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WAVELET AND GEOMETRY

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ABSTRACT

In this paper the focus is on the vector space and signaling geometrical dimension of wavelets. There exists a strong analogy between what we do in vector algebra and signal processing which helps us to visualize and interpret the process and output of the process that we do on signals. Developing a mental geometrical paradigm of this analogy is the main aim of this paper.

Keywords: *Wavelet, Geometry of Haar wavelet.*

1. INTRODUCTION:

Wavelet based analysis of signals is an interesting and relatively recent new tool. In this paper we won't try to be mathematically precise in our statements regarding wavelets and vector space. Instead our aim is to relate geometry and wavelet in an intuitive way. Fourier is not always the best device for analyzing the signals because we may come across a signal that is not periodic (speech).

2. PRELIMINARIES:

In this section, we present some concepts and definitions that are needed for our subsequent discussion. This is based on Leon, Steven [1] and Strang, Gilbert [2].

2.1 Vector space: Any vector in three dimension al space can be represented as $\overline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ unit are vectors in three orthogonal directions so that $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$. The orthogonality condition ensures that the representation of any vector using $\hat{i}, \hat{j}, \hat{k}$ is unique.

2.2 Bases: we call $\hat{i}, \hat{j}, \hat{k}$ as bases of space $\Re^3$. By this we mean $\hat{i}, \hat{j}, \hat{k}$ span the space $\Re^3$. Let $\overline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a vector in $\Re^3$.Continuously changing a_1, a_2, a_3; we will go on get new vectors in $\Re^3$. We imagine that, set of all such vectors constitute the vector space $\Re^3$. We express this truth by $\underset{a_1,a_2,a_3}{span}\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \equiv \Re^3$.

2.3 Projection: Given a vector $\overline{a}$ we can find its normal component s a_1, a_2, a_3. We project $\overline{a}$ on to bases $\hat{i}, \hat{j}, \hat{k}$ to get a_1, a_2, a_3. This is the direct result of orthogonality of our basis vectors. Since $\overline{a}.\hat{i} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\hat{i} = a_1$, $\overline{a}.\hat{j} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\hat{j} = a_2$ and $\overline{a}.\hat{k} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\hat{k} = a_3$.

So projecting a vector $\overline{a}$ on to a base of space $\Re^3$ gives the corresponding coefficient (components) of the base .There is a parallel for this in wavelet analysis.

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3. FUNCTIONS AND FUNCTIONS SPACES:

In this section we relate wavelets from geometrical view point. This is based on earlier work by K.P. Soman and K.I. Ramchandran [3].

In vector space ,we represent a vector in terms of orthogonal unit vectors called bases. Wavelet is an extension of this idea for functions. (Electronics engineers call them as signals). A function is expressed in terms of a set of orthogonal functions. To correlate with ideas in geometry, we now need to introduce the concept of orthogonality and norm with respect to functions. Note that function is a quantity which varies with respect to one or more running parameters, usually time and space. Orthogonality of functions depends on multiplication and integration of functions. Multiplication and integration of function must be thought of as equivalent to projection of vector on to another.

3.1 Orthogonal functions: Two real functions $f(t)$ and $g(t)$ are said to be orthogonal if and only if $\int_{-\infty}^{\infty} f(t).g(t)\,dt = 0$.Here first we are doing point by point multiplication of two function then we are summing up the area under the resulting function (obtained after multiplication). If this area is zero, we say the two functions are orthogonal. We also interpret it as similarity of functions $f(t)$ and $g(t)$.If $f(t)$ and $g(t)$ vary synchronously, that is if $f(t)$ goes up $g(t)$ also goes up and if $f(t)$ goes down $g(t)$ also goes down then we say the two functions have high similarity. Otherwise, they are dissimilar. Magnitude or absolute value of $\int_{-\infty}^{\infty} f(t).g(t)\,dt$ will be high if they vary synchronously and zero if they vary perfectly asynchronous way.

The geometrical analog of $\int_{-\infty}^{\infty} f(t).g(t)\,dt = 0$ is the dot product $\hat{i}.\hat{j} = 0$. For the given interval when $f(t)$ is projected on $g(t)$ (projection here means multiplying and integrating), if the result is zero, we say the two functions are orthogonal in the given interval. Clearly look at the analogy. Taking a dot product in vector space (multiplying corresponding terms and adding) is equivalent to multiplying and integrating in function space (point wise multiplication and integration). The dot product is maximum when two vectors are collinear and zero when they are at 90^0. This analogy will help us to easily visualize and interpret various results in signal processing.

3.2 Orthonormal functions: Two real functions $f(t)$ and $g(t)$ are said to be orthonormal if and only

If $\int_{-\infty}^{\infty} f(t).g(t)\,dt = 0$ and $\int_{-\infty}^{\infty} f(t).f(t)\,dt = \int_{-\infty}^{\infty} g(t).g(t)\,dt = 1$. (2.1)

3.3 Function spaces: Just like unit orthogonal set of vectors span vector spaces, orthogonal set of functions can span function spaces. In vector algebra, we know that the unit orthogonal vectors $\hat{i}, \hat{j}, \hat{k}$ span the space $\Re^3$.

Consider the haar wavelets $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ then $\psi_{0,0}(t) = \psi(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ -1 & \frac{1}{2} \le t \le 1 \\ 0 & otherwise \end{cases}$ (2.2)

is haar mother wavelet other wavelets are produced by translation and dilation of the mother wavelet, which are infinite number of functions. These set of functions are mutually orthonormal. Let us verify the truth through some representative examples.

$$\psi_{0,0}(t)=\psi(t)=\begin{cases}1 & 0\le t\le \frac{1}{2}\\ -1 & \frac{1}{2}\le t\le 1\\ 0 & otherwise\end{cases},\ \psi_{0,1}(t)=\psi(t-1)=\begin{cases}1 & 1\le t\le \frac{3}{2}\\ -1 & \frac{3}{2}\le t\le 2\\ 0 & otherwise\end{cases}\ \text{and}\ \psi_{2,1}(t)=2\psi(4t-1)=\begin{cases}2 & \frac{1}{4}\le t\le \frac{3}{8}\\ -2 & \frac{3}{8}\le t\le \frac{1}{2}\\ 0 & otherwise\end{cases}$$

Let us take function $\psi_{0,0}(t)$ in (0, 1).This function and its product with itself is shown in figure 2.1. The resulting curve is after point wise multiplication.

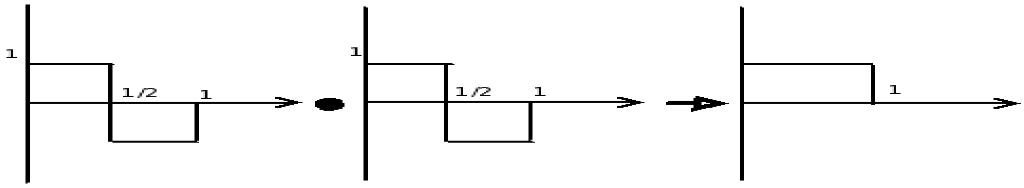

Figure: 2.1 $\int_{-\infty}^{\infty}\psi_{0,0}(t).\psi_{0,0}(t)\,dt\ =1$

Area under resulting curve (0, 1) is obviously one square unit. Mathematically $\int_{-\infty}^{\infty}\psi_{0,0}(t).\psi_{0,0}(t)\,dt\ =1$

We say $\psi_{0,0}(t)$ has unit norm. Similarly $\psi_{0,1}(t)$ and $\psi_{2,1}(t)$ has unit norm as shown in figure 2.2 and 2.3 below

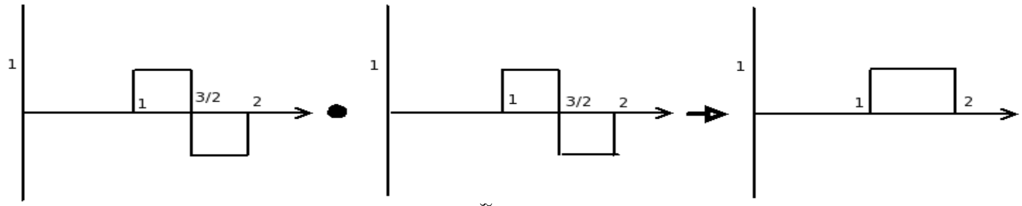

Figure: 2.2 $\int_{-\infty}^{\infty}\psi_{0,1}(t).\psi_{0,1}(t)\,dt\ =1$

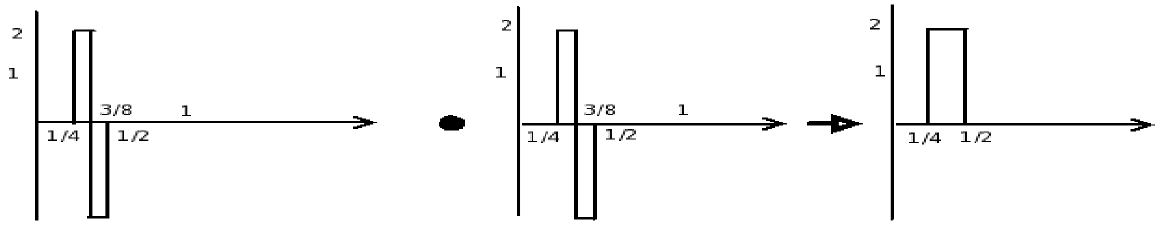

Figure: 2.3 $\int_{-\infty}^{\infty}\psi_{2,1}(t).\psi_{2,1}(t)\,dt\ =1$

Consider $\psi_{0,0}(t)$ and $\psi_{2,1}(t)$ in (0, 1) . Figure 2.4 shows the two functions and corresponding function obtained after point wise multiplication.

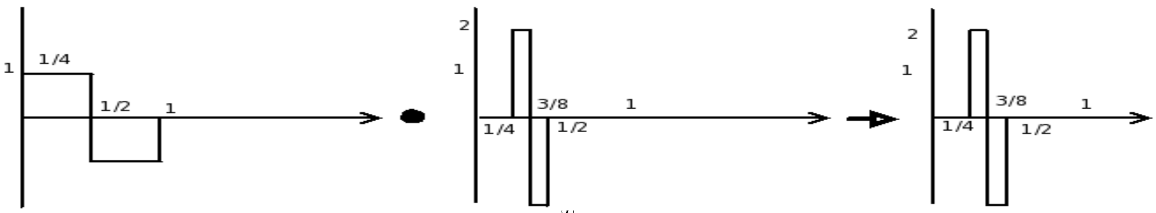

Figure: 2.4 $\int_{-\infty}^{\infty}\psi_{0,0}(t).\psi_{2,1}(t)\,dt\ =0$

Area above and below of the resulting curve is same implying that the net area is zero. Mathematically $\int_{-\infty}^{\infty} \psi_{0,0}(t).\psi_{2,1}(t)\,dt = 0$

This in turn imply that $\psi_{0,0}(t)$ and $\psi_{2,1}(t)$ are orthogonal in (0, 1). Similarly Figure 2.5 shows $\psi_{0,0}(t)$ and $\psi_{0,1}(t)$ are orthogonal in (0, 1)

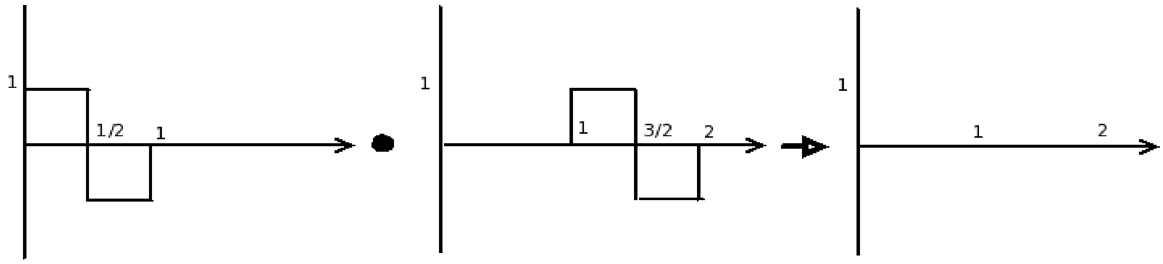

Figure: 2.5 $\int_{-\infty}^{\infty} \psi_{0,0}(t).\psi_{0,1}(t)\,dt = 0$

Generalizing $\int_{-\infty}^{\infty} \psi_{j,k}(t).\psi_{l,m}(t)\,dt = \begin{cases} 1 & j = k,\ l = m \\ 0 & otherwise \end{cases}$. Thus, we have set of functions $\psi_{j,k}(t)$ defined over (0, 1) such that they are mutually orthonormal similar to $\hat{i}, \hat{j}, \hat{k}$ in the space $\Re^3$. This seems quite obvious with respect to Haar wavelet system, when we go on other wavelet systems this geometrical part may not be that obvious since most of other wavelet systems are fractal in nature. We cannot draw directly the functions for visualization.

4. CONCLUSION:

The theory of wavelet has a strong connection with geometry or at least we can understand the wavelet theory from a geometrical view point.

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