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On Continued Fractions of Period Five and Real Quadratic Fields of Class Number Even

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ABSTRACT

In this paper, by using the methods of T. Azuhata in $[1]$ *and K. Tomita in* $[6]$, *in the case of* $k_d = l(w_d) = 5$ *for* $d = a^2 + b \equiv 2(\bmod 4)$ *where* $(a, b \in Z^+, 0 \langle b \le 2a)$ *that is not investigated in the papers of R.A. Mollin in* $[2]$ *and K. Tomita in* $[6]$, *the general forms of the continued fraction expansions of* $w_d = \sqrt{d}$ *and* t_d, u_d *explicitly in the fundamental unit* $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right) \rangle 1$ *of* $Q\left(\sqrt{d}\right)$ *are determined.Furthermore, the necessary and suffficent conditions are given for Yokoi's invariant value of* n_d *which is defined in terms of coefficent of fundamental unit. Also, it is denoted that the class number* h_d *is always even.Finally, the real quadratic fields* $Q\left(\sqrt{d}\right)$ *with* $d \equiv 2(\bmod 4)$ *and* $h_d = 2$ *are given in the Table 3.1.*

Mathematics Subject Classification: *11A55, 11R11, 11R27,11R29.*

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1. INTRODUCTİON AND NOTATİONS:

K. Tomita in $[6]$, for all real quadratic fields $Q\left(\sqrt{d}\right)$ such that the period k_d of continued fraction expansions $w_d = \dfrac{1+\sqrt{d}}{2}$ is equal to 5 $(i.e; in\, the\, case\, of\;\; d \equiv 1(\bmod 4))$, determined t_d, u_d explicity and uniformily in the fundamental unit $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right) \rangle 1$ of $Q\left(\sqrt{d}\right)$ and general forms of continued fraction expansions of $w_d = \dfrac{1+\sqrt{d}}{2}$.Also, he described d itself by using at most four parameters appering in the continued fraction expansions and gived some results on Yokoi's invariant value of n_d, m_d by connected with class number one problem.

Furthermore, R. A. Mollin in $[2]$, in the case of $k_d = l(w_d) = 5$ for $d \equiv 1(\bmod 4)$, described some results on all real quadratic fields $Q\left(\sqrt{d}\right)$ of class number one by using a specific Rabinowitch polynomial.

In this article, by using the methods of T. Azuhata in $[1]$ and K. Tomita in $[6]$, in the case of $k_d = l(w_d) = 5$ for $d = a^2 + b \equiv 2(\bmod 4)$ where $a, b \in Z^+$, $0 \langle b \le 2a$ (Since $k_d = 5$ odd integer, d is not equal to 3 modulo 4 which is showed in section 3 of this article) that is not investigated in the papers of R.A. Mollin in $[2]$ and K. Tomita in $[6]$, the general forms of the continued fraction expansions of $w_d = \sqrt{d}$ and t_d, u_d explicitly in the fundamental unit $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right) \rangle 1$ of $Q\left(\sqrt{d}\right)$ are determined, and some results are obtained on Yokoi's invariant value of n_d which is defined in terms of coefficent of fundamental unit of real quadratic fields $Q\left(\sqrt{d}\right)$

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with period $k_d = 5$. Furthermore, it is denoted that the class number h_d is always even and the real quadratic fields $Q\left(\sqrt{d}\right)$ with $d \equiv 2 \left(\mod 4\right)$ and $h_d = 2$ are given in the Table 3.1.

Throughout this paper, Let d be a positive square - free integer and put $d = a^2 + b$ where $a, b \in Z^+$, $0 \langle b \leq 2a$. Here a, b are the integers uniquely determined by d such that $\sqrt{d} - 1 \langle a \langle \sqrt{d}$ .Also, Δ, $\varepsilon_d = \left(t_d + u_d \sqrt{d}\right) \rangle 1$, h_d , k_d be the diskriminant, the fundamental unit, the class number of real quadratic fields $Q\left(\sqrt{d}\right)$ and the period of continued fraction expansion of $w_d = \sqrt{d}$, respectively.

Let $I\left(d\right)$ be the set of all quadratic irrationals with discriminant Δ. An element w_d of $I\left(d\right)$ is called "reduced" if $w_d \rangle 1$, $-1 \langle w_d' \langle 0$ where w_d' is conjugate of w_d with respect to Q.

Let $R\left(d\right)$ be the set of all reduced quadratic irrationals with discriminant Δ and continued fraction with period k_d is generally denoted by $w_d = \left[q_0; \overline{q_1, ..., q_{k_d}}\right]$ and $\lfloor x \rfloor$ means the greatest integer not greater than x.

2. LEMMAS AND THEOREMS:

We need the following Lemmas nad Theorems in order to prove our main results.

Lemma 2.1: $\left(\left[6\right]\right)$For a square-free integer $d = a^2 + b \equiv 2\left(\mod 4\right)$ where $a, b \in Z^+$, $0 \langle b \leq 2a$ we put

$$w_d = \sqrt{d} \ , \ q_0 = \lfloor w_d \rfloor \ , \ w_R = \sqrt{d} + a = w_d + \lfloor w_d \rfloor$$

Then $w_d \notin R(d)$ and $w_R \in R(d)$ holds. Moreover ,for the period k of w_R we get

$$w_R = \left[\overline{2q_0, q_1, ..., q_{k-1}}\right] \text{ and } w_d = \left[q_0; \overline{q_1, ..., q_{k-1}, 2q_0}\right]$$

Furthermore , let $w_R = \dfrac{P_k . w_R + P_{k-1}}{Q_k . w_R + Q_{k-1}} = \left[2q_0; q_1, ..., q_{k-1}, w_R\right]$ be a modular otomorphism of w_R , then the fundamental unit ε_d of $Q\left(\sqrt{d}\right)$ is given by the following formula :

$$\varepsilon_d = \left(t_d + u_d \sqrt{d}\right) \rangle 1$$
$$t_d = q_0 Q_k + Q_{k-1} \ , \ u_d = Q_k$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = q_i . Q_i + Q_{i-1}$, $i \geq 1$

Proof:. it is easy to see that the proof of this Lemma is from Lemma 1 in $\left(\left[6\right]\right)$.

Lemma 2.2: $\left(\left[1\right]\right)$ Let d be a square free integer and put $d = a^2 + b$ where $a, b \in Z^+$, $0 \langle b \leq 2a$ we put $w_d \in R(d)$ .Let $w_i = l_i + \dfrac{1}{w_{i+1}}$ $\left(k_{i+1} = l_i = \lfloor w_i \rfloor, \ i \geq 0\right)$ be the continued fraction expansion of $w_d = w_0$ .Then each w_i is expressed in the form $w_i = \dfrac{a - r_i + \sqrt{D}}{c_i}$ $(c_i, r_i \in Z^+)$ and l_i, c_i, r_i can be obtained from the following recurrence formula :

$$w_0 = \frac{a - r_0 + \sqrt{D}}{c_0}$$
$$2a - r_i = c_i . l_i + r_{i+1}$$
$$c_{i+1} = c_{i-1} + (r_{i+1} - r_i) . l_i \ \ (i \geq 0)$$

where $0 \le r_{i+1} \langle c_i$, $c_{-1} = \dfrac{(b + 2ar_0 - r_0^2)}{c_0}$. Moreover, for the period $k_d \ge 1$ of w_d , we get

$$l_i = l_{k_d - i} \ (1 \le i \le k_d - 1)$$
$$r_i = r_{k_d - i + 1} \ (1 \le i \le k_d)$$
$$c_i = c_{k_d - i} \ (1 \le i \le k_d)$$

Furthermore if $w_d = w_R$ and $d \equiv 2,3(\bmod 4)$ then we have

$$r_0 = r_1 = 0,\ c_0 = 1,\ c_1 = b,\ l_0 = k_1 = 2a$$

Proof: it is easy to see that the proof of this Lemma is from Proposition 1 and Proposition 2 in $([1])$.

Theorem 2.3: $([5])$ If $l(N)$ is odd, then the following two (equivalent) conditions hold:

(a) $N = u^2 + v^2$ where $(u,v) = 1$

(b) N has no prime factors of the form $4k+3$ and is not divisible by 4.

Here, N is a positive integer which is not perfect – square. The continued fraction for $\sqrt{N}$ has periodic form $\sqrt{N} = \left[a_0; \overline{a_1, \ldots\ldots, a_{l-1}, 2a_0}\right]$ where $a_1, a_2, \ldots. a_{l-1}$ is palindrome and the period $l(N)$ is minimal length.

Proof: For the proof of this theorem, see Theorem C in $([5])$.

Corollary 2.4: $([4])$ Let Δ be the fundamental diskriminant where $\Delta = d$ if $d \equiv 1(\bmod 4)$ otherwise $\Delta = 4d$. If $\Delta \rangle 0$ then the class number h_d is odd if and only if $d = p,\ 2p_1\ or\ p_1.p_2$ where p is prime , $p_1 \equiv p_2 \equiv 3(\bmod 4)$ are primes.

Proof: For the proof of this corollary, see Corollary 1.3.2. in $([4])$.

3. MAİN RESULTS AND APPLİCATİONS:

Theorem 3.1:(MainTheorem) For a positive square – free integer $d = a^2 + b \equiv 2(\bmod 4)$ where $a, b \in Z^+$, $0 \langle b \le 2a$, we assume that $k_d = 5$ .Then, we get

$$w_d = \left[a, \overline{l_1, l_2, l_2, l_1, 2a}\right] = \begin{cases} \left[a, \overline{l, 2k+1, 2k+1, l, 2a}\right] & for\ an\ odd\ integer \quad if\ a\ is\ even \\ & l_2 \ge 1 \\ \left[a, \overline{2l, 2v, 2v, 2l, 2a}\right] & for\ two\ even\ integer \quad if\ a\ is\ odd \\ & l_1, l_2 \ge 2 \end{cases}$$

and then

$$(t_d, u_d) = \begin{cases} \left((Ar + t\,l_1)(A^2 + l_1^2) + (Al_2 + l_1),\ A^2 + l_1^2 \right) & if\ a\ is\ even \\ \left(\left(Ar + s\dfrac{l_1}{2} \right)(A^2 + l_1^2) + (Al_2 + l_1),\ A^2 + l_1^2 \right) & if\ a\ is\ odd \end{cases}$$

and

$$d = A^2 r^2 + 2Br + C$$

holds where A, B, C and r are determined uniquely as follows :

(i) In the case where a is even ;

$$A=l_1l_2+1,\ \ B=Atl_1+l_2\,,\ C=t\left(\ 2+tl_1^2\right),$$

r is the non–negative integer determined uniquely by $a=Ar+tl_1$.

(ii) In the case where a is odd ;

$$A=l_1l_2+1,\ \ B=Asl+l_2\,,\ C=s\left(1+sl\right),$$

r is the non–negative integer determined uniquely by $a=Ar+sl$.

Now, we define generally the set

$$S_{\beta}^{\alpha}=\left\{d\in Z^{+}\ \middle|\ d\equiv\alpha\left(\bmod 8\right)\ ,\ b\equiv\beta\left(\bmod 8\right)\right\}$$

where Z^{+} is the set of all positive integers.

Remark 3.2: For four parameters l,v,r and s in Theorem 3.1. satisfy the following conditions :

(i) In the case where a is even ;

$$l_1\equiv r\left(\bmod 2\right),\ l_2\equiv 1\left(\bmod 2\right),\ s\equiv 0\left(\bmod 2\right)$$

(ii) In the case where a is odd ;

$$\left(\frac{l_1}{2},r\right)\equiv\left(0,1\right),\left(1,0\right)\left(\bmod 2\right)$$

Remark 3.3: The set of all positive square-free integers congruent to 2 modulo 8 is union of S_2^2, S_6^2 and S_1^2 .The sets are represented as follows:

$$S_2^2=\left\{\ d\in Z^{+}\ \middle|\ d=a^2+8m+2,\ a\equiv 0\left(\ \bmod 4\right),\ 0\langle 4m\langle a\right\}$$

$$S_6^2=\left\{\ d\in Z^{+}\ \middle|\ d=a^2+8m+2,\ a\equiv 2\left(\ \bmod 4\right),\ 0\langle 4m\langle a-2\right\}$$

$$S_1^2=\left\{\ d\in Z^{+}\ \middle|\ d=a^2+8m+1,\ a\equiv 1\left(\ \bmod 2\right),\ 0\langle 4m\langle a\right\}.$$

Moreover, because of the Theorem 2.3, there is not any set S_5^2 .

Remark 3.4: For $k_d=l\left(w_d\right)=5$, there is not any real quadratic field $Q\left(\sqrt{d}\right)$ where $d\equiv 6\left(\bmod 8\right)$ or $d\equiv 3\left(\bmod 8\right)$.

Because of the Theorem 2.3. , d is no prime factors of the form $p\equiv 3\left(\bmod 4\right)$ and not divisible by 4. Therefore, there is not any real quadratic field $d\equiv 6\left(\bmod 8\right)$.Also, again using the Theorem 2.3. , $d=u^2+v^2$ where $\left(u,v\right)=1$ implies d is not congruent to 3 modulo 8.

Remark 3.5: Since Remark 3.4., for $k_d=l\left(w_d\right)=5$, S_{β}^{α} is not defined where $\alpha\equiv 6\left(\bmod 8\right)$ and $\beta\equiv 1,2,5\,or\,6\left(\bmod 8\right)$.

Remark 3.6: For $k_d=l\left(w_d\right)=5$, in the case of $w_d=\sqrt{d}$, the class number h_d of real quadratic field $Q\left(\sqrt{d}\right)$ is always even.

Since $d\neq 2$, $d\equiv 2\left(\bmod 4\right)$ implies $d\neq p,\ 2p_1$ *and* $p_1.p_2$ where p is prime , $p_1\equiv p_2\equiv 3\left(\bmod 4\right)$ are primes , it holds that h_d is always even because of Corollary 2.4.

For the set S of all square-free positive integers, we define the set

$$\Gamma_{k_d}\left(S\right)=\left\{w_d\ \middle|\ d\in S\ \ and\ \ k_d\ \ is\ the\ period\ of\ \ w_d=\sqrt{d}\ \right\}$$

and we put $w_0 = q_0 + w_d$ for $w_d = \left[q_0; \overline{q_1, \ldots\ldots, q_{k-1}, q_{k_d}}\right]$ in $\Gamma_{k_d}(S)$, then $w_0 \in R(d)$. For w_0 in $R(d)$, let $w_i = l_i + \dfrac{1}{w_{i+1}}$, $(k_{i+1} = l_i = \lfloor w_i \rfloor,\ i \geq 0)$ be the continued fraction expansion of w_0. Also, each w_0 is expressed in the form $w_i = \dfrac{a - r_i + \sqrt{D}}{c_i}$ $(c_i, r_i \in Z^+)$ in Lemma 2.2.

Proof of Main Theorem: .

(a) In the case where a is even , we first assume that d in $S_2^2 \cup S_6^2$. It follows from $q_0 = \lfloor w_d \rfloor = a$ and Lemma 2.2. implies

$$r_0 = r_1 = 0 \ ,\ c_0 = 1 \ ,\ c_1 = b \ ,\ l_0 = k_1 = 2a \ .$$

(i) We assume that w_d belongs to $\Gamma_5(S_2^2)$. Then $c_1 = 8m+2$, $m \in Z^+$ holds and Lemma 2.2. implies $2a = (8m+2)l_1 + r_2$. Hence, we can put $r_2 = 2r$, $r \in Z^+$ and get $a = (4m+1)l_1 + r$. Moreover, from Lemma 2.2. we get $c_2 = 1 + r_2.l_1$ and $2a = c_2 l_2 + r_2 + r_3$. Hence, we get

$$(8m+2)l_1 = (1+2rl_1)l_2 + r_3 \tag{1}$$

On the other hand, $c_2 = 1 + r_2.l_1$ and $c_3 = c_1 + (r_3 - r_2)l_2$ imply

$$(8m+2) = 2rl_1 + (2r - r_3)l_2 + 1 \tag{2}$$

because of $c_3 = c_2$.

If we assume $l_2 \equiv 0 \pmod 2$, then in all case of integer l_1, we get $r_3 \equiv 0 \pmod 2$ from (1). Hence, $0 \equiv 1 \pmod 2$ holds in (2), which is a contradiction. Therefore, we have $l_2 \equiv 1 \pmod 2$ and from (1) and (2), we can determine $r_3 \equiv 1 \pmod 2$ for in all case of integer l_1. Moreover, from (1), $l_2 + r_3 \equiv 0 \pmod{l_1}$ holds. Thus, there exists a positive even integer s such that $r_3 = sl_1 - l_2$ because of $r_3 \equiv 1 \pmod 2$. By substitution of this r_3 in (1), we get $4m+1 = rl_2 + t$ and because of $a = (4m+1)l_1 + r$, we get $a = Ar + tl_1$ where $A = l_1 l_2 + 1$ and $s = 2t$, $t \in Z^+$.

On the other hand, (2) implies $2(rl_2 + t) = 2rl_1 + (2r - 2tl_1 + l_2)l_2 + 1$ and hence we get $2rl_1 - sA = -l_2^2 - 1$. Therefore, because of $A^2 - l_1^2 \neq 0$ such integers r, s are uniquely determined.

Now, we consider $A = l_1 l_2 + 1$. Then, since $w_d = \left[a, \overline{l_1, l_2, l_2, l_1, 2a}\right]$,

$$Q_3 = A,\ Q_4 = Al_2 + l_1,\ Q_5 = A^2 + l_1^2$$

hold in Lemma 2.1. Therefore, we have that

$$t_d = (Ar + tl_1)(A^2 + l_1^2) + (Al_2 + l_1) \ \text{and}\ u_d = A^2 + l_1^2$$

Moreover, if we put $B = Atl_1 + l_2$, $C = t(2 + tl_1^2)$, then $d = A^2 r^2 + 2Br + C$ holds.

(ii) Next, we assume that w_d belongs to $\Gamma_5(S_6^2)$, then we have only to replace $(8m+2)$ with $(8m+6)$ in the case that w_d belongs to $\Gamma_5(S_6^2)$. Hence, (1) and (2) are replaced by

$$(8m+6)l_1 = (1+2rl_1)l_2 + r_3 \quad \text{and} \quad (8m+6) = 2rl_1 + (2r - r_3)l_2 + 1$$

respectively. Then, there exists a positive even integer $s = 2t$, $t \in Z^+$ such that $r_3 = sl_1 - l_2$. The proof of this case is obtained as the proof of previous case.

As an application of the first part of the case a is even integer of this theorem we get $d=74=8^2+8.1+2$, since $a=(4m+1)l_1+r$ and $s=(8m+2)-2rl_2$, we have $l_1=1,\ l_2=1,\ r=3,\ s=4,\ t=2$ and $A=2$. Hence w_d is easily determined as follows:

$$w_d=\sqrt{74}=\left[8,\overline{1,1,1,1,16}\right].$$

Moreover, the fundamental unit of $Q\left(\sqrt{74}\right)$ is immediately seen as $\varepsilon_d=43+5\sqrt{74}$ by using $t_d=43$ and $u_d=5$.

As an application of the second part of the case a is even integer of this theorem we get $d=218=14^2+8.2+6$, since $a=(4m+3)l_1+r$ and $s=(8m+6)-2rl_2$, we have $l_1=1,\ l_2=3,\ r=3,\ s=4,\ t=2$ and $A=4$.

Hence w_d is easily determined as follows:

$$w_d=\sqrt{218}=\left[14,\overline{1,3,3,1,28}\right].$$

Moreover, the fundamental unit of $Q\left(\sqrt{74}\right)$ is immediately seen as $\varepsilon_d=251+17\sqrt{218}$ by using $t_d=251$and $u_d=17$.

(b)In the case where a is odd integer , we have only to consider d in S_1^2 and w_d belongs to $\Gamma_5\left(S_1^2\right)$. Then $q_0=\lfloor w_d\rfloor=a$ holds and Lemma 2.2. implies $r_0=r_1=0,\ c_0=1,\ c_1=b=8m+1,\ m\in Z^+,\ l_0=k_1=2a.$ By using Lemma 2.2. , we get $2a=(8m+1)l_1+r_2$.

If we assume $l_1\equiv1(\mathrm{mod}\,2)$ $i.e.\ l_1=2l+1,\ l\in Z^+$, then $r_2=2r+1,\ r\in Z^+$ holds. Hence, we can put $r_2=2r+1$ and $a=4ml_1+l+r+1$. Moreover, from Lemma 2.2. we get $c_2=1+r_2.l_1$ and $2a=(8m+1)l_1+r_2$ and so

$$(8m+1)l_1=(1+r_2l_1)l_2 \tag{3}$$

holds. If we consider $l_1\equiv1(\mathrm{mod}\,2)$ and $r_2\equiv1(\mathrm{mod}\,2)$, then $0\equiv1(\mathrm{mod}\,2)$holds in (3), which is a contradiction.Hence, we have $l_1\equiv r_2=0(\mathrm{mod}\,2)$.Therefore, from (3), we can determine $l_1=2l,\ r_2=2r,\ l,r\in Z^+$ and Lemma 2.2. implies $a=(8m+1)l+r.$Moreover, from Lemma 2.2. we get $c_2=1+r_2.l_1$ and $2a=(8m+1)l_1+r_2$ .Hence, we get

$$(8m+1)2l=(1+4rl)l_2+r_3 \tag{4}$$

On the other hand, $c_2=1+r_2.l_1$ and $c_3=c_1+(r_3-r_2)l_2$ imply

$$(8m+1)=4rl+(2r-r_3)l_2+1 \tag{5}$$

because of $c_3=c_2$.

If we assume $l_2\equiv1(\mathrm{mod}\,2)$, then we get $r_3=(8m+1)2l-(1+4rl)l_2$ is odd integer from (4).Hence, $0\equiv1(\mathrm{mod}\,2)$holds in (5), which is a contradiction. Hence, we have $l_2\equiv0(\mathrm{mod}\,2)$ $i.e.\ l_2=2v$, $v\in Z^+$.Therefore, from (4) and (5), we can determine $r_3\equiv0(\mathrm{mod}\,2)$. Moreover, from (4), $l_2+r_3\equiv0(\mathrm{mod}\,l_1)$ holds.Thus, there exists a positive odd integer s such that $r_3=sl_1-l_2=2(sl-v)$. By substitution of this r_3 in (4), we get $8m+1=4rv+s$ and because of $a=(4rv+s)l+r,$ we get $a=Ar+sl$ where $A=l_1l_2+1=4vl+1$. On the other hand, (5) implies $4rl-sA=-4v^2-1.$.Therefore, because of $A^2-4l^2\neq0$ such integers r,s are uniquely determined.

Now, we consider $A=4vl+1$ .Then, since $w_d=\left[a,\overline{2l,2v,2v,2l,2a}\right]$, $q_0=a=(4vl+1)r+sl$

$$Q_3=A,\ Q_4=Al_2+l_1=2(Av+l),\ Q_5=A^2+l_1^2=A^2+4l^2$$

hold in Lemma 2.1. Therefore, we have that

$$t_d=(Ar+sl)(A^2+4l^2)+(2Av+2l) \text{ and } u_d=A^2+l_1^2=A^2+4l^2$$

Moreover, if we put $A=4vl+1$ $B=Asl+2v$, $C=s(1+sl^2)$, then $d=A^2r^2+2Br+C$ holds.

As an application of the case a is odd integer of this theorem we get $d=1378=37^2+8.1+1$,

since $a=(8m+1)l+r$ and $s=(8m+1)-4rv$,

we have
$l_1=8$, $l_2=4$, $r=1$, $s=1$, and $A=33$. Hence w_d is easily determined as follows:

$$w_d=\sqrt{1378}=\left[37,\overline{8,4,4,8,74}\right].$$

Moreover, the fundamental unit of $Q\left(\sqrt{1378}\right)$ is immediately seen as $\varepsilon_d=85602+2306\sqrt{1378}$ by using $t_d=85602$ and $u_d=2306$.

For any square-free integer d in $[7]$, Yokoi defined some new invariants by taking the fundamental unit of $Q\left(\sqrt{d}\right)$ as

$$n_d=\left\lfloor\frac{t_d}{u_d^2}\right\rfloor,\ m_d=\left\lfloor\frac{u_d^2}{t_d}\right\rfloor$$

etc…and studied relationship between these new invariants and class number of real quadratic fields $Q\left(\sqrt{d}\right)$.

In this section, we apply our results to these invariants, and consider the class number h_d of real quadratic fields $Q\left(\sqrt{d}\right)$ for d in S^2 where S^2 is the set of all positive square-free integers congruent to 2 modulo 8.

Now, we apply Yokos's invariant $n_d=\left\lfloor\frac{t_d}{u_d^2}\right\rfloor$ to main theorem above, see $[7]$. Then, we get following theorem.

Theorem 3.2: Let $d=a^2+b\equiv 2(\bmod 4)$ where $a,b\in Z^+$, $0\langle b\leq 2a$ be a square free integer , $k_d=5$ and $\varepsilon_d=\left(t_d+u_d\sqrt{d}\right)\rangle 1$ be a the fundamental unit of $Q\left(\sqrt{d}\right)$. Then, for the obtained values t_d , u_d in Theorem 3.1. the following statements are hold :

(a) $a\langle u_d \Leftrightarrow n_d=0$

(b) $w_d=\sqrt{d}=\left[a,\overline{1,1,1,1,2a}\right] \Leftrightarrow u_d=5$ *and* $n_d=\frac{a-3}{5}$ $\quad(i.e.\ m_d=0)$

(c) $w_d=\sqrt{d}=\left[a,\overline{2,1,1,2,2a}\right] \Leftrightarrow u_d=13$ *and* $n_d=\frac{a-5}{13}$ $\quad(i.e.\ m_d=0)$

Proof: We note that $n_d=\left\lfloor\frac{t_d}{u_d^2}\right\rfloor=0 \Leftrightarrow u_d^2-t_d\rangle 0$.

(a) Firstly, we assume that d belongs to $S_2^2\cup S_6^2$. In this case, for $\varepsilon_d=\left(t_d+u_d\sqrt{d}\right)\rangle 1$, $t_d=a.(A^2+l_1^2)+(Al_2+l_1)$ and $u_d=A^2+l_1^2$ hold.

$(:\Rightarrow)$We assume that $a \langle u_d$. Since

$$u_d - (Al_2 + l_1) = (A^2 + l_1^2) - (Al_2 + l_1) \geq 2(2l_2 + 1)(l_2 + 1)$$

and $l_2 \geq 1$,we get $u_d \rangle Al_2 + l_1$. Hence, we get also

$$n_d = \left\lfloor \frac{t_d}{u_d^2} \right\rfloor = \left\lfloor \frac{a(A^2 + l_1^2) + (Al_2 + l_1)}{(A^2 + l_1^2)^2} \right\rfloor$$
$$= \left\lfloor \frac{a}{A^2 + l_1^2} \right\rfloor = 0$$

by using the $a \langle u_d$.

$(\Leftarrow:)$Conversely, we suppose that $n_d = 0$.By using the $u_d^2 - t_d \rangle 0$, we get $(A^2 + l_1^2) \rangle a$ and so $a \langle u_d$.

Next, we assume that d belongs to S_1^2.In this case is proved in the same way as previous case. Therefore, $a \langle u_d$ is necessary and sufficent condition for $n_d = 0$.

(b) $(:\Rightarrow)$We assume that the continued fraction expansion of w_d is the form of $w_d = \sqrt{d} = [a, \overline{1,1,1,1,2a}]$.

Then, we get $A=2$ and $u_d = 5$ because of $A = l_1 l_2 + 1$ and $u_d = (A^2 + l_1^2)$. Since $l_1 = 1$ is odd, d does not belong to S_1^2 and so

$$t_d = a.u_d + (Al_2 + l_1) \text{ and } u_d = 5$$

hold. By using the equivalent

$$t_d = a.u_d^2 + (Al_2 + l_1)u_d + (Al_2 + l_1)$$

we get $n_d = \dfrac{a-3}{5}$.

$(\Leftarrow:)$Conversely, we assume that $u_d = 5$ and $n_d = \dfrac{a-3}{5}$. Using the values $u_d = (A^2 + l_1^2)$ and $A = l_1 l_2 + 1$, we have $l_1 = l_2 = 1$. Hence, we get

$$w_d = \sqrt{d} = [a, \overline{1,1,1,1,2a}].$$

As an application of this case, we get $d = 74 = 8^2 + 8.1 + 2$. By using the Theorem 3.1. it is easily seen that $a = 8$, $l_1 = l_2 = 1$, and $A = 2$. Therefore, we have $u_d = 5$ and $n_d = 1$.

(c)The proof of this case is obtained as the proof of (b).

As an application of this case, we get $d = 1970 = 44^2 + 8.4 + 2$. By using the Theorem 3.1. it is easily seen that $a = 44$, $l_1 = 2$, $l_2 = 1$, and $A = 3$. Therefore, we have $u_d = 13$ and $n_d = 3$.

Corollary 3.3: Let $d = a^2 + b \equiv 2 (\bmod 4)$ where $a, b \in Z^+$, $0 \langle b \leq 2a$ be a square free integer and $k_d = 5$. If a is even integer, then there exist exactly three real quadratic fields $Q(\sqrt{d})$ with class number $h_d = 2$ which are given in Table 3.1. (with one posible exeption of d)Moreover, there is not any real quadratic field $Q\left(\sqrt{d}\right)$ with class number $h_d = 2$ where a is odd integer.

d	a	m	n_d	h_d	w_d
$74 \in S_2^2$	8	1	1	2	$\sqrt{74} = \left[8, \overline{1,1,1,1,16}\right]$
$218 \in S_6^2$	14	2	0	2	$\sqrt{218} = \left[14, \overline{1,3,3,1,28}\right]$
$2138 \in S_6^2$	46	2	0	2	$\sqrt{2138} = \left[46, \overline{4,5,5,4,92}\right]$

Table 3.1

Proof: It is easy to see that the proof of this corollary is from Corollary 2.4. All of the fields with class number two in Table 2.1. in $([4])$ are obtained from Corollary 3.3.

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