

THE MAXIMAL SUBGROUPS OF THE UNITARY GROUP PSU (8, q), where $q = 2^k$

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ABSTRACT.

The purpose of this paper is to study maximal subgroups of the Unitary group PSU (8, q), where $q = 2^k$. The main result is a list of maximal subgroups called "the main theorem" which has been proved by using Aschbacher's Theorem ([1]). Thus, this work is divided into two main parts:

Part (1): In this part, we will find the maximal subgroups in the classes $C_1 - C_8$ of Aschbacher's Theorem ([1]).

Part (2): In this part, we will find the maximal subgroups in the class C_9 of Aschbacher's Theorem ([1]), which are the maximal primitive subgroups H of G that have the property that the minimal normal subgroup M of H is not abelian group and simple, thus, we divided this part into two cases:

Case (1): M is generated by transvections: In this case, we will use result of Kantor ([9]).

Case (2): M is a finite primitive subgroup of rank three: In this case, we will use the classification of Kantor and Liebler ([8]).

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1. INTRODUCTION: The purpose of this paper is to prove the following main theorem:

Theorem (1.1): Let $G = \text{PSU}(8, q)$, $q = 2^k$. If H is a maximal subgroup of G , then H isomorphic to one of the following subgroups:

1. $H_1 = q^{13} : (\text{PGL}(1, q^2) \times \text{PSU}(6, q))$.
2. $H_2 = q^{20} : (\text{PGL}(2, q^2) \times \text{PSU}(4, q))$.
3. $H_3 = q^{21} : (\text{PGL}(3, q^2) \times \text{PSU}(2, q))$.
4. $H_4 = q^{16} : \text{PGL}(4, q^2)$.
5. $H_5 = \text{PSU}(1, q) \times \text{PSU}(7, q)$.
6. $H_6 = \text{PSU}(2, q) \times \text{PSU}(6, q)$.
7. $H_7 = \text{PSU}(3, q) \times \text{PSU}(5, q)$.
8. $H_8 = \text{PSU}(2, q) : S_4$.
9. $H_9 = \text{PSU}(4, q) : S_2$.
10. $H_{10} = \text{PSU}(4, q^2) : 2$.
11. $H_{11} = \text{PSU}(2, q) \circ \text{PSU}(4, q)$.
12. $H_{12} = \text{PSU}(8, q')$, where $q' = 2^{k'}$ and k' is a prime number divides k .
13. $H_{13} = \text{PSU}(2, q) : S_3$.
14. $H_{14} = \text{PSp}(8, q)$;

We will prove this theorem by Aschbacher's theorem (Result 2.9) {see [1]}:

If $x \in GF(q^2)$, we set $\bar{x} = x^q$ and assume that X is a matrix, then the matrix \bar{X}^t is obtained from X by replacing each entry x with \bar{x} and then taking the transpose. Thus, in a matrix form, The unitary group $U(n, q)$ consists of all matrices $X \in GL(n, q)$ such that $\bar{X}^t X = J$, where

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$$J = \begin{pmatrix} & & & & & & & 1 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 1 & & & & & & & \end{pmatrix}$$

The special unitary group $SU(n, q) = \{X \in U(n, q) : \det X = 1\}$, and the projective special unitary group $PSU(n, q)$ is the factor group $SU(n, q)/(SU(n, q) \cap Z)$, where the center $Z = \{\text{diag}(\lambda, \lambda, \dots, \lambda) : \lambda \in F, \lambda^n = 1, \lambda \bar{\lambda} = 1\}$. $PSU(n, q)$ is simple except for $PSU(2, 2)$, $PSU(2, 3)$ and $PSU(3, 2)$.

The group $U(8, q)$, q even can generate by the two elements:

$$\left\{ \begin{pmatrix} \alpha & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & & & & 1 & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{pmatrix} \right\},$$

where α is a generator element of the multiplicative group of $GF(q^2)$ {see [15]}. Throughout this article, G will denote $PSU(8, q)$, $q = 2^k$, unless otherwise stated. G is a simple group of order $q^{28}(q^8 - 1)(q^7 + 1)(q^6 - 1)(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)$. G acts as a doubly transitive permutation group on the points of the projective space $PG(7, q)$. {see [2]}.

2. ASCHBACHER'S THEOREM:

A classification of the maximal subgroups of $GL(n, q)$ by Aschbacher's theorem {see [1]}, is a very strong tool in the finite groups for finding the maximal subgroups of finite linear groups. There are many good works in finite groups which simplify this theorem, see for example [12]. But before giving a brief description of this theorem, we will give the following definitions:

Definition 2.1: A split extension (a semidirect product) $A : B$ is a group G with a normal subgroup A and a subgroup B such that $G = AB$ and $A \cap B = 1$. A non-split extension $A.B$ is a group G with a normal subgroup A and $G/A \cong B$, but with no subgroup B satisfying $G = AB$ and $A \cap B = 1$. A group $G = A \circ B$ is a central product of its subgroups A and B if $G = AB$ and $[A, B]$, the commutator of A and $B = \{1\}$, in this case A and B are normal subgroups of G and $A \cap B \leq Z(G)$. If $A \cap B = \{1\}$, then $A \circ B = AB$.

Definition 2.2: Let V be a vector space of dimensional n over a finite field q , a subgroup H of $GL(n, q)$ is called reducible if it stabilizes a proper nontrivial subspace of V . If H is not reducible, then it is called irreducible. If H is irreducible for all field extension F of F_q , then H is absolutely irreducible. An irreducible subgroup H of $GL(n, q)$ is called imprimitive if there are subspaces $V_1, V_2, \dots, V_k, k \geq 2$, of V such that $V = V_1 \oplus \dots \oplus V_k$ and H permutes the elements of the set $\{V_1, V_2, \dots, V_k\}$ among themselves. When H is not imprimitive then it is called primitive.

Definition 2.3: A group $H \leq GL(n, q)$ is a superfield group of degree s if for some s divides n with $s > 1$, the group H may be embedded in $GL(n/s, q^s)$.

Definition 2.4: If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes V_2$ with $\dim(V_1) \neq \dim(V_2)$, then H is a tensor product group.

Definition 2.5: Suppose that $n = r^m$ and $m > 1$. If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes \dots \otimes V_m$ with $\dim(V_i) = r$ for $1 \leq i \leq m$, then H is a tensor induced group.

Definition 2.6: A group $H \leq GL(n, q)$ is a subfield group if there exists a subfield $F_{q_0} \subset F_q$ such that H can be embedded in $GL(n, q_0).Z$, where Z is the centre group of H .

Definition 2.7: A p -group H is called a special group if $Z(H) = H'$ and is called an extraspecial group if also $|Z(H)| = p$.

Definition 2.8: Let Z denote the centre group of H . Then H is *almost simple modulo scalars* if there is a non-abelian simple group T such that $T \leq H/Z \leq \text{Aut}(T)$, the automorphism group of T .

A classification of the maximal subgroups of $GL(n, q)$ by Aschbacher's theorem [1], can be summarized as follows:

Result 2.9: (Aschbacher's theorem)

Let H be a subgroup of $GL(n, q)$, $q = p^e$ with the centre Z and let V be the underlying n -dimensional vector space over a field q . If H is a maximal subgroup of $GL(n, q)$, then one of the following holds:

- C_1 :- H is a reducible group.
- C_2 :- H is an imprimitive group.
- C_3 :- H is a superfield group.
- C_4 :- H is a tensor product group.
- C_5 :- H is a subfield group.
- C_6 :- H normalizes an irreducible extraspecial or symplectic-type group.
- C_7 :- H is a tensor induced group.
- C_8 :- H normalizes a classical group in its natural representation.
- C_9 :- H is absolutely irreducible and $H/(H \cap Z)$ is almost simple.

3. CLASSES $C_1 - C_8$ OF RESULT 2.9:

In this section, we will find the maximal subgroups in the classes $C_1 - C_8$ of Result 2.9:

Lemma 3.1: There are seven reducible maximal subgroups of C_1 in G which are:

1. $H_1 = q^{13} : (\text{PGL}(1, q^2) \times \text{PSU}(6, q))$.
2. $H_2 = q^{20} : (\text{PGL}(2, q^2) \times \text{PSU}(4, q))$.
3. $H_3 = q^{21} : (\text{PGL}(3, q^2) \times \text{PSU}(2, q))$.
4. $H_4 = q^{16} : \text{PGL}(4, q^2)$.
5. $H_5 = \text{PSU}(1, q) \times \text{PSU}(7, q)$.
6. $H_6 = \text{PSU}(2, q) \times \text{PSU}(6, q)$.
7. $H_7 = \text{PSU}(3, q) \times \text{PSU}(5, q)$.

Proof: Let H be a reducible subgroup of the unitary group $SU(n, q)$ and W be an invariant subspace of H . Let $r = \dim(W)$, $1 \leq r \leq n/2$ and let $G_r = G_{(W)}$ denote the subgroup of $SU(n, q)$ containing all elements fixing W as a whole and $H \subseteq G_{(W)}$. with a suitable choice of a basis, $G_{(W)}$ consists of all matrices of the form $\begin{pmatrix} A & D & E \\ & B & F \\ & & C \end{pmatrix}$ where D and F are two elementary abelian groups of order $q^{r(n-2r)}$, A is a p - group of upper triangular matrix of identity diagonal of order $q^{\frac{r(r-1)}{2}}$, C is a p - group of upper triangular matrix of order $q^{\frac{r(r+1)}{2}}$, $E \in GL(r, q^2)$, $B \in SU(n-2r, q)$ such that $\bar{A}^t J C = J$, and J is anti-diagonal identity matrix. Thus the maximal parabolic subgroups are the stabilizers of totally isotopic subspaces $\langle e_1, e_2, \dots, e_r \rangle$ is isomorphic to a group of the form $q^{r(2n-3r)} : (\text{GL}(r, q^2) \times \text{SU}(n-2r, q))$.

Thus, we have the following reducible maximal subgroups of $PSU(8, q)$:

1. If $r = 1$, then we get a group, stabilizing a point is isomorphic to a group of the form $H_1 = q^{13} : (\text{PGL}(1, q^2) \times \text{PSU}(6, q))$.
2. If $r = 2$, then we get a group, stabilizing a line is isomorphic to a group of the form $H_2 = q^{20} : (\text{PGL}(2, q^2) \times \text{PSU}(4, q))$.
3. If $r = 3$, then we get a group, stabilizing a plane is isomorphic to a group of the form $H_3 = q^{21} : (\text{PGL}(3, q^2) \times \text{PSU}(2, q))$.
4. If $r = 4$, then we get a group, stabilizing a 3-space is isomorphic to a group of the form $H_4 = q^{16} : \text{PGL}(4, q^2)$.

Also, H is the maximal reducible subgroup of the unitary group $SU(n, q)$ which stabilizers of non-singular subspaces of dimension d which have the shape $H = \text{SU}(d, q) \times \text{SU}(b, q)$ where $n = d + b$ and $1 \leq d < b$. Thus, we have the following reducible maximal subgroups of $PSU(8, q)$:

5. If $d = 1$ and $b = 7$, then we get the group $H_5 = \text{PSU}(1, q) \times \text{PSU}(7, q)$.
6. If $d = 2$ and $b = 6$, then we get the group $H_6 = \text{PSU}(2, q) \times \text{PSU}(6, q)$.
7. If $d = 3$ and $b = 5$, then we get the group $H_7 = \text{PSU}(3, q) \times \text{PSU}(5, q)$.

Which prove the points (1), (2), (3), (4), (5), (6) and (7) of the main theorem 1.1.

Lemma 3.2: There are two imprimitive group of C_2 in G which are

1. $H_8 = \text{PSU}(2, q):S_4$.
2. $H_9 = \text{PSU}(4, q):S_2$.

Proof: If H is imprimitive of the unitary group $SU(n, q)$, then H preserves a decomposition of V as orthogonal direct sum $V = V_1 \oplus \dots \oplus V_t$ with $t \geq 2$, of t subspaces of V each of dimension $m = n/t$, which are permuted transitively by H , thus H are isomorphic to $SU(m, q):S_t$ with $0 < m < n = mt, t \geq 2$.

Thus, we have the following two imprimitive maximal subgroups of $\text{PSU}(8, q)$:

1. If $m = 2$ and $t = 4$, then we get the group $H_8 = \text{PSU}(2, q):S_4$.
2. If $m = 4$ and $t = 2$, then we get the group $H_9 = \text{PSU}(4, q):S_2$.

Which prove the points (8) and (9) of the main theorem 1.1.

Lemma 3.3: There is one semilinear group of C_3 in G which is $H_{10} = \text{PSU}(4, q^2):2$.

Proof: Let H is (superfield group) a semilinear groups of $\text{PSU}(n, q)$ over extension field F_r of $\text{GF}(q)$ of prime degree $r > 1$ where r prime number divide n . Thus V is an F_r -vector space in a natural way, so there is an F -vector space isomorphism between n -dimensional vector space over F and the m -dimensional vector space over F_r , where $m = n/r$, thus H embeds in $\text{PSU}(m, q^r):r$. Consequently, there is one C_3 group in G which is the group $H_{10} = \text{PSU}(4, q^2):2$. This proves the point (10) of the main theorem 1.1.

Lemma 3.4: There is one tensor product group $H_{11} = \text{PSU}(2, q) \circ \text{PSU}(4, q)$ of C_4 in G .

Proof: If H is a tensor product group of $SU(n, q)$, then H preserves a decomposition of V as a tensor product $V_1 \otimes V_2$, where $\dim(V_1) \neq \dim(V_2)$ of spaces of dimensions k and m over $\text{GF}(q)$ and $n = km, k \neq m$. So, H stabilizes the tensor product decomposition $F^k \otimes F^m$. Thus, H is a subgroup of the central product of $SU(k, q) \circ SU(m, q)$. Consequently, there is one C_4 group $H_{11} = \text{PSU}(2, q) \circ \text{PSU}(4, q)$ in $\text{PSU}(8, q)$. This proves the point (11) of the main theorem 1.1.

Lemma 3.5: There are subfield groups of C_5 in G which are $H_{12} = \text{PSU}(8, q')$, where $q' = 2^{k'}$ and k' is a prime number divides k .

Proof: If H is a subfield group of the unitary group $SU(n, q)$ and $q = p^k$, then H is the unitary group over subfield of $\text{GF}(q)$ of prime index. Thus H can be embedded in $SU(n, p^f)$, where f is prime number divides k . Consequently, since $q = 2^k$, then there are subfield groups in $\text{PSU}(8, 2^k)$ which are the groups $H_{12} = \text{PSU}(8, q')$, where $q' = 2^{k'}$ and k' is a prime number divides k . This proves the point (12) of the main theorem 1.1.

Lemma 3.6: There are no C_6 groups in G .

Proof: For the dimension $n = r^m$, if $r = 2$ and 4 divides $q-1$, then $H = 2^{2m+1}.O(2m, 2)$ normalizes a 2-group of symplectic type of order 2^{2m+2} {see [12]}, Consequently, since $r = 2$ and $q=2^k$ but 4 not divides $q-1$, then there is no subfield group of C_6 in G .

Lemma 3.7: There is one tensor induced group of C_7 in G which is $H_{13} = \text{PSU}(2, q):S_3$.

Proof: If H is a tensor induced of the unitary group $SU(n, q)$, then H preserves a decomposition of V as $V_1 \otimes V_2 \otimes \dots \otimes V_r$, where V_i are isomorphic, each V_i has dimension m , $\dim V = n = m^r$, and the set of V_i is permuted by H , so H stabilize the tensor product decomposition $F^m \otimes F^m \otimes \dots \otimes F^m$, where $F = F_q$. Thus, $H/Z \leq \text{PSU}(m, q):S_r$. Consequently, there is one tensor induced group of C_7 in G which is the group $H_{13} = \text{PSU}(2, q):S_3$. This proves the point (13) of the main theorem 1.1.

Lemma 3.8: There is one maximal C_8 group in G which is the group $H_{14} = \text{PSp}(8, q)$.

Proof: The groups in this class are stabilizers of forms, this means H is the normalizers of one classical groups $\text{PSL}(n, q)$, $\text{PO}^\epsilon(n, q)$ or $\text{PSp}(n, q)$ as a subgroup of $\text{PSU}(n, q)$. Thus, from [4], if n is even, then the normalize of $\text{PSp}(n, q)$ is a maximal subgroup of $\text{PSU}(n, q)$ and from [5] the normalizer of $\text{PSp}(8, q)$ in $\text{PSU}(8, q)$ is $\text{PSp}(8, q)$. Also, from

[6], if n is even and q is even, then the normalizers of $PO^+(n, q)$ and $PO^-(n, q)$ are maximal subgroups of $PSp(n, q)$ except when $n = 4$ and $\epsilon = -$. Consequently, $PSGO^+(8, q)$ and $PSGO^-(8, q)$ are two irreducible maximal subgroups of

$PSU(8, q)$ but not maximal in $PSU(8, q)$ since they also are subgroups of the group $PSp(8, q)$. This proves the point (14) of theorem 1.1.

In the following, we will find the maximal subgroups of class C_9 of Result 2.9:

4. THE MAXIMAL SUBGROUPS OF C_9 :

In Corollary 4.1, we will find the primitive non abelian simple subgroups of G and in Theorem 4.2, we will find the maximal primitive subgroups H of G which have the property that the minimal normal subgroup M of H is not abelian group and simple. We will prove this Theorem 4.2 by finding the normalizers of the groups of Corollary 4.1 and determine which of them are maximal.

Corollary 4.1: If M is a non abelian simple group of a primitive subgroup H of G , then M is isomorphic to one of the following groups:

- (i) $PSU(8, q')$ where $q' = 2^{k'}$ and k' is a prime number divides k ;
- (ii) $PSp(8, q)$;
- (iii) $PSO^-(8, q)$;
- (iv) $PSO^+(8, q)$.
- (v) $PSp(4, q)$, where $q = 2^k$, $k > 1$;
- (vi) $P\Omega^+(8, q)$, where $q = 2^k$;
- (vii) $PSp(6, 2)$;

Proof: Let H be a primitive subgroup of G with a minimal normal subgroup M of H which is not abelian and simple. So, we will discuss the possibilities of M of H according to:

- (I) M contains transvections, {section 4.1}.
- (II) M is a finite primitive subgroup of rank three, {section 4.2}.

4.1 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections:

Definition 4.1.1: An element $T \in GL(n, q)$ is called a *transvection* if T satisfies $\text{rank}(T - I_n) = 1$ and $(T - I_n)^2 = 0$. The collineation of projective space induced by a transvection is called *elation*. The *axis* of the transvection is the hyperplane $\text{Ker}(T - I_n)$; this subspace is fixed elementwise by T , Dually, the *centre* of T is the image of $(T - I_n)$.

To find the primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian and is generated by transvections, we will use the following result of Kantor [9]:

Result 4.1.2: Let H be a proper irreducible subgroup of $SU(n, q^i)$ generated by transvections. Then H is one of:

1. $SU(n, q)$;
2. $Sp(n, q)$ for n even;
3. $O^\pm(n, q)$ for n and q even;
4. S_n for n even;
5. S_{n+1} for n even;
6. $SL(2, 5) < Sp(2, 9^i)$;
7. $3.P\Omega^-(6, 3) < SU(6, 2^i)$;
8. $A:S_n < SU(n, s)$, $|A|=a^{n-1}$, $a|s+1$, s even;
9. Dihedral subgroups of $Sp(2, 2^i)$.

In the following corollary, we will find the primitive subgroups of $PSU(8, q)$ which is generated by transvections:

Corollary 4.1.3: If M is a non abelian simple group and contains some transvections, then M is isomorphic to one of the groups:

- (i) $PSU(8, q')$ where $q' = 2^{k'}$ and k' is a prime number divides k ;
- (ii) $PSp(8, q)$;
- (iii) $PSO^-(8, q)$;
- (iv) $PSO^+(8, q)$.

Proof: We will discuss the different possibilities of Result (4.1.2), so, M is isomorphic to one of the following groups:

1. From Lemma 3.5, PSU(8, q') is an irreducible subgroup of G, where $q' = 2^{k'}$ and k' is a prime number divides k.
2. From Lemma 3.8, PSp(8, q) is an irreducible subgroup of G.
3. From Lemma 3.8, PSO⁻(8, q) and PSO⁺(8, q) are irreducible subgroups of G.
4. The irreducible 2-modular characters for S_8 by GAP are: [[1, 1], [6, 1], [8, 1], [14, 1], [40, 1], [64, 1]]
(gap> CharacterDegrees(CharacterTable("S8")mod 2);)

Thus there is one irreducible character of degree 8, thus $S_8 \subset G$, but the symmetric group S_8 is not a simple group.

5. The irreducible 2-modular characters for S_9 by GAP are: [[1, 1], [8, 1], [16, 1], [26, 1], [40, 1], [48, 1], [78, 1], [160, 1]] (gap> CharacterDegrees(CharacterTable("S9")mod 2);)

Thus there is one irreducible character of degree 8, thus $S_9 \subset G$, but the symmetric group S_8 is not simple group.

6. $SL(2, 5) \not\subset G$, since the irreducible 2-modular characters for $SL(2, 5)$ by GAP are: [[1, 1], [2, 2], [4, 1]]
(gap> CharacterDegrees(CharacterTable("L2(5)") mod 2);) And non of these characters of degree 8.
7. $3.P\Omega(6, 3) \not\subset G$ since G is not $SU(6, 2^i)$;
8. If $a|q+1$, then $a^7:S_8$ is a subgroup of G but not simple.
9. If M is a Dihedral subgroups of $Sp(2, 2^i)$, then $M \not\subset G$, since M is not a simple group.

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup M of H which is not abelian is a finite primitive subgroup of rank three:

A group G has rank 3 in its permutation representation on the cosets of a subgroup K if there are exactly 3 (K, K)-double cosets and the rank of a transitive permutation group is the number of orbits of the stabilizer of a point, thus we may consider PSU(n, q) as group of permutations of the absolute points of the corresponding projective space, then PSU(n, q) is a transitive group of rank 3.

In this section, we will consider the minimal normal subgroup M of H is not abelian and a finite primitive subgroup of rank three, so will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. The following Corollary is the main result of this section:

Corollary 4.2.1: If M is a non abelian simple group which is a finite primitive subgroup of rank three group of H, then M is isomorphic to one of the following groups:

1. PSp(4, q), where $q = 2^k$, $k > 1$;
2. $P\Omega^+(8, q)$, where $q = 2^k$;
3. PSp(6, 2);

Proof: Let M is not an abelian finite primitive subgroup of rank three of H, and will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three [8]. So, we will prove Corollary 4.2.1 by series of Lemmas 4.2.3 through Lemmas 4.2.18 and Result 4.2.2.

Result 4.2.2: If Y acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of $Sp(2n-2, q)$, $\Omega^\pm(2n, q)$, $\Omega(2n-1, q)$ or $SU(n, q)$. Then for $n \geq 3$, Y has a simple normal subgroup M^* , and $M^* \subseteq Y \subseteq \text{Aut}(M^*)$, where M^* as follows:

- (i) $M = Sp(4, q)$, $SU(4, q)$, $SU(5, q)$, $\Omega^-(6, q)$, $\Omega^+(8, q)$ or $\Omega^+(10, q)$.
- (ii) $M = SU(n, 2)$, $\Omega^\pm(2n, 2)$, $\Omega^\pm(2n, 3)$ or $\Omega(2n-1, 3)$.
- (iii) $M = \Omega(2n-1, 4)$ or $\Omega(2n-1, 8)$;
- (iv) $M = SU(3, 3)$;
- (v) $SU(3, 5)$;
- (vi) $SU(4, 3)$;
- (vii) $Sp(6, 2)$;
- (viii) $\Omega(7, 3)$;
- (ix) $SU(6, 2)$;

In the following, we will discuss the different possibilities of Result 4.2.2;

Lemma 4.2.3: $\text{PSp}(4, q) \subset G$, where $q = 2^k, k > 1$.

Proof: From Corollary 4.1.3, we have $\text{PSp}(8, q) \subset G$, but $\text{PSp}(4, q) \subset \text{PSp}(8, q)$, thus $M \subset G$. Which prove the point (1) of Corollary 4.2.1.

Lemma 4.2.4: If $M = \text{PSU}(4, q) \cong \text{P}\Omega^-(6, q)$, where $q = 2^k$, then $M \not\subset G$.

Proof: $\text{PSU}(4, 2) \not\subset G$ since the irreducible 2-modular characters for $\text{PSU}(4, 2)$ by GAP:

`[[1, 1], [4, 2], [6, 1], [14, 1], [20, 2], [64, 1]].` (`gap> CharacterDegrees(CharacterTable("U4(2)")mod 2);`)

And non of these characters of degree 8. But $\text{PSU}(4, 2) \subset \text{PSU}(4, q)$, where $q = 2^k$ and $\text{PSU}(4, q) \cong \text{P}\Omega^-(6, q)$, thus, $\text{PSU}(4, q) \cong \text{P}\Omega^-(6, q) \not\subset G$.

Lemma 4.2.5: $\text{PSU}(5, q) \not\subset G$.

Proof: $\text{PSU}(5, 2) \not\subset G$ since the irreducible 2-modular characters for $\text{PSU}(5, 2)$ by GAP: `[[1, 1], [5,2], [10, 2],`

`[24, 1], [40, 4], [74, 1], [160, 2], [280, 2], [1024, 1]]` (`gap> CharacterDegrees(CharacterTable("U5(2)")mod 2);`)

And non of these characters of degree 8. But $\text{PSU}(5, 2) \subset \text{PSU}(5, q)$, where $q = 2^k$, thus, $\text{PSU}(5, q) \not\subset G$.

Lemma 4.2.6: $\text{P}\Omega^+(8, q) \subset G$, where $q = 2^k$.

Proof: See Lemma 3.8. Which prove the point (2) of Corollary 4.2.1.

Lemma 4.2.7: $\text{P}\Omega^+(10, q) \not\subset G$.

Proof: $\text{P}\Omega^+(2n, q), n \geq 4, q \neq 2, 3, 5$, has no projective representation in G of degree less than $(q^{n-1} - 1)(q^{n-2} + 1)$, and $\text{P}\Omega^+(2n, q), n \geq 4, q = 2, 3$ or 5 , has no projective representation in G of degree less than $q^{n-2}(q^{n-1} - 1)$, {see [13] and [14]}, but these bounds are greater than 8 for all $n \geq 4$, thus $\text{P}\Omega^+(10, q) \not\subset G$.

Lemma 4.2.8: $\text{PSU}(6, 2) \not\subset G$.

Proof: $\text{PSU}(6, 2) \not\subset G$ since the irreducible 2-modular characters for $\text{PSU}(6, 2)$ by GAP:

`[[1, 1], [20, 1], [34, 1], [70, 2], [154, 1], [400, 1], [896, 2], [1960, 1], [3114, 1], [32768, 1]]`

(`gap> CharacterDegrees(CharacterTable("U6(2)")mod 2);`)

And non of these characters of degree 8.

Lemma 4.2.9: If $M = \text{P}\Omega^\pm(2n, 2)$, then $M \not\subset G$.

Proof: In our case $n = 8$, thus we need to consider $\text{P}\Omega^\pm(16, 2)$:

- $\text{P}\Omega^+(2n, q), n \geq 4, q = 2$ has no projective representation in G of degree less than $q^{n-2}(q^{n-1} - 1)$, {see [13] and [14]}, but this bound is greater than 8 for all $n \geq 4$ and $q = 2$, thus $\text{P}\Omega^+(12, 2) \not\subset G$.
- $\text{P}\Omega^-(2n, q), n \geq 4, q = 2$, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [14]}, but this bound is greater than 8 for all $n \geq 4$ and $q = 2$, thus $\text{P}\Omega^-(12, 2) \not\subset G$.

Lemma 4.2.10: If $M = \text{P}\Omega^\pm(2n, 3)$, then $M \not\subset G$.

Proof: In our case $n = 8$, thus we need to consider $\text{P}\Omega^\pm(16, 3)$:

- $\text{P}\Omega^+(2n, q), n \geq 4, q = 3$ has no projective representation in G of degree less than $q^{n-2}(q^{n-1} - 1)$, {see [13] and [14]}, but this bound is greater than 8 for all $n \geq 4$ and $q = 3$, thus $\text{P}\Omega^+(16, 3) \not\subset G$.
- $\text{P}\Omega^-(2n, q), n \geq 4, q = 3$, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [14]}, but this bound is greater than 8 for all $n \geq 4$ and $q = 3$, thus $\text{P}\Omega^-(16, 3) \not\subset G$.

Lemma 4.2.11: If $M = P\Omega(2n-1, 3)$, then $M \not\subset G$.

Proof: In our case $n = 8$, thus, we have $P\Omega(15, 3) \not\subset G$, since $P\Omega(2n+1, q)$, $n \geq 3$, $q = 3$, has no projective representation in G of degree less than $q^{n-1}(q^{n-1}-1)$, {[13] and [14]}, which is greater than 8 for all $n \geq 3$ and $q = 3$.

Lemma 4.2.12: If $M = P\Omega(2n-1, 4)$, then $M \not\subset G$.

Proof: In our case $n = 8$, thus we have $P\Omega(15, 4) \not\subset G$. since, $P\Omega(2n+1, q) \cong PSp(2n, q)$ for q even, then $P\Omega(15, 4) \cong PSp(14, 4)$, and $PSp(2n, q)$, $n \geq 2$, has no projective representation in G of degree less than $\frac{1}{2}(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even {see [13] and [14]}, which is greater than 8 for all $n \geq 2$ and $q = 4$.

Lemma 4.2.13: If $M = P\Omega(2n-1, 8)$, then $M \not\subset G$.

Proof: In our case $n = 8$, thus we have $P\Omega(15, 8) \not\subset G$. since, $P\Omega(2n+1, q) \cong PSp(2n, q)$ for q even, then $P\Omega(15, 8) \cong PSp(14, 8)$, and $PSp(2n, q)$, $n \geq 2$, has no projective representation in G of degree less than $\frac{1}{2}(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even {see [13] and [14]}, which is greater than 8 for all $n \geq 2$ and $q = 8$.

Lemma 4.2.14: $PSU(3, 3) \not\subset G$.

Proof: Since the irreducible 2-modular characters for $PSU(3, 3)$ by GAP are: [[1, 1], [6, 1], [14, 1], [32, 2]],
{gap> CharacterDegrees(CharacterTable("U3(3)")mod 2);} And non of these characters of degree 8.

Lemma 4.2.15: $PSU(3, 5) \not\subset G$.

Proof: Since the irreducible 2-modular characters for $PSU(3, 5)$ by GAP are:[[1, 1], [20, 1], [28, 3], [104, 1],
[144, 2]] (gap> CharacterDegrees(CharacterTable("U3(5)")mod 2);) And non of these of degree 8.

Lemma 4.2.16: $PSU(4, 3) \not\subset G$.

Proof: Since the irreducible 2-modular characters for $PSU(4, 3)$ by GAP are: [[1, 1], [20, 1], [34, 2], [70, 4], [120, 1], [640, 2], [896, 1]] (gap> CharacterDegrees(CharacterTable("U4(3)")mod 2);) And non of these of degree 8.

Lemma 4.2.17: $PSp(6, 2) \subset G$.

Proof: $PSp(6, 2) \subset G$ since the irreducible 2-modular characters for $PSp(6, 2)$ by GAP are: [[1, 1], [6, 1], [8, 1],
[14, 1], [48, 1], [64, 1], [112, 1], [512, 1]] (gap> CharacterDegrees(CharacterTable("S6(2)")mod 2);)

Which prove the point (3) of Corollary 4.2.1.

Lemma 4.2.18: $P\Omega(7, 3) \not\subset G$.

Proof: $P\Omega(7, 3) \not\subset G$, since $P\Omega(2n+1, q)$, $n \geq 3$, $q = 3$, has no projective representation in G of degree less than $q^{n-1}(q^{n-1}-1)$, {see [13] and [14]}, which is greater than 8 for all $n \geq 3$ and $q = 3$.

Now, we will determine the maximal primitive group of C_9 :

Theorem 4.2: If H is a maximal primitive subgroup of G which has the property that a minimal normal subgroup M of H is not abelian group, then H is isomorphic to one of the following subgroups of G :

- (i) $PSU(8, q')$ where $q' = 2^{k'}$ and k' is a prime number divides k ;
- (ii) $PSp(8, q)$.

Proof: We will prove this theorem by finding the normalizers N of the groups of **Corollary 4.1** and determine which of them are maximal:

From [5], the normalizer of $Sp(2n, k)$ in $SL(2n, k)$ is $SGSp(2n, k) = GSp(2n, k) \cap SL(2n, k)$. From [4], the normalizer of $SU(n, k)$ in $SL(n, k)$ is $SGU(n, k) = GU(n, k) \cap SL(n, k)$. From [10], the normalizer of $SO(n, k)$ in $SL(n, k)$ is $SGO(n, k) = GO(n, k) \cap SL(n, k)$. Thus,

- If $Y = PSU(8, q')$ where $q' = 2^k$ and k' is a prime number divides k , then $N = PSGU(8, q)$ but in $PSU(8, q)$, $PSGU(8, q') = PSU(8, q')$, which prove the point (i) of this theorem. But this result is the same result H_{12} that was proved by Lemma 3.5.
- If $Y = PSp(8, q)$, then $N = PSGSp(8, q)$ but in $PSU(8, q)$, $PSGSp(8, q) = PSp(8, q)$, which prove the point (ii) of this theorem. But this result is the same result H_{14} that was proved by Lemma 3.8.
- If $Y = PSO^-(8, q)$, then $N = PSGO^-(8, q)$, but in $PSU(8, q)$, $PSGO^-(8, q) = PSO^-(8, q)$, in this case Y is a subgroup of $PSp(8, q)$, thus Y is not a maximal subgroup of G .
- If $Y = PSO^+(8, q)$, then $N = PSGO^+(8, q)$, but in $PSU(8, q)$, $PSGO^+(8, q) = PSO^+(8, q)$, in this case Y is a subgroup of $PSp(8, q)$, thus Y is not a maximal subgroup of G .
- If $Y = PSp(4, q)$, where $q = 2^k$ and $k > 1$, then $N = PSGSp(4, q)$ but in $PSU(8, q)$, $PSGSp(4, q) = PSp(4, q)$, in this case Y is a subgroup of $PSp(8, q)$, thus Y is not a maximal subgroup of G .
- If $Y = P\Omega^+(8, q)$, where $q = 2^k$, then the normalizer in G is $N = PSGO^+(8, q)$, but in $PSU(8, q)$, $PSGO^+(8, q) = PSO^+(8, q)$, in this case Y is a subgroup of $PSp(8, q)$, thus Y is not a maximal subgroup of G .
- If $Y = PSp(6, 2)$, then $N = PSGSp(6, 2)$ but in $PSU(6, q)$, $PSGSp(6, 2) = PSp(6, 2)$, in this case Y is a subgroup of $PSp(8, q)$, thus Y is not a maximal subgroup of G .

This completes the proof of the main theorem 1.1.

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