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A MATHEMATICAL SYN-ECOLOGICAL MODEL COMPRISING OF PREY-PREDATOR, HOST-COMMENSAL, MUTUALISM AND NEUTRAL PAIRS

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ABSTRACT

This investigation deals with a mathematical model of a four species (S_1, S_2, S_3 and S_4) Syn-Ecological system. S_2 is a predator surviving on the prey S_1. The predator S_2 is a commensal to the host S_3. The pairs S_2 and S_4, S_1 and S_3 are neutral. The mathematical model equations characterizing the syn-ecosystem constitute a set of four first order non-linear coupled differential equations. There are in all sixteen equilibrium points. Criteria for the asymptotic stability of the sixteen equilibrium points are established. The linearised equations for the perturbations over the equilibrium point are analyzed to establish the criteria for stability. Trajectories of the perturbations have been illustrated. Also global stability is discussed by Lyapunov's function. Further analytical stability criteria are supported by numerical simulations using Mat lab.

Key words: *Equilibrium state, stability, Mutualism, Co-Existent State.*

1. INTRODUCTION:

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmocokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [6] and by Volterra [12] followed by several mathematicians and ecologists. They contributed their might to the growth of this area of knowledge as reported in the treatises of Meyer [7], Paul Colinvaux [8], Freedman [2], Kapur [3, 4]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [11] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [5] has investigated the two species prey-predator models and stability analysis of competitive species was investigated by Archana [1]. Further Local stability analysis for a two-species ecological mutualism model has been investigated by B. Ravindra Reddy et al. [9, 10]. In this connection here we constructed a four species (S_1, S_2, S_3 and S_4) mathematical model based on the system of non-linear equations. S_2 is a predator surviving on the prey S_1. The predator S_2 is a commensal to the host S_3. The pairs S_2 and S_4, S_1 and S_3 are neutral. Equilibrium points of the system are identified and the stability analysis is carried out. Example for S_1, S_2, S_3 and S_4 are Insects, Insectivorous Plants (nephantis, drosera etc.), VAM associated with the plant roots, Soil bacteria respectively.

2. BASIC EQUATIONS:

The model equations for a four species multi-system are given by a set of four non-linear ordinary differential equations as

(i) For S_1**:** The Prey of S_1 and Neutral to S_3

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 \tag{2.1}$$

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(ii) For S_2**:** The Predator surviving on S_1 and Commensal to S_3

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}{N_2}^2 + a_{21}N_2N_1 + a_{23}N_2N_3 \tag{2.2}$$

(iii) For S_3**:** The Host of S_2 and Mutual to S_4

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}{N_3}^2 + a_{34}N_3N_4 \tag{2.3}$$

(iv) For S_4**:** Mutual to S_3 and Neutral to S_2

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}{N_4}^2 + a_{43}N_4N_3 \tag{2.4}$$

with the following notation.

N_i (t): Population strengths of the species S_i at time t, i=1, 2, 3, 4.
a_i : The natural growth rates of S_i, i = 1,2,3,4
a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13} : Coefficient for commensal for S_1 due to the Host S_3
a_{34}, a_{43} : Mutually interaction between S_3 and S_4
K_i: $\frac{a_i}{a_{ii}}$: Carrying capacities of S_i, i=1, 2, 3, 4.

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

1. EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1,2,3,4 \tag{3.1}$$

are given in the following table.

I. Fully washed out state:

E_1: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

II. States in which three of the four species are washed out and fourth is surviving

E_2: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

E_3: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

E_4: $\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$

E_5: $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

III. States in which two of the four species are washed out while the other two are surviving

E_6: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_4a_{34} + a_3a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \frac{a_3a_{43} + a_4a_{33}}{a_{33}a_{44} - a_{34}a_{43}}$

This state exists only when $a_{33}a_{44} - a_{34}a_{43} > 0$

E_7: $\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

E_8: $\overline{N_1}=0, \overline{N_2}=\frac{a_3}{a_{22}}\frac{a_{23}}{a_{33}}+\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

E_9: $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

E_{10}: $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

E_{11}: $\overline{N_1}=\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_2}=\frac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_3}=0, \overline{N_4}=0$

This state exists only when $a_1a_{22}-a_2a_{12}>0$

IV. States in which one of the four species is washed out while the other three are surviving

E_{12}: $\overline{N_1}=0, \overline{N_2}=\frac{a_{23}(a_4a_{34}+a_3a_{44})}{a_{22}(a_{33}a_{44}-a_{34}a_{43})}+\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}},$
$\overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

This state exists only when $a_{33}a_{44}-a_{34}a_{43}>0$

E_{13}: $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

This state exists only when $(a_{33}a_{44}-a_{34}a_{43})>0$

E_{14}: $\overline{N_1}=\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_2}=\frac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

This state exists only when $a_1a_{22}-a_2a_{12}>0$

E_{15}: $\overline{N_1}=\frac{\beta_4}{\beta_1}, \overline{N_2}=\frac{\beta_5}{\beta_1}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

Where

$$\beta_1=a_{33}(a_{11}a_{22}+a_{12}a_{21})$$
$$\beta_4=a_{33}(a_1a_{22}-a_2a_{12})-a_3a_{23}a_{12}$$
$$\beta_5=a_{33}(a_1a_{21}+a_2a_{11})+a_3a_{23}a_{11}$$

This state exists only when $\beta_4>0$

V. The co-existent state (or) Normal steady state

E_{16}: $\overline{N_1}=\frac{\gamma_1+a_{12}a_{23}\gamma_2}{\gamma_3(a_{33}a_{44}-a_{34}a_{43})}, \overline{N_2}=\frac{\gamma_4+a_{11}a_{23}\gamma_2}{\gamma_3(a_{33}a_{44}-a_{34}a_{43})},$
$\overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

Where

$$\gamma_1=(a_1a_{22}+a_2a_{12})(a_{33}a_{44}-a_{34}a_{43}), \gamma_2=a_3a_{44}+a_4a_{34}$$
$$\gamma_3=a_{11}a_{22}+a_{12}a_{21}, \gamma_4=(a_1a_{21}-a_2a_{11})(a_{33}a_{44}-a_{34}a_{43})$$

This state exists only when $(a_1a_{21}-a_2a_{11})>0$ *and* $(a_{33}a_{44}-a_{34}a_{43})>0$.

4. STABILITY OF THE EQUILIBRIUM STATES:

Let $N = \left(N_1, N_2, N_3, N_4\right) = \bar{N} + U$ (4.1)

where $U = \left(u_1, u_2, u_3, u_4\right)$ is a perturbation over the equilibrium state $\bar{N} = \left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4\right)$.

The basic equations (2.1), (2.2), (2.3), (2.4) are linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 & -a_{12}\bar{N}_1 & 0 & 0 \\ a_{21}\bar{N}_2 & a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 + a_{23}\bar{N}_3 & a_{23}\bar{N}_2 & 0 \\ 0 & 0 & a_3 - 2a_{33}\bar{N}_3 + a_{34}\bar{N}_4 & a_{34}\bar{N}_3 \\ 0 & 0 & a_{43}\bar{N}_4 & a_4 - 2a_{44}\bar{N}_4 + a_{43}\bar{N}_3 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is $\det\left[A - \lambda I\right] = 0$ (4.4)

The equilibrium state is stable if both the roots of the equation (4.4) are negative in case they are real or have negative real parts in case they are complex. The equilibrium states $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_{10}, E_{11}, E_{13}, E_{14}, E_{15}$ are noticed to be unstable (The detailed investigation of these states is not included here for treatment). The stability criteria of states E_{12}, E_{16} are discussed below.

4.1 Stability of Prey (S_1) Washed out Equilibrium State: (E_{12})

$$\overline{N_1} = 0, \overline{N_2} = \frac{a_{23}(a_4 a_{34} + a_3 a_{44})}{a_{22}(a_{33}a_{44} - a_{34}a_{43})} + \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}} \tag{4.1.1}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = w_1 u_1 \tag{4.1.2}$$

$$\frac{du_2}{dt} = a_{21}\overline{N_2}u_1 - w_3 u_2 + a_{23}\overline{N_2}u_3 \tag{4.1.3}$$

$$\frac{du_3}{dt} = -a_{33}\overline{N_3}u_3 + a_{34}\overline{N_3}u_4 \tag{4.1.4}$$

$$\frac{du_4}{dt} = a_{43}\overline{N_4}u_3 - a_{44}\overline{N_4}u_4 \tag{4.1.5}$$

Here $w_1 = a_1 - a_{12}\overline{N_2}$, $w_3 = a_2 + a_{23}\overline{N_3}$ (4.1.6)

The characteristic equation of which is

$$(\lambda - w_1)(\lambda + w_3)\left[\lambda^2 + (a_{33}\overline{N_3} + a_{44}\overline{N_4})\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N_3}\,\overline{N_4}\right] = 0 \tag{4.1.7}$$

The characteristic roots of (4.1.7) are

$$\lambda = w_1, \lambda = -w_3, \lambda = \frac{-(a_{33}\bar{N}_3 + a_{44}\bar{N}_4) \pm \sqrt{(a_{33}\bar{N}_3 - a_{44}\bar{N}_4)^2 + 4a_{34}a_{43}\bar{N}_3\bar{N}_4}}{2}$$

Case (A): If $w_1 < 0$ [i.e. $a_1 < a_{12}\overline{N_2}$]

Here $w_1, -w_3$ are negative and the other two roots are also negative.

Hence the equilibrium state is **stable**.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$$u_1 = u_{10}e^{w_1 t} \tag{4.1.8}$$

$$u_2 = \left[u_{20} - \frac{a_{21}\overline{N_2}u_{10}}{(w_1+w_3)} - \frac{a_{23}\overline{N_2}(P_1+P_2)}{(\lambda_3+w_3)}\right]e^{-w_3 t} + \frac{a_{21}\overline{N_2}u_{10}}{(w_1+w_3)}e^{w_1 t} + \frac{a_{23}\overline{N_2}(P_1e^{\lambda_3 t}+P_2e^{\lambda_4 t})}{(\lambda_3+w_3)} \tag{4.1.9}$$

$$u_3 = \left[\frac{u_{30}\left(\lambda_3 + a_{44}\overline{N}_4\right) + u_{40}a_{34}\overline{N}_3}{\lambda_3-\lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{30}\left(\lambda_4 + a_{44}\overline{N}_4\right) + u_{40}a_{34}\overline{N}_3}{\lambda_4-\lambda_3}\right]e^{\lambda_4 t} \tag{4.1.10}$$

$$u_4 = \left[\frac{u_{40}\left(\lambda_3 + a_{33}\overline{N}_3\right) + u_{30}a_{43}\overline{N}_4}{\lambda_3-\lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{40}\left(\lambda_4 + a_{33}\overline{N}_3\right) + u_{30}a_{43}\overline{N}_4}{\lambda_4-\lambda_3}\right]e^{\lambda_4 t} \tag{4.1.11}$$

Where $P_1 = \dfrac{u_{30}\left(\lambda_3 + a_{44}\overline{N}_4\right) + u_{40}a_{34}\overline{N}_3}{\lambda_3-\lambda_4}$, $P_2 = \dfrac{u_{30}\left(\lambda_4 + a_{44}\overline{N}_4\right) + u_{40}a_{34}\overline{N}_3}{\lambda_4-\lambda_3}$

where u_{10}, u_{20}, u_{30}, u_{40} are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$, $u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_3 < a_1 < A_2 < a_4$

In this case initially S_4 dominates the Prey (S_1) and the Predator (S_2) till the time instant t^*_{14}, t^*_{24} respectively and thereafter the dominance is reversed. And u_1, u_2, u_3, u_4 are converging asymptotically to the equilibrium point. Hence the equilibrium point is **stable**.

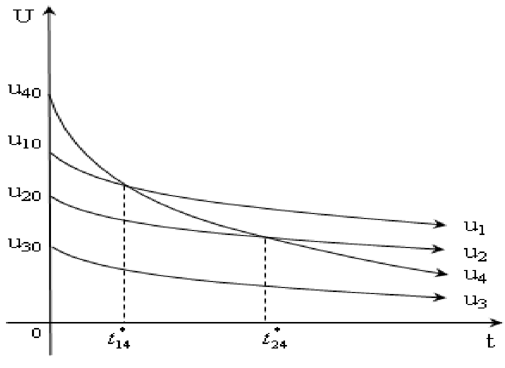

Fig. 1

Case (ii): If $u_{40} < u_{10} < u_{30} < u_{20}$ and $A_2 < a_3 < a_1 < a_4$

In this case initially the Predator (S_2) dominates the Prey (S_1) till the time instant t^*_{12} and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Prey (S_1) and S_4 till the time instant t^*_{13}, t^*_{43} respectively and the dominance gets reversed thereafter.

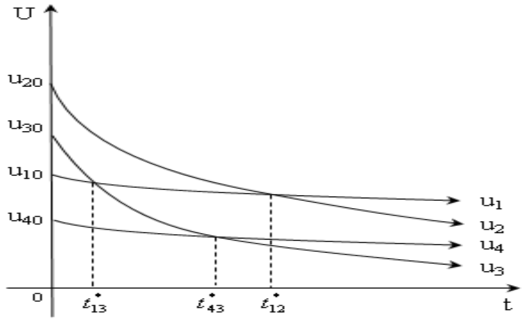

Fig. 2

Case (B): If $w_1 > 0$ [i.e. $a_1 > a_{12}\overline{N_2}$]

Here the root w_1 is positive and the other three roots are negative.

Hence the equilibrium state is **unstable** and the solutions in this case are same as in Case (A).

4.2 Stability of Co-Existing State : (E_{16})

$$\overline{N_1} = \frac{\gamma_1 + a_{12}a_{23}\gamma_2}{\gamma_3(a_{33}a_{44} - a_{34}a_{43})}, \overline{N_2} = \frac{\gamma_4 + a_{11}a_{23}\gamma_2}{\gamma_3(a_{33}a_{44} - a_{34}a_{43})}, \overline{N_3} = \frac{a_4a_{34} + a_3a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \quad \overline{N_4} = \frac{a_4a_{33} + a_3a_{43}}{a_{33}a_{44} - a_{34}a_{43}} \tag{4.2.1}$$

Where

$$\gamma_1 = (a_1a_{22} + a_2a_{12})(a_{33}a_{44} - a_{34}a_{43}), \gamma_2 = a_3a_{44} + a_4a_{34}$$
$$\gamma_3 = (a_{11}a_{22} + a_{12}a_{21}), \gamma_4 = (a_1a_{21} - a_2a_{11})(a_{33}a_{44} - a_{34}a_{43})$$

This can exist only when $(a_1a_{21} - a_2a_{11}) > 0$ and $(a_{33}a_{44} - a_{34}a_{43}) > 0$ (4.2.2)

Let us consider small deviations $u_1(t), u_2(t), u_3(t), u_4(t)$ from the steady state

i.e. $N_i(t) = \overline{N}_i + u_i(t), i = 1,2,3,4$ (4.2.3)

Substituting (4.2.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\overline{N}_1u_1 - a_{12}\overline{N}_1u_2 \tag{4.2.4}$$

$$\frac{du_2}{dt} = -a_{22}\overline{N}_2u_2 + a_{21}\overline{N}_2u_1 + a_{23}\overline{N}_2u_3 \tag{4.2.5}$$

$$\frac{du_3}{dt} = -a_{33}\overline{N}_3u_3 + a_{34}\overline{N}_3u_4 \tag{4.2.6}$$

$$\frac{du_4}{dt} = -a_{44}\overline{N}_4u_4 + a_{43}\overline{N}_4u_3 \tag{4.2.7}$$

The characteristic equation of which is

$$\left[\lambda^2 + (a_{11}\overline{N}_1 + a_{22}\overline{N}_2)\lambda + (a_{11}a_{22} + a_{12}a_{21})\overline{N}_1\overline{N}_2\right] \times \left[\lambda^2 + (a_{33}\overline{N}_3 + a_{44}\overline{N}_4)\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N}_3\overline{N}_4\right] = 0 \tag{4. 2.8}$$

The characteristic roots of (4.2.8) are

$$\lambda = \frac{-(a_{11}\overline{N}_1 + a_{22}\overline{N}_2) \pm \sqrt{(a_{11}\overline{N}_1 - a_{22}\overline{N}_2)^2 - 4a_{12}a_{21}\overline{N}_1\overline{N}_2}}{2}, \quad \lambda = \frac{-(a_{33}\overline{N}_3 + a_{44}\overline{N}_4) \pm \sqrt{(a_{33}\overline{N}_3 - a_{44}\overline{N}_4)^2 + 4a_{34}a_{43}\overline{N}_3\overline{N}_4}}{2} \tag{4. 2.9}$$

$$\Rightarrow \lambda = \frac{-(a_{11}\overline{N}_1 + a_{22}\overline{N}_2) \pm \sqrt{\Delta_1}}{2}, \lambda = \frac{-(a_{33}\overline{N}_3 + a_{44}\overline{N}_4) \pm \sqrt{\Delta_2}}{2} \tag{4.2.10}$$

Where

$$\Delta_1 = (a_{11}\overline{N}_1 - a_{22}\overline{N}_2)^2 - 4a_{12}a_{21}\overline{N}_1\overline{N}_2, \ \Delta_2 = (a_{33}\overline{N}_3 - a_{44}\overline{N}_4)^2 + 4a_{34}a_{43}\overline{N}_3\overline{N}_4$$

Case (i): When $\Delta_1 > 0 \; and \; \Delta_2 > 0$

In this case the roots are real and negative.

Hence the equilibrium state is **stable.**

Case (ii): When $\Delta_1 < 0 \; and \; \Delta_2 < 0$

In this case the roots are complex with negative real parts.

Hence the equilibrium state is **stable.**

Case (iii): When $\Delta_1 = 0 \; and \; \Delta_2 = 0$

In this case the roots are repeated, which are negative.

Hence the equilibrium state is **stable.**

The trajectories are given by

$$u_1 = \left[\frac{a_{12}\bar{N}_1\left(u_{10} - u_{20}\right) + \mu_1\lambda_2 + \mu_2 a_{13}\bar{N}_1 - \mu_3}{\lambda_1 - \lambda_2}\right]e^{\lambda_1 t} + \left[\frac{\left(\mu_1 - u_{10}\right)\lambda_1 + u_{10}\lambda_2 + (a_{12}u_{10} - a_{12}u_{20} + \mu_2 a_{13})\bar{N}_1 - \mu_3}{\lambda_2 - \lambda_1}\right]e^{\lambda_2 t} + \sigma_1 e^{\lambda_3 t} + \sigma_2 e^{\lambda_4 t} \tag{4.2.11}$$

$$u_2 = \left[\frac{a_{12}\bar{N}_1\left(u_{10} - u_{20}\right) + \mu_1\lambda_2 + \mu_2 a_{13}\bar{N}_1 - \mu_3}{\lambda_1 - \lambda_2}\right]\delta_1 e^{\lambda_1 t} + \left[\frac{\left(\mu_1 - u_{10}\right)\lambda_1 + u_{10}\lambda_2 + (a_{12}u_{10} - a_{12}u_{20} + \mu_2 a_{13})\bar{N}_1 - \mu_3}{\lambda_2 - \lambda_1}\right]\delta_2 e^{\lambda_2 t} + \sigma_3 e^{\lambda_3 t} + \sigma_4 e^{\lambda_4 t} \tag{4.2.12}$$

$$u_3 = \left[\frac{u_{30}\left(\lambda_3 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{30}\left(\lambda_4 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \tag{4.2.13}$$

$$u_4 = \left[\frac{u_{40}\left(\lambda_3 + a_{33}\bar{N}_3\right) + u_{30}a_{43}\bar{N}_4}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{40}\left(\lambda_4 + a_{33}\bar{N}_3\right) + u_{30}a_{43}\bar{N}_4}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \tag{4.2.14}$$

Here

$$\mu_1 = \sigma_1 + \sigma_2 \,;\, \mu_2 = p_1 + p_2 \,;\; \mu_3 = \sigma_1\lambda_3 + \sigma_2\lambda_4 \,;$$

$$\sigma_1 = \frac{\alpha_2}{{\lambda_3}^2 + \left(a_{11}\bar{N}_1 + a_{22}\bar{N}_2\right)\lambda_3 + \alpha_1} \,;\; \sigma_2 = \frac{\alpha_3}{{\lambda_4}^2 + \left(a_{11}\bar{N}_1 + a_{22}\bar{N}_2\right)\lambda_4 + \alpha_1} \,;$$

$$\alpha_1 = (a_{11}a_{22} + a_{12}a_{21})\bar{N}_1\bar{N}_2 \,;\; \alpha_2 = p_1 a_{13}\bar{N}_1\left(\lambda_3 + a_{22}\bar{N}_2\right);\; \alpha_3 = p_2 a_{13}\bar{N}_3\left(\lambda_4 + a_{22}\bar{N}_2\right);$$

$$p_1 = \frac{u_{30}\left(\lambda_3 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_3 - \lambda_4} \;;\; p_2 = \frac{u_{30}\left(\lambda_4 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_4 - \lambda_3};\;\; \delta_1 = \frac{a_{11}}{a_{12}} - \frac{\lambda_1}{a_{12}\bar{N}_1} \,;$$

$$\delta_2 = \frac{a_{11}}{a_{12}} - \frac{\lambda_2}{a_{12}\bar{N}_1}\ ;\ \sigma_3 = \frac{a_{13}p_1}{a_{12}} + \left(a_{11} - \frac{\lambda_3}{\bar{N}_1}\right)\frac{\sigma_1}{a_{12}}\ ;\sigma_4 = \frac{a_{13}p_2}{a_{12}} + \left(a_{11} - \frac{\lambda_4}{\bar{N}_1}\right)\frac{\sigma_2}{a_{12}}$$

and u_{10}, u_{20}, u_{30}, u_{40} are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

The solutions are illustrated in figures.

Case (i): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_3 < a_4 < a_1 < a_2$

In this case initially S_4 dominates the Prey (S_1) till the time instant t^*_{14} and the dominance gets reversed thereafter. And u_1, u_2, u_3, u_4 are converging asymptotically to the equilibrium point.

Hence the equilibrium point is **stable**.

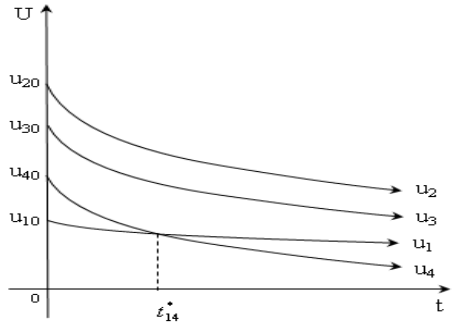

Fig. 3

Case (ii): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $a_2 < a_3 < a_1 < a_4$

In this case initially S_4 dominates the Prey (S_1) and the Predator (S_2) till the time instant $t^*_{14,}$ t^*_{24} respectively and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Predator (S_2) and the Prey (S_1) till the time instant $t^*_{23,}$ t^*_{13} respectively and the dominance gets reversed thereafter. And the Predator (S_2) dominates the Prey (S_1) till the time instant t^*_{12} and thereafter the dominance is reversed.

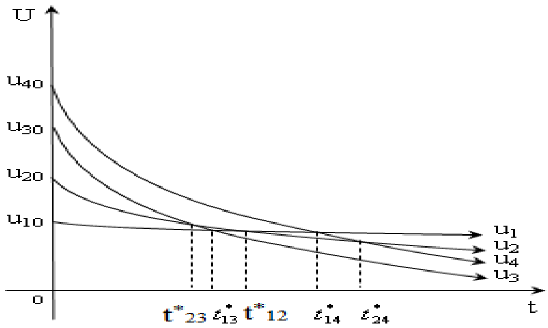

Fig. 4

5. LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

We discussed the local stability of the state of co-existence. We now examine the global stability of the dynamical system (2.1), (2.2), (2.3) and (2.3). We have already noted that this system has a unique, stable non-trivial co-existent equilibrium state at $\overline{N_1} = \dfrac{\gamma_1 + a_{13}a_{22}\gamma_2}{\gamma_3}, \overline{N_2} = \dfrac{\gamma_4 + a_{13}a_{21}\gamma_2}{\gamma_3}, \overline{N_3} = \dfrac{a_4a_{34} + a_3a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \dfrac{a_4a_{33} + a_3a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$

We define a Liapunov function

$$V(N_1, N_2, N_3, N_4) = N_1 - \overline{N}_1 - \overline{N}_1 \log\left(\frac{N_1}{\overline{N}_1}\right) + l_1\left\{N_2 - \overline{N}_2 - \overline{N}_2 \log\frac{N_2}{\overline{N}_2}\right\}$$

$$+\ l_2\left\{N_3 - \overline{N}_3 - \overline{N}_3 \log\left(\frac{N_3}{\overline{N}_3}\right)\right\} + l_3\left\{N_4 - \overline{N}_4 - \overline{N}_4 \log\left(\frac{N_4}{\overline{N}_4}\right)\right\} \qquad (5.1)$$

where l_1, l_2 and l_3 are suitable constants to be determined in the subsequent steps.

Now, the time derivative of V along the solution of (2.1), (2.2), (2.3) and (2.4) is

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N}_1}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_2-\overline{N}_2}{N_2}\right)\frac{dN_2}{dt}+l_2\left(\frac{N_3-\overline{N}_3}{N_3}\right)\frac{dN_3}{dt}+l_3\left(\frac{N_4-\overline{N}_4}{N_4}\right)\frac{dN_4}{dt} \tag{5.2}$$

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N}_1}{N_1}\right)N_1\left\{a_1-a_{11}N_1-a_{12}N_2\right\}+l_1\left(\frac{N_2-\overline{N}_2}{N_2}\right)N_2\left\{a_2-a_{22}N_2+a_{21}N_1+a_{23}N_3\right\}$$
$$+l_2\left(\frac{N_3-\overline{N}_3}{N_3}\right)N_3\left\{a_3-a_{33}N_3+a_{34}N_4\right\}+l_3\left(\frac{N_4-\overline{N}_4}{N_4}\right)N_4\left\{a_4-a_{44}N_4+a_{43}N_3\right\} \tag{5.3}$$

$$=\left(N_1-\overline{N}_1\right)\left\{a_1-a_{11}N_1-a_{12}N_2\right\}+l_1\left(N_2-\overline{N}_2\right)\left\{a_2-a_{22}N_2+a_{21}N_1+a_{23}N_3\right\}$$
$$+l_2\left(N_3-\overline{N}_3\right)\left\{a_3-a_{33}N_3+a_{34}N_4\right\}+l_3\left(N_4-\overline{N}_4\right)\left\{a_4-a_{44}N_4+a_{43}N_3\right\} \tag{5.4}$$

$$\frac{dV}{dt}=\left(N_1-\overline{N}_1\right)\left\{a_{11}\overline{N}_1+a_{12}\overline{N}_2-a_{11}N_1-a_{12}N_2\right\}+l_1\left(N_2-\overline{N}_2\right)\left\{a_{22}\overline{N}_2-a_{21}\overline{N}_1-a_{23}\overline{N}_3-a_{22}N_2+a_{21}N_1+a_{23}N_3\right\}$$
$$+l_2\left(N_3-\overline{N}_3\right)\left\{a_{33}\overline{N}_3-a_{34}\overline{N}_4-a_{33}N_3+a_{34}N_4\right\}+l_3\left(N_4-\overline{N}_4\right)\left\{a_{44}\overline{N}_4-a_{43}\overline{N}_3-a_{44}N_4+a_{43}N_3\right\} \tag{5.5}$$

$$=\left(N_1-\overline{N}_1\right)\left\{-a_{11}\left(N_1-\overline{N}_1\right)-a_{12}\left(N_2-\overline{N}_2\right)\right\}+l_1\left(N_2-\overline{N}_2\right)\left\{-a_{22}\left(N_2-\overline{N}_2\right)+a_{21}\left(N_1-\overline{N}_1\right)+a_{23}\left(N_3-\overline{N}_3\right)\right\}$$
$$+l_2\left(N_3-\overline{N}_3\right)\left\{-a_{33}\left(N_3-\overline{N}_3\right)+a_{34}\left(N_4-\overline{N}_4\right)\right\}+l_3\left(N_4-\overline{N}_4\right)\left\{-a_{44}\left(N_4-\overline{N}_4\right)+a_{43}\left(N_3-\overline{N}_3\right)\right\} \tag{5.6}$$

$$\frac{dV}{dt}=-a_{11}\left(N_1-\overline{N}_1\right)^2-a_{12}\left(N_1-\overline{N}_1\right)\left(N_2-\overline{N}_2\right)$$
$$+l_1\left\{(-a_{22})\left(N_2-\overline{N}_2\right)^2+a_{21}\left(N_1-\overline{N}_1\right)\left(N_2-\overline{N}_2\right)+a_{23}\left(N_2-\overline{N}_2\right)\left(N_3-\overline{N}_3\right)\right\}$$
$$+l_2\left\{(-a_{33})\left(N_3-\overline{N}_3\right)^2+(a_{34})\left(N_3-\overline{N}_3\right)\left(N_4-\overline{N}_4\right)\right\}$$
$$+l_3\left\{(-a_{44})\left(N_4-\overline{N}_4\right)^2+(a_{43})\left(N_3-\overline{N}_3\right)\left(N_4-\overline{N}_4\right)\right\} \tag{5.7}$$

Choosing $l_1=\dfrac{a_{12}}{a_{21}}$, l_2 and l_3 are any positive constants, (5.7) becomes,

$$\frac{d\ V}{dt}=-a_{11}\left(N_1-\overline{N}_1\right)^2-\frac{a_{12}a_{22}}{a_{21}}\left(N_2-\overline{N}_2\right)^2+\frac{a_{12}a_{23}}{a_{21}}\left(N_2-\overline{N}_2\right)\left(N_3-\overline{N}_3\right)-l_2a_{33}\left(N_3-\overline{N}_3\right)^2$$
$$+l_2a_{34}\left(N_3-\overline{N}_3\right)\left(N_4-\overline{N}_4\right)-l_3a_{44}\left(N_4-\overline{N}_4\right)^2+l_3a_{43}\left(N_3-\overline{N}_3\right)\left(N_4-\overline{N}_4\right) \tag{5.8}$$

$$<-a_{11}\left(N_1-\overline{N}_1\right)^2-\frac{a_{12}a_{22}}{a_{21}}\left(N_2-\overline{N}_2\right)^2+\frac{a_{12}a_{23}}{2a_{21}}\left\{\left(N_2-\overline{N}_2\right)^2+\left(N_3-\overline{N}_3\right)^2\right\}$$
$$-a_{33}l_2\left(N_3-\overline{N}_3\right)^2-a_{44}l_3\left(N_4-\overline{N}_4\right)^2+\frac{(a_{34}l_2+a_{43}l_3)}{2}\left\{\left(N_3-\overline{N}_3\right)^2+\left(N_4-\overline{N}_4\right)^2\right\} \tag{5.9}$$

$$<-a_{11}\left(N_1-\overline{N}_1\right)^2+\left[\frac{a_{23}}{2}-a_{22}\right]\frac{a_{12}}{a_{21}}\left(N_2-\overline{N}_2\right)^2+\left[\frac{(a_{34}l_2+a_{43}l_3)}{2}-\frac{a_{12}a_{23}}{2a_{21}}-a_{33}l_2\right]\left(N_3-\overline{N}_3\right)^2$$
$$+\left[\frac{(a_{34}l_2+a_{43}l_3)}{2}-a_{44}l_3\right]\left(N_4-\overline{N}_4\right)^2 \tag{5.10}$$

<0, Provided $\dfrac{a_{23}}{2}<a_{22}$, $\dfrac{(a_{34}l_2+a_{43}l_3)}{2}<\dfrac{a_{12}a_{23}}{2a_{21}}+a_{33}l_2$ and $\dfrac{(a_{34}l_2+a_{43}l_3)}{2}<a_{44}l_3$

Hence the co-existent is globally asymptotically stable.

6. NUMERICAL EXAMPLES:

(1) Let a_1=1,a_{11}=0.1,a_{12}=0.1,a_{23}=0.1,a_2=1,a_{22}=0.2,a_{21}=0.2,a_3=1,a_{33}=0.3, a_{34}=0.2,a_4=1,a_{44}=0.3,a_{43}=0.3,N_1 =20, N_2 =30, N_3 =40, N_4 =50

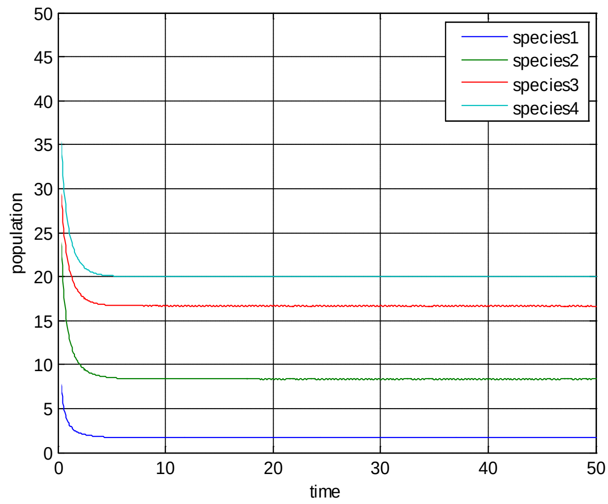

Fig.1

Fig.1: Graph Represents Variations in the growth rate of the populations against time.

(2) Let a_1=2,a_{11}=0.1,a_{12}=0.02,a_{23}=0.1,a_2=2,a_{22}=0.4,a_{21}=0.3,a_3=1,a_{33}=0.3, a_{34}=0.001,a_4=2,a_{44}=0.2,a_{43}=0.001,N_1 =40, N_2 =30, N_3 =20, N_4 =50

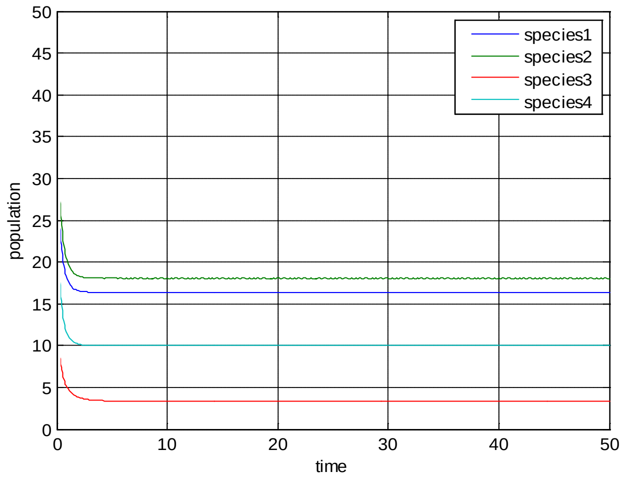

Fig.2

Fig. 2: Graph Represents Variations in the growth rate of the populations against time.

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