}\rho^j \rho^i dv + n(n+2)\alpha_1 \int_{Kn} \rho_i \rho^i dv + n[4\alpha_0 - \frac{(n+6)R}{n-1}\alpha_1] \int_{Kn} \rho^2 \omega dv \geq 0\ ,$$

Which proves (2.12). It is easily proved from Lemma (2.6) and our assumption that the equality in (2.12) holds if and only if K_n is isometric to a sphere.

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