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INEQUALITIES CONCERNING THE B-OPERATORS

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ABSTRACT

In this paper we consider an operator B which carries a polynomial P(z) of degree n into $B[P(z)]= \lambda_0 P(z) + \lambda_1 (nz/2)P'(z)/1! + \lambda_2 (nz/2)^2 P''(z)/2!$ *Where* λ_0, λ_1 *and* λ_2 *are such that all the zeros of* $U(z)= \lambda_0 + C(n, 1)\lambda_1 z + C(n, 2)\lambda_2 z^2$ *lie in the half plane* $|z|\leq|z-n/2|$ *and investigate the dependence of* $|B[P(Rz)] - \alpha B[P(rz)]|$ *on the minimum and the maximum modulus of P(z) on* $|z|=1$ *for every real or complex number α with* $|\alpha|\leq 1$, $R > r \geq 1$ *with restriction on the zeros of the polynomial P(z) and establish some new operator preserving inequalities between polynomials.*

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1. INTRODUCTION TO THE STATEMENT OF RESULTS.

Let $P_n(z)$ denote the space of all complex polynomials $P(z)=\sum_{j=1}^{n} a_j z^j$ of degree n. If $P \in P_n$, then according to a famous result known as Bernstein's inequality (for reference see[4, 7,10]),

(1) $$\underset{|z|=1}{Max}|P^{/}(z)| \leq n\underset{|z|=1}{Max}|P(z)|$$

where as concerning the maximum modulus of P(z) on a larger circle |z| = R >1, we have

(2) $$\underset{|z|=R>1}{Max}|P(z)| \leq R^n\underset{|z|=1}{Max}|P(z)|$$

(for reference see [8, p. 158 problem 269] or [11, p. 346]) Equality in (1) and (2) holds for $P(z)=\lambda z^n, \lambda \neq 0$.

For the class of polynomials $P \in P_n$ having all their zero in $|z| \leq 1$, we have

(3) $$\underset{|z|=1}{Min}|P^{/}(z)| \geq n\underset{|z|=1}{Min}|P(z)|$$

and

(4) $$\underset{|z|=R>1}{Min}|P(z)| \geq R^n\underset{|z|=1}{Min}|P(z)|.$$

Inequalities (3) and (4) are due to A. Aziz and Q. M. Dawood [2]. Both the results are sharp and equality in (3) and (4) holds for $P(z)=\lambda z^n, \lambda \neq 0$. For the class of polynomials $P \in P_n$ having no zero in $|z|<1$, we have

(5) $$\underset{|z|=1}{Max}|P^{/}(z)| \leq \frac{n}{2}\underset{|z|=1}{Max}|P(z)|$$

and

(6) $$|P(z)| \leq \frac{R^n+1}{2}\underset{|z|=1}{Max}|P(z)|.$$

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Equality in (5) and (6) holds for $P(z) = \lambda z^n + \mu,\ |\lambda| = |\mu| = 1.$ Inequality (5) was conjectured by P. Erdös and later verified by P. D. Lax [5]. Ankeny and Rivilin [1] used (5) to prove (6).

A. Aziz and Q.M. Dawood [2] improved inequalities (5) and (6) by showing that if $P(z) \neq 0$ in |z| < 1, then

(7) $\underset{|z|=1}{Max} |P'(z)| \leq \frac{n}{2}\left(\underset{|z|=1}{Max}|P(z)| - \underset{|z|=1}{Min}|P(z)| \right)$

and

(8) $\underset{|z|=R>1}{Max} |P(z)| \leq \frac{R^n+1}{2} \underset{|z|=1}{Max} |P(z)| - \frac{R^n-1}{2} \underset{|z|=1}{Min} |P(z)|.$

As a compact generalization of inequalities (5) and (6), Aziz and Rather [3] have shown that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for every real or complex number α with |α|≤1 and $R \geq 1$,

(9) $|P(Rz) - \alpha P(z)| \leq \frac{1}{2}\left\{ R^n - |\alpha| |z|^n + |1-\alpha| \right\} \underset{|z|=1}{Max}|P(z)|$ for $|z| \geq 1.$

The result is sharp and equality in (7) holds for $P(z) = az^n + b,\ |a| = |b| = 1$.

Rahman [9] (see also Rahman and Schmeisser[10, p.538]) introduced a class B_n of operators B that carries a polynomial $P \in P_n$ into

(10) $B[P(z)] := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2} \right) \frac{P'(z)}{1\,!} + \lambda_2 \left(\frac{nz}{2} \right)^2 \frac{P''(z)}{2\,!}$

where λ_0, λ_1 and λ_2 are such that all the zeros of

(11) $u(z) = \lambda_0 + \lambda_1 C(n,1)z + \lambda_2 C(n,2)z^2\ ,\ C(n,r) = n!/r!(n-r)!,\ 0 \leq r \leq n,$

lie in the half plane

(12) $|z| \leq |z - n/2|.$

As a generalization of the inequalities (1) and (2), Q.I. Rahman [9] proved that if $P \in P_n$, then

(13) $|B[P(z)]| \leq |B[z^n]| \underset{|z|=1}{Max}|P(z)|$ for |z| ≥ 1

(see [9],inequality (5.1)) and if $P \in P_n,\ P(z) \neq 0$ for $|z| < 1$, then

(14) $|B[P(z)]| \leq \frac{1}{2}\left\{ |B[z^n]| + |\lambda_0| \right\} \underset{|z|=1}{Max}|P(z)|$ for |z| ≥ 1,

where $B \in B_n$ (see [8], inequality (5.2) and (5.3)).

In this paper we investigate the dependence of $|B[P(Rz)] - \alpha B[P(rz)]|$ on the minimum and the maximum of modulus of P(z) on $|z| = 1$ for every real or complex number α with |α|≤ 1, $R > r \geq 1$ and obtain certain compact generalizations of some well-known polynomial inequalities. In this direction we first present the following interesting result which is a compact generalization of inequalities (1), (2) and (13).

Theorem 1: *If $F \in P_n$ has all its zeros in |z| ≤ 1 and P(z) is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z|=1,$$

then for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$

(15) $\left|B[P(Rz)]-\alpha B[P(rz)]\right| \le \left|B[F(Rz)]-\alpha B[F(rz)]\right|$ for $|z| \ge 1$,

where $B \in B_n$.

The following result immediately follows from Theorem 1 by taking $F(z) = Mz^n$ where $M = \underset{|z|=1}{Max} |P(z)|$.

Corollary 1: *If* $P \in P_n$*, then for every real or complex number α with* $|\alpha| \le 1, R > r \ge 1$,

(16) $\left|B[P(Rz)]-\alpha B[P(rz)]\right| \le \left|R^n - \alpha r^n\right| \left|B[z^n]\right| \underset{|z|=1}{Max}|P(z)|$ for $|z| \ge 1$

where $B \in B_n$ *.The result is best possible and equality in (16) holds for* $P(z) = az^n$, $a \ne 0$.

Remark 1: For α = 0, Corollary 1 reduces to the inequality (13). Next if we choose $\lambda_1 = \lambda_2 = 0$ in (16) and note that in this case all the zeros of u(z) defined by (11) lie in region defined by (12), we obtain for every real or complex number α with $|\alpha| \le 1, R > r \ge 1$,

(17) $\left|P(Rz)-\alpha P(rz)\right| \le \left|R^n - \alpha r^n\right| |z|^n \underset{|z|=1}{Max}|P(z)|$ for $|z| \ge 1$.

For α = 0, inequality (17) includes inequality (2) as a special case. Further, if we divide both sides of the inequality (17) by R - r with α = 1 and make R →r, we get

$$\left|P'(rz)\right| \le nr^{n-1}|z|^{n-1} \underset{|z|=1}{Max}|P(z)| \quad \text{for} \quad |z| \ge 1,$$

which, in particular, yields inequality (1) as a special case.

Next we present the following result, which is a compact generalization of the inequalities (3) and (4) .

Theorem 2:. *If P ∈ P_n and P(z) has all its zeros in |z|≤ 1, then for every real or complex number α with* $|\alpha| \le 1$ *and* $R > r \ge 1$

(18) $\left|B[P(Rz)]-\alpha B[P(rz)]\right| \ge \left|R^n - \alpha r^n\right| \left|B[z^n]\right| \underset{|z|=1}{Min}|P(z)|$ for $|z| \ge 1$,

where $B \in B_n$ *. The result is best possible and equality in (18) holds for* $P(z) = az^n$, $a \ne 0$.

Remark 2: For $\alpha = 0$, from inequality (18), we have for $|z| \ge 1$ and $R > 1$,

(19) $\left|B[P(Rz)]\right| \ge R^n \left|B[z^n]\right| \underset{|z|=1}{Min}|P(z)| = \left|B[R^n z^n]\right| \underset{|z|=1}{Min}|P(z)|$,

where $B \in B_n$ .The result is sharp.

Next, taking $\lambda_0 = \lambda_2 = 0$ in (18) and noting that all the zeros of $u(z)$ defined by (11) lie in the half plane (12), we get

Corollary 2: *If* $P \in P_n$ *has all its zeros in |z| ≤1, then for every real or complex number α with* $|\alpha| \le 1, R > r \ge 1$,

(20) $\left|RP'(Rz)-\alpha rP'(rz)\right| \ge n\left|R^n - \alpha r^n\right| |z|^n \underset{|z|=1}{Min}|P(z)|$ for $|z| \ge 1$.

The result is sharp and the extremal polynomial is $P(z) = \lambda z^n, \lambda \ne 0$

If we divide the two sides of (20) by R - r with α = 1 and let R → r, we get for |z| ≥1,

$$\left|P'(rz)+rzP''(z)\right| \geq n^2 r^{n-1}|z|^{n-1} \underset{|z|=1}{Min}|P(z)|.$$

The result is sharp.

For the choice $\lambda_1=\lambda_2=0$ in (18), we obtain for every real or complex number α with $|\alpha|\leq 1, R>r\geq 1$,

$$(21)\quad \left|P(Rz)-\alpha P(rz)\right| \geq \left|R^n-\alpha r^n\right||z|^n \underset{|z|=1}{Min}|P(z)| \quad \text{for} \quad |z|\geq 1.$$

For α = 0, inequality (21) includes inequality (4) as a special case. If we divide both sides of the inequality (21) by R - r with α =1 and make R →r, we get

$$(22)\quad \left|P'(rz)\right| \geq nr^{n-1}|z|^{n-1} \underset{|z|=1}{Min}|P(z)| \quad \text{for} \quad |z|\geq 1,$$

which, in particular, yields inequality (3) as a special case.

Corollary 1 can be sharpened if we restrict ourselves to the class of polynomials $P\in P_n$, having no zero in |z| < 1. In this direction, we next present the following compact generalization of the inequalities (7), (8) and (9), which also include refinements of the inequalities (13) and (14) as special cases.

Theorem 3: *If* $P\in P_n$ *and* $P(z)\neq 0$ *for* $|z|<1$, *then for every real or complex number α with* $|\alpha|\leq 1, R>r\geq 1$ *and* $|z|\geq 1$,

(23)

$$\left|B[P(Rz)]-\alpha B[P(rz)]\right| \leq \frac{1}{2}\left[\left\{\left|R^n-\alpha r^n\right|\left|B[z^n]\right|+|1-\alpha||\lambda_0|\right\}\underset{|z|=1}{Max}|P(z)|-\left\{\begin{matrix}\left|R^n-\alpha r^n\right|\left|B[z^n]\right| \\ -|1-\alpha||\lambda_0|\end{matrix}\right\}\underset{|z|=1}{Min}|P(z)|\right]$$

where $B\in B_n$ .The result is sharp and equality in (23) holds for $P(z)=az^n+b,\ |a|=|b|=1.$

Remark 3: For α = 0, inequality (23) yields refinement of Inequality (14). If we choose $\lambda_0=\lambda_2=0$ in (23) and note that all the zeros of u (z) defined by (11) lie in the half plane defined by (12), we get for $|z|\geq 1, R>r\geq 1$ and |α|≤1,

$$(24)\quad \left|RP'(Rz)-\alpha rP'(rz)\right| \leq \frac{n}{2}\left|R^n-\alpha r^n\right||z|^{n-1}\left(\underset{|z|=1}{Max}|P(z)|-\underset{|z|=1}{Min}|P(z)|\right).$$

Setting α = 0 in (24), we obtain for |z|≥1 and R>1,

$$\left|P'(Rz)\right| \leq \frac{n}{2}R^{n-1}|z|^{n-1}\left(\underset{|z|=1}{Max}|P(z)|-\underset{|z|=1}{Min}|P(z)|\right)$$

which ,in particular, gives inequality (7).

Next choosing $\lambda_1=\lambda_2=0$ in (23), we immediately get the following result, which is a refinement of inequality (9).

Corollary 3: *If* $P\in P_n$ *and* $P(z)\neq 0$ *for* $|z|<1$, *then for every real or complex number α with* $|\alpha|\leq 1, R>r\geq 1$ *and* $|z|\geq 1$,

$$(25)\quad \left|P(Rz)-\alpha P(rz)\right| \leq \frac{1}{2}\left[\left\{\left|R^n-\alpha r^n\right||z|^n+|1-\alpha|\right\}\underset{|z|=1}{Max}|P(z)|-\left\{\left|R^n-\alpha r^n\right||z|^n-|1-\alpha|\right\}\underset{|z|=1}{Min}|P(z)|\right].$$

The result is sharp and equality in (25) holds for $P(z)=az^n+b, |a|=|b|=1$. Inequality (25) is a compact generalization of the inequalities (7) and (8).

2. LEMMAS

For the proofs of these theorems, we need the following lemmas.

Lemma 1: If $P \in P_n$ *and* $P(z)$ *has all its zeros in* $|z| \le k$ where $k \le 1$, *then for every* $R \ge r \ge 1$ and $|z| = 1$,

(26) $| P(Rz)| \ge \left(\dfrac{R+k}{r+k}\right)^n | P(rz)|.$

Proof of Lemma 1: Since all the zeros of P(z) lie in $|z| \le k$ where $k \le 1$, we write

$$P(z) = Cos\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right),$$

where $r_j \le k$, $j = 1, 2, \cdots, n$. Now for $0 \le \theta < 2\pi$, $R \ge r \ge 1$, we have

$$\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| = \left\{\frac{R^2 + r_j^2 - 2Rr_j Cos(\theta - \theta_j)}{r^2 + r_j^2 - 2rr_j Cos(\theta - \theta_j)}\right\}^{1/2} \ge \left(\frac{R + r_j}{r + r_j}\right) \ge \left(\frac{R+k}{r+k}\right).$$

Hence

$$\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| \ge \left(\frac{R+k}{r+k}\right)^n$$

for $0 \le \theta < 2\pi$, which implies for |z|= 1 and $R \ge r \ge 1$,

$$| P(Rz)| \ge \left(\frac{R+k}{r+k}\right)^n | P(rz)|.$$

This completes the proof of Lemma 1.

The next lemma follows from Corollary 18.3 of [6 , p. 65].

Lemma 2: If $P \in P_n$ *and* $P(z)$ *has all its zeros in* $|z| \le 1$, *then all the zeros of B[P(z)] also lie in* $| z | \le 1$.

Lemma 3: If $P \in P_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for every real or complex number* α *with* $| \alpha | \le 1$, $R > r \ge 1$, *and* $| z | = 1$,

(27) $\left|B[P(Rz)] - \alpha B[P(rz)]\right| \le \left|B[Q(Rz)] - \alpha B[Q(rz)]\right|$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.*The result is sharp and equality in* (27) *holds for* $P(z) = az^n + b$, |a|=|b|=1.

Proof of Lemma 3: Let $Q(z) = z^n \overline{P(1/\bar{z})}$. Since all the zeros of nth degree polynomial $P(z)$ lie in $| z | \ge 1$, therefore, Q(z) is a polynomial of degree n having all its zeros in $| z | \le 1$. Applying Theorem 1 with F(z) replaced by $Q(z)$, we obtain for every $R > r \ge 1$ and |z|≥ 1,

(28) $| B[P(Rz)] - \alpha B[P(rz)]| \le | B[Q(Rz)] - \alpha B[Q(rz)]|.$

This proves Lemma 3.

Lemma 4: *If* $P \in P_n$, *then for every real or complex number α with* $|\alpha| \le 1, R > r \ge 1$ *and* $|z| \ge 1$,

(29) $\left|B[P(Rz)]-\alpha B[P(rz)]\right|+\left|B[Q(Rz)]-\alpha B[Q(rz)]\right|\le\left\{\left|R^n-\alpha r^n\right|\left|B[z^n]\right|+\left|1-\alpha\right|\left|\lambda_0\right|\right\}\underset{|z|=1}{Max}\left|P(z)\right|$

where $Q(z)=z^n\overline{P(1/\overline{z})}$ *. The result is sharp and equality in (29) holds for* $P(z)=\lambda z^n$ *, α≠0.*

Proof of Lemma 4: Let $M=\underset{|z|=1}{Max}|P(z)|$, then $|P(z)|\le M$ for $|z|=1$. If μ is any real or complex number with $|\mu|>1$, then by Rouche's theorem , the polynomial $F(z)=P(z)-\mu M$ does not vanish in $|z|<1$. Applying Lemma 3 to the polynomial $F(z)$ and using the fact that B is a linear operator, it follows that for every real or complex number α with $|\alpha|\le 1$, $R>r\ge 1$,

$$|B[F(Rz)]-\alpha B[F(rz)]|\le|B[H(Rz)]-\alpha B[H(rz)]| \qquad \text{for } |z|\ge 1,$$

where

$$H(z)=z^n\overline{F(1/\overline{z})}=z^n\overline{P(1/\overline{z})}-\overline{\mu}Mz^n=Q(z)-\overline{\mu}Mz^n.$$

Again using the linearity of B and the fact $B[1]=\lambda_0$, we obtain

(30) $\left|\left(B[P(Rz)]-\alpha B[P(rz)]\right)-\mu(1-\alpha)\lambda_0 M\right|\le\left|\left(B[Q(Rz)]-\alpha B[Q(rz)]\right)-\overline{\mu}\left(R^n-\alpha r^n\right)B[z^n]M\right|$

for every real or complex number α with $|\alpha|\le 1$, $R>r\ge 1$ and $|z|\ge 1$. Now choosing the argument of μ on the right hand side of (30) such that

$$\left|\left(B[Q(Rz)]-\alpha B[Q(rz)]\right)-\overline{\mu}\left(R^n-\alpha r^n\right)B[z^n]M\right|=|\mu|\left|R^n-\alpha r^n\right|\left|B[z^n]\right|M-\left|B[Q(Rz)]-\alpha B[Q(rz)]\right|,$$

which is possible by Corollary 1, we get, from (30),

(31) $\left|B[P(Rz)]-\alpha B[P(rz)]\right|-|\mu||1-\alpha|\,|\lambda_0|\,M\le|\mu|\left|R^n-\alpha r^n\right|\left|B[z^n]\right|M-\left|B[Q(Rz)]-\alpha B[Q(rz)]\right|$

for $|\alpha|\le 1$, $R>r\ge 1$ and $|z|\ge 1$. Letting $|\mu|\to 1$ in (31), we obtain

$$\left|B[P(Rz)]-\alpha B[P(rz)]\right|+\left|B[Q(Rz)]-\alpha B[Q(rz)]\right|\le\left\{\left|R^n-\alpha r^n\right|\,\left|B[z^n]\right|+|\lambda_0||1-\alpha|\,|\lambda_0|\right\}M\ .$$

This proves Lemma 4.

2. PROOFS OF THE THEOREM

Proof of Theorem 1: By hypothesis $F(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$ and $P(z)$ is a polynomial of degree at most n such that

(32) $|P(z)|\le|F(z)|$ for $|z|=1$,

Therefore, if $F(z)$ has a zero of multiplicity m at $z=e^{i\theta_0}$, then $P(z)$ must have a zero of multiplicity at least m at $z=e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (15) is obvious. We assume that $P(z)/F(z)$ is not a constant, so that by maximum modulus principle , it follows that

$$|P(z)|<|F(z)| \qquad \text{for } |z|>1.$$

Suppose $F(z)$ has m zeros on $|z|=1$ where $0\le m\le n$ so that we write $F(z)=F_1(z)F_2(z)$ where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z|=1$ and $F_2(z)$ is a polynomial of degree exactly n – m having all its zeros in $|z|<1$. This gives with the help of inequality (32) that

$$P(z)=P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most n – m. Now, from inequality (32), we get

$$|P_1(z)| \le |F_2(z)| \qquad \text{for} \quad |z| = 1$$

where $F_2(z) \ne 0$ for |z| = 1. Therefore, for every real or complex number λ with | λ|> 1, a direct application of Rouche's theorem shows that all the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree n – m ≥ 1 lie in |z| < 1. Hence the polynomial

$$G(z) = F_1(z)\big(P_1(z) - \lambda F_2(z)\big) = P(z) - \lambda F(z)$$

has all its zeros in |z| ≤ 1 with at least one zero in |z| < 1, so that we can write

$$G(z) = (z - te^{i\beta})H(z)$$

where $t < 1$ and H(z) is a polynomial of degree n -1 having all its zeros in |z|≤ 1. Hence with the help of Lemma 1 with k = 1, we obtain for every $R > r \ge 1$ and $0 \le \theta < 2\pi$,

$$\begin{aligned}
|G(Re^{i\theta})| &= |Re^{i\theta} - te^{i\beta}|\ |H(Re^{i\theta})| \\
&\ge |Re^{i\theta} - te^{i\beta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})| \\
&= \left(\frac{R+1}{r+1}\right)^{n-1} \left|\frac{Re^{i\theta} - te^{i\beta}}{re^{i\theta} - te^{i\beta}}\right| |(re^{i\theta} - te^{i\beta})H(re^{i\theta})| \\
&\ge \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |G(re^{i\theta})|.
\end{aligned}$$

This implies for R > r ≥ 1 and $0 \le \theta < 2\pi$,

$$(33) \quad \left(\frac{r+t}{R+t}\right) |G(Re^{i\theta})| \ge \left(\frac{R+1}{r+1}\right)^{n-1} |G(re^{i\theta})|.$$

Since R > r ≥ 1 > t so that $G(Re^{i\theta}) \ne 0$ for $0 \le \theta < 2\pi$ and $1 > \frac{1+r}{1+R} > \frac{r+t}{R+t}$, from inequality (33), we obtain

$$|G(Re^{i\theta})| > \left(\frac{R+1}{r+1}\right)^{n} |G(re^{i\theta})|$$

for $R > r \ge 1$ and $0 \le \theta < 2\pi$, which leads to

$$|G(re^{i\theta})| < \left(\frac{r+1}{R+1}\right)^{n} |G(Re^{i\theta})| < |G(Re^{i\theta})|$$

for $0 \le \theta < 2\pi$ and $R > r \ge 1$. Equalivalently, we have

$$(34) \quad |G(rz)| < |G(Rz)| \quad \text{for} \quad |z| = 1 \text{ and } R > r \ge 1.$$

Since all the zeros of $G(Rz)$ lie in |z| ≤ (1/R) < 1, a direct application of Rouche's theorem shows that the polynomial $G(Rz) - \alpha G(rz)$ has all its zeros in |z| < 1 for every real or complex number α with |α| ≤ 1. Applying Lemma 2 and using the linearity of B, it follows that all the zeros of the polynomial

$$T(z) = B[G(Rz) - \alpha G(rz)] = B[G(Rz) - \alpha B[G(rz)]$$

lie in |z| < 1 for every real or complex number α with |α|≤ 1 and $R > r \geq 1$. Replacing G(z) by P(z) – λF(z), we conclude that all the zeros of the polynomial

(35) $T(z) = \left(B[P(Rz)] - \alpha B[P(rz)]\right) - \lambda\left(B[F(Rz) - \alpha B[F(rz)]\right)$

lie in |z| < 1 for all real or complex numbers α, λ with |α| ≤ 1, |λ| >1 and $R > r \geq 1$. This implies

(36) $\left|B[P(Rz)] - \alpha B[P(rz)]\right| \leq \left|B[F(Rz) - \alpha B[F(rz)]\right|$ for |z| ≥ 1 and R > r ≥ 1.

If inequality (36) is not true , then there a point z = w with |w| ≥ 1 such that

$$\left|\left(B[P(Rz)] - \alpha B[P(rz)]\right)_{z=w}\right| > \left|\left(B[F(Rz) - \alpha B[F(rz)]\right)_{z=w}\right| , \qquad R > r \geq 1.$$

Since all the zeros of F(z) lie in |z| ≤ 1, it follows (as in the case of G(z)) that all the zeros of $B[F(Rz) - \alpha B[F(rz)]$ lie in |z| ≤ 1. Hence

$$\left(B[F(Rz) - \alpha B[F(rz)]\right)_{z=w} \neq 0, \quad R > r \geq 1.$$

We choose

$$\lambda = \frac{\left(B[P(Rz)] - \alpha B[P(rz)]\right)_{z=w}}{\left(B[F(Rz)] - \alpha B[F(rz)]\right)_{z=w}}$$

so that λ is well defined real or complex number with |λ| > 1, and with choice of λ, from (35), we get, T(w) = 0 with |w| ≥ 1. This is clearly a contradiction to the fact that all the zeros of T(z) lie in |z| < 1. Thus for every real or complex number α with |α| ≤ 1 and R > r ≥ 1,

$$\left|B[P(Rz)] - \alpha B[P(rz)]\right| \leq \left|B[F(Rz) - \alpha B[F(rz)]\right|.$$

This completes the proof of Theorem 1.

Proof of Theorem 2: The result is clear if P(z) has a zero on |z| = 1, for then $m = \underset{|z|=1}{Min} | P(z) | = 0$. We now assume that P(z) has all its zeros in |z| < 1 so that m > 0 and

$$m \leq |P(z)| \qquad \text{for } |z|=1.$$

This gives for every λ with |λ|< 1,

$$|\lambda z^n| m < |P(z)| \qquad \text{for } |z|=1$$

By Rouche's theorem , it follows that all the zeros of polynomial $F(z) = P(z) - \lambda m z^n$ lie in |z|< 1 for every real or complex number λ with | λ |< 1. Therefore, (as before) we conclude that all the zeros of polynomial $G(z) = F(Rz) - \alpha F(rz)$ lie in |z| < 1 for every real or complex α with number |α|≤ 1 and R > r≥ 1 . Hence by Lemma 2, all the zeros of the polynomial

(37) $S(z) = B[G(z)] = B[F(Rz)] - \alpha B[F(rz)]$

$$= B[P(Rz)] - \alpha B[P(rz)] - \lambda(R^n - \alpha r^n)B[z^n]m$$

lie in |z| < 1 for all real or complex numbers α, λ with |α| ≤ 1, | λ |< 1 and R > r ≥ 1.This implies

(38) $\left|B[P(Rz)] - \alpha B[P(rz)]\right| \geq \left|R^n - \alpha r^n\right| \left|B[z^n]\right| m$ for |z|≥ 1 and R >r ≥1.

If inequality (38) is not true, then there is a point z = w with |w| ≥ 1 such that

$$\left|\left\{B[P(Rz)]-\alpha B[P(rz)]\right\}_{z=w}\right| < \left|R^n-\alpha r^n\right| \left|\left\{B[z^n]\right\}_{z=w}\right| m .$$

Since $\left\{B[z^n]\right\}_{z=w} \neq 0$, we take

$$\lambda = \left\{B[P(Rz)]-\alpha B[P(rz)]\right\}_{z=w} / m\left(R^n-\alpha r^n\right)\left\{B[z^n]\right\}_{z=w}$$

so that λ is a well defined real or complex number with | λ |< 1 and with choice of λ, from (37), we get S(w) = 0 with |w|≥ 1. This contradicts the fact that all the zeros of S(z) lie in |z| < 1. Thus for every real or complex number α with |α|≤1 and R > r ≥1,

$$\left|B[P(Rz)]-\alpha B[P(rz)]\right| \geq \left|R^n-\alpha r^n\right| \left|B[z^n]\right| m \qquad \text{for} \quad |z|\geq 1 .$$

This completes the proof of Theorem 2.

Proof of Theorem 3: By hypothesis , the polynomial P(z) does not vanish in |z|< 1, therefore, if $m = \underset{|z|=1}{Min} |P(z)|$, then $m \leq |P(z)|$ for |z|≤ 1. We first show that for every real or complex number δ with |δ|≤ 1 , the polynomial $F(z) = P(z) + m\delta z^n$ does not vanish in |z| < 1. This is obvious if m = 0 and for m > 0, we prove it by a contradiction. Assume that *F*(z) has a zero in |z| < 1 say at z = w with |w| < 1,
then we have $P(w) + m\delta\ w^n = F(w) = 0.$ This gives

$$|P(w)| = |m\delta\ w^n| \leq m|w|^n < m,$$

which is clearly a contradiction(to the minimum modulus principle). Hence F(z) has no zero in |z| < 1 for every δ with |δ| ≤ 1. Applying Lemma 3 to the polynomial F(z), we obtain for every real or complex number α with number |α|≤ 1 and R > r ≥ 1,

$$|B[F(Rz)]-\alpha B[F(rz)]| \leq |B[G(Rz)]-\alpha B[G(rz)]|, \quad |z| \geq 1,$$

where $G(z) = z^n\overline{F(1/\overline{z}\ )} = z^n\overline{P(1/\overline{z})} - m\overline{\delta} = Q(z) - m\overline{\delta}$. Equivalently,

$$\text{(39)} \quad \left|B[P(Rz)]-\alpha B[P(rz)]-m\delta\left(R^n-\alpha r^n\right)B[z^n]\right| \leq \left|B[Q(Rz)]-\alpha B[Q(rz)]-m\overline{\delta}(1-\alpha)\lambda_0\right|$$

for all real or complex numbers α, δ with number |α|≤ 1, |δ|≤ 1 and R > r ≥ 1. Now choosing the argument of δ such that

$$\left|B[P(Rz)]-\alpha B[P(rz)]-m\delta\left(R^n-\alpha r^n\right)B[z^n]\right| = \left|B[P(Rz)]-\alpha B[P(rz)]\right| + m|\delta|\,|1-\alpha|\,|B[z^n]|\,|,$$

We obtain from (39), for |α|≤ 1, |δ| ≤ 1 and R > r ≥ 1,

$$|B[P(Rz)]-\alpha B[P(rz)]| + m|\delta|\,|R^n-\alpha r^n|\,|B[z^n]| \leq \left|B[Q(Rz)]-\alpha B[Q(rz)]\right| + m|\delta|\,|1-\alpha|\,|\lambda_0|,$$

for |z| ≥ 1, or equivalently,

$$\left|B[P(Rz)]-\alpha B[P(rz)]\right| + |\delta|\left(|R^n-\alpha r^n|\,|B[z^n]| - |1-\alpha|\,|\lambda_0|\right)m \leq \left|B[Q(Rz)]-\alpha B[Q(rz)]\right|,$$

for |α|≤ 1, |δ|≤ 1 and R > r ≥ 1. Letting |δ| → 1, we get

$$\left|B[P(Rz)]-\alpha B[P(rz)]\right| + \left(|R^n-\alpha r^n|\,|B[z^n]| - |1-\alpha|\,|\lambda_0|\right)m \leq \left|B[Q(Rz)]-\alpha B[Q(rz)]\right|,$$

for $|\alpha| \leq 1$ and $R > r \geq 1$. Combining this inequality with Lemma 4, we get , for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} 2\left|B[P(Rz)] - \alpha B[P(rz)]\right| + \left(| R^n - \alpha r^n || B[z^n] | - | 1-\alpha || \lambda_0 |\right)m \\ &\leq \left|B[P(Rz)] - \alpha B[P(rz)]\right| + \left|B[Q(Rz)] - \alpha B[Q(rz)]\right| \\ &\leq \left(\left|R^n - \alpha r^n\right| \left|B[z^n]\right| + \left|1-\alpha\right| \left|\lambda_0\right|\right)M , \end{aligned}$$

which is equivalent to (23) and this completes the proof of Theorem 3.

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