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MATHEMATICAL INEQUALITIES ON FRACTIONAL CALCULUS: A SURVEY

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ABSTRACT

The study of mathematical inequalities play very important role in classical differential and integral equations which has applications in many fields. Fractional inequalities are important in studding the existence, uniqueness and other properties of fractional differential equations. In this paper, we present some fractional inequalities, Generalized Gronwall inequalities, Hilbert-Pachpatte inequalities, Minkowski fractional inequalities, Harmite-Hadamard integral inequalities and Opial-Type inequalities.

Keywords: *Generalized Gronwall inequalities, Hilbert-Pachpatte inequalities, Minkowski fractional inequalities, Harmite-Hadamard integral inequalities, Opial-Type inequalities, fractional derivative and fractional integration.*

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1. INTRODUCTION AND PRELIMINARIES

The main objective of the paper is to survey of some recent development in the field of fractional inequalities which is currently receiving considerable attention. We prepare some material which will be needed later since we will be dealing with different result and application in different paper, it will be also task here in this section to unify the notations. We denote $C^m[(0,x)]$ the space of all functions of all $[0,x]$ which have continuous derivative up to order m, $AC([0,x])$ is the space of all absolutely continuous function on $[0,x]$. $AC^m([0,x])$ denote the space of all $g \in C^{m-1}([0,x])$ with $g^{(m-1)} \in A([0,x])$ for any $\alpha \in R$ we denoted by $[\alpha]$ the integral part of α (the integer K satisfying $K \le \alpha \le K+1$). let $\alpha > 0$, and $m = [\alpha]+1$, the space $J^{\alpha}(L_1)$ consist of all functions of f on $[0,x]$ of the form $f \in J^{\alpha}\psi$ for some $\psi \in L[0,x]$.

We shall introduce the following definitions and properties which are used through our this paper.

Definition 1.1: *A function* $f(t)$, *is convex function if*

$$f(\alpha t + (1-\alpha)s) \le \alpha f(t) + (1-t) f(s), \tag{1}$$

where $0 \le \alpha \le 1$.

Definition 1.2: *A function* $f(t)$, *is concave function if*

$$f(\alpha t + (1-\alpha)s) \ge \alpha f(t) + (1-t) f(s), \tag{2}$$

where $0 \le \alpha \le 1$.

Definition 1.3: *A real valued function* $f(t)$, $t \ge 0$ *is said to in space* C_{μ}, $\mu \in R$ *if there exist real number* $p \ge \mu$ *such that*

$f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty[)$.

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Definition 1.4: *A function* $f(t)$, $t \geq 0$ *is said to in space* C_{μ}^{n}, $\mu \in R$ *if* $f^{n}(t) \in C([0,\infty[)$.

Definition 1.5: *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *for function* $f \in C_{\mu}(\mu \geq -1)$ *is defined as:*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, \quad t > 0, \tag{3}$$

$$J^{0} f(t) = f(t),$$

where $\Gamma(\alpha) = \int_{0}^{\infty} e^{-u} u^{\alpha-1} du$.

For our convenience of establishing the result, we give the semigroup property:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \quad \alpha \geq 0, \quad t \geq 0, \tag{4}$$

Definition 1.6: *The Riemann-Liouville fractional derivative of* f *of order* α, *is defined as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^{m} \int_{0}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau, \tag{5}$$

where $m = [\alpha]+1$. In addition, we stipulate

$$D^{0} f := f := J^{0} f, J^{-\beta} f := D^{\beta} f, \quad if \quad \beta > 0, D^{-\alpha} f := J^{\alpha} f, if \quad 0 \leq \alpha \leq 1. \tag{6}$$

If α is positive integer, then $D^{\alpha} f = (\frac{d}{dt})^{\alpha} f$.

We also need the following result on law of indices for fractional integral and differentiation.

Lemma 1.1: (Chapter 1 Theorem 2.5) *The law of indices*

$$J^{u} J^{v} f(t) = J^{u+v} f(t) \tag{7}$$

is valid in the following case:

1. $v > 0, u+v > 0$ and $f \in L[0,t]$;
2. $v < 0, u > 0$ and $f \in J^{v}(L_1)$;
3. $u < 0, u+v < 0$ and $f \in J^{-u-v}(L_1)$.

Lemma 1.2: (5.Lemma 3.1) *Let* $\alpha \geq 0$, $\beta \geq \alpha$, *let* $f \in L(0,x)$ *have an* L^{∞} *fractional derivative* $D^{\beta} f$ *in [0,x], and let* $D^{\beta-k} f(0) = 0$, *for* $k = 1...[\beta]+1$. *Then*

$$D^{\beta} f(t) = \frac{1}{\Gamma(\beta-\alpha)} \int_{0}^{t} (t-\tau)^{\beta-\alpha-1} D^{\beta} f(\tau) d\tau, t \in [0,x]. \tag{8}$$

2. GENERALIZED GRONWALL INEQUALITY

The Gronwall inequality, which play a very important role in classical differential equations. We first recall standard Gronwall inequality which found in [3].

Theorem 2.1: *If* $x(t) \leq h(t) + \int_{t_0}^{t} k(s) x(s) ds \quad t \in [t_0, T)$,

where all functions involved are continuous on $[t_0, T)$, $T \leq +\infty$, and $k(t) \geq 0$, then $x(t)$ satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)\, k(s)\, exp[\int_{s}^{t} k(u) du] ds, \; t \in [t_0, T).$$

If, in addition, $h(t)$ nondecreasing, then

$$x(t) \le h(t)\ exp\left(\int_{t_0}^{t} k(s)ds\right),\ t \in [t_0, T).$$

In [11] author establish A Generalized fractional Gronwall inequality on the base of iteration argument.

Theorem 2.2: *Suppose that* $\beta \ge 0$, $a(t)$ *is nonnegative function locally integrable on* $0 \le t \le T$ *(some* $T \le +\infty$*) and g(t) is nonnegative, nondecreasing continuous function defined on* $0 \le t \le T$, $g(t) \le M(constant)$, *and suppose* $u(t)$ *is nonnegative and locally integrable on* $0 \le t \le T$ *with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1} u(s)ds.$$

on the interval. Then

$$u(t) \le a(t) + \int_0^t \left[\left(\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}\right)(t-s)^{n\beta-1} a(s) \right] ds,\ 0 \le t \le T.$$

Proof: Let $B\phi(t) = g(t)\int_0^t (t-s)^{\beta-1} u(s)ds$, $t \ge 0$, for locally integrable function ϕ. Then

$$u(t) \le a(t) + Bu(t)$$

implies

$$u(t) = \sum_{k=0}^{n-1} B^k a(t) + u(t).$$

Let us prove

$$B^n u(t) \le \int_0^t \left(\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}\right)(t-s)^{n\beta-1} u(s))ds, \tag{9}$$

and $B^n u(t) \to 0$ as $n \to \infty$ for each t in $0 \le t \le T$. We know this relation (2.1) is true for $n = 1$. Assume that it is true for some $n = k$. If $n = k+1$ then the induction hypothesis implies

$$B^{k+1}u(t) = B(B^k u(t)) \le g(t)\int_0^t (t-s)^{\beta-1}\left[\int_0^s \left(\frac{(g(s)\Gamma(\beta))^k}{\Gamma(k\beta)}\right)(s-\tau)^{k\beta-1} u(\tau)d\tau\right]ds$$

since $g(t)$ is nondecreasing , it follows that

$$B^{k+1}u(t) \le (g(t))^{k+1}\int_0^t (t-s)^{\beta-1}\left[\int_0^s \left(\frac{(\Gamma(\beta))^k}{\Gamma(k\beta)}\right)(s-\tau)^{k\beta-1} u(\tau)d\tau\right]ds.$$

By interchanging the order of integration, we have

$$B^{k+1}u(t) \le (g(t))^{k+1}\int_0^t \left[\int_0^t \left(\frac{(\Gamma(\beta))^k}{\Gamma(k\beta)}\right)(t-s)^{\beta-1}(s-\tau)^{k\beta-1} ds\right] u(\tau)d\tau$$
$$= \int_0^t \left(\frac{(\Gamma(\beta))^{k=1}}{\Gamma((k+1)\beta)}\right)(t-s)^{(k+1)\beta-1} u(s)ds,$$

where the integral

$$\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{k\beta-1} ds = (t-\tau)^{k\beta+\beta-1}\int_0^1 (1-z)\ z^{k\beta-1}dz$$
$$= (t-\tau)^{(k+1)\beta-1}\beta(k\beta,\beta)$$
$$= \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta}(t-\tau)^{(k+1)\beta-1},$$

is evaluated with the help of substitution $s = (\tau + z(t-\tau))$ and the definition of beta function see [8. PP.67].

Hence the relation (2.1) is proved, since

$$B^n u(T) \le \int_0^t \left(\frac{(M\Gamma(\beta))^n}{\Gamma(n\beta)}\right)(t-s)^{n\beta-1}u(s))ds \to 0 \text{ as } n \to +\infty \text{ for } t \in [0,T),$$

hence theorem is proved.

3. HILBERT-PACHPATTE TYPE INTEGRAL INEQUALITY FOR FRACTIONAL DERIVATIVE

Recall the original Hilbert double integral inequality see [6.Theorem 316]

Theorem 3.1: *If* $p > 1$, $q = p/p-1$ *and*

$$\int_0^\infty f^p(t)dt < F, \qquad \int_0^\infty g^q(s)ds < G,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(t)g(s)}{t+s}dt\, ds < \frac{\pi}{sin\pi/p}F^{1/p} \; G^{1/q},$$

where f, g are nonmeasurable function not identically zero.

In [7. theorem 1] Pachpatte obtained analogues of Hilbert inequality

Theorem 3.2: *If* $n \ge 1$, *and* $0 \le k \le n-1$ *be integers. Let* $u \in C^n([0,x])$ *and* $v \in C^n([0,y])$, *where* $x > 0$, $y > 0$ *and let* $u^{(i)}(0) = v^{(j)}(0) = 0$ *for* $j \in \{0,...n-1\}$. *Then*

$$\int_0^y \frac{|u^{(k)}(t)||v^{(k)}(s)|}{t^{2n-2k-1}+s^{2n-2k-1}}dt\, ds \le M(n.k,x,y)\left(\int_0^x (x-t)|u^n(t)|^2\, dt\right)^{1/2} \times \left(\int_0^y (y-s)|v^n(s)|^2\, ds\right)^{1/2} \tag{10}$$

where

$$M(n.k,x,y) = \frac{1}{2}\frac{\sqrt{xy}}{[(n-k-1)!]^2[2n-2k-1]}. \tag{11}$$

In [5] to drive the theorem for Hilbert-Pachpatte Type Integral Inequality for Fractional Derivative they used some facts about fractional derivative in[9. Chapter 1].

Theorem 3.3: *For each* $i \in \{1,...n\}$ *let* $x_i > 0, u_i \in L(0,x_i)$ *and* $\Phi_i \in L^\infty(0,x_i)$ *be non negative,* $r_i > -1$, *let* $p_i, q_i > 1$ *satisfy* $1/p_i + 1/q_i = 1$, $w_i > 0$ *satisfy* $\sum_{i=1}^n w_i = 1$, *and* $a_i, b_i \in [0,1]$ *satisfy* $a_i + b_i = 1$; *in addition,* $b_i > (r_i+1)/(1-q_i)r_i$ *for those* i *for for which* $r_i < 0$. *If*

$$|u_i(t_i)| \le \int_0^{t_i}(t_i-\tau_i)^{r_i}\Phi(\tau_i)d\tau_i, \qquad t_i \in [0,x_i], i = 1,...n, \tag{12}$$

then

$$\int_0^{x_1}...\int_0^{x_n} \frac{\prod_{i=1}^n |u_i(t_i)|}{\sum_{i=1}^n w_i t_i^{((a_i+b_iq_i)r_i+1)/q_iw_i}}dt_1...dt_n \le \Omega \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i}(x_i-t_i)^{a_ir_i+1}\Phi_i(t_i)dt_i\right)^{1/p_i} \tag{13}$$

Where $$\Omega = \frac{1}{\prod_{i=1}^n \left[((a_i+b_iq_i)r_i+1)^{1/q_i}(a_ir_i+1)^{1/p_i}\right]} \tag{14}$$

using above theorem [3.3]. They drive Hilbert-Pachpatte type integral inequality for fractional derivative as follows:

Theorem 3.4: *Let* $n \geq N$. *For each* $i \in \{1,...n\}$ *let* $x_i > 0$, $\alpha_i > 0$, $\beta_i > \alpha_i$. *Let* p_i, q_i *satisfy* $1/p_i + 1/q_i = 1$, $w_i > 0$ *satisfy* $\sum_{i=1}^{n} w_i = 1$, *and* a_i, $b_i \in [0,1]$ *satisfy* $a_i + b_i = 1$; *if* $\beta_i < \alpha_i + 1$, *let in addition* $b_i > (\beta_i - \alpha_i)/(q_i - 1)(1 - \beta_i + \alpha_i)$. *Write* $r_i = \beta_i + \alpha_i - 1$. *If, for each* $i \in \{1,...n\}$, $f_i \in L[0, x_i]$ *has an* L^∞ *fractional derivative* $D^{\beta_i} f_i$ *and* $D^{\beta_i - j} f_i(1) = 0$ *for* $j = 1,...[\beta_j] + 1$, *then*

$$\int_0^{x_1} ... \int_0^{x_n} \frac{\prod_{i=1}^{n} |D^{\alpha_i} f_i(t_i)|}{\sum_{i=1}^{n} w_i t_i^{((a_i + b_i q_i) r_i + 1)/q_i w_i}} dt_1 ... dt_n \leq \Omega \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - t_i)^{a_i r_i + 1} |D^{\beta_i} f_i(t_i)|^{p_i} \, dt_i \right)^{1/p_i} \tag{15}$$

Where

$$\Omega = \frac{1}{\prod_{i=1}^{n} \left[\Gamma(r_i + 1)((a_i + b_i q_i) r_i + 1)^{1/q_i} (a_i r_i + 1)^{1/p_i} \right]}. \tag{16}$$

Proof: By lemma(1.2),

$$D^{\alpha_i} f_i(t_i) = \frac{1}{\Gamma(r_i + 1)} \int_0^{t_i} (t_i - \tau_i)^{r_i} D^{\beta_i} f_i(\tau_i) d\tau_i, \quad t_i \in [0, x_i]. \tag{17}$$

Set

$$\Phi_i(t_i) = \frac{|D^{\beta_i} f_i(t_i)|}{\Gamma(r_i + 1)}.$$

Then theorem [3.5] applies with $r_i = \beta_i + \alpha_i - 1 > -1$.

4. MINKOWSKI INTEGRAL INEQUALITY

The well-know Minkowski Integral Inequality is given in [2].

Theorem 4.1: *Let* $p \geq 1$, $0 < \int_a^b f^p(t) dt \leq \infty$ *and* $0 < \int_a^b g^q(t) dt \leq \infty$. *Then*

$$\left(\int_a^b (f(t) + g(t))^p dt \right)^{1/p} \leq \left(\int_a^b f^p(t) dt \right)^{1/p} + \left(\int_a^b g^q(t) dt \right)^{1/p}. \tag{18}$$

and using this in author established the reverse Minkowski integral inequality as:

Theorem 4.2: *Let* f *and* g *be positive function satisfying*

$$0 < m \leq \frac{f(t)}{g(t)} \leq M, \text{ for all } \quad t \in [a,b]. \tag{19}$$

Then

$$\left(\int_a^b f^p(t) dt \right)^{1/p} + \left(\int_a^b g^q(t) dt \right)^{1/p} \leq c \left(\int_a^b (f(t) + g(t))^p dt \right)^{1/p}. \tag{20}$$

where

$$c = \frac{M(m+1) + (M+1)}{(m+1)(M+1)}.$$

In [4] author established reverse Minkowski fractional integral inequality.

Theorem 4.3: *Let* $\alpha > 0$, $p \geq 1$ *and let* f, g *be two positive function on* $[0, \infty[$, *such that for all* $t > 0$,

$J^{\alpha} f^{p}(t) < \infty$, $J^{\alpha} g^{p}(t) < \infty$. *If* $0 < m \le \frac{f(\tau)}{g(\tau)} \le M$, $\tau \in [0,t]$, *then we have*

$$\left[J^{\alpha} f^{p}(t)\right]^{1/p} + \left[J^{\alpha} g^{p}(t)\right]^{1/p} \le \frac{1+M(m+2)}{(m+1)(M+1)} \left[J^{\alpha}(f+g)^{p}(t)\right]^{1/p}. \tag{21}$$

Proof: Using the condition $\frac{f(\tau)}{g(\tau)} \le M$, $\tau \in [0,t]$, $t > 0$, we can write $(M+1)^{p} f(\tau) \le M^{p}(f+g)^{p}(t)$. (22)

Multiplying both side of [4.5] by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $\tau \in (0,t)$, then integrating the resulting inequality with respect to τ, over $(0,t)$, we obtain,

$$\frac{(M+1)^{p}}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f^{p}(\tau) d\tau \le \frac{M^{p}}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} (f+g)^{p}(\tau) d\tau,$$

which is equivalent to

$$J^{\alpha} f^{p}(t) \le \frac{(M+1)^{p}}{M^{p}} J^{\alpha}(f+g)^{p}(t).$$

Hence we can write

$$\left[J^{\alpha} f^{p}(t)\right]^{1/p} \le \frac{M}{(M+1)} \left[J^{\alpha}(f+g)^{p}(t)\right]^{1/p}. \tag{23}$$

On other hand, using the condition $mg(t) \le f(\tau)$, we can write

$$\left(1+\frac{1}{m}\right) g(\tau) \le \frac{1}{m}\left(f(\tau)+g(\tau)\right).$$

Therefore,

$$\left(1+\frac{1}{m}\right)^{p} g^{p}(\tau) \le (\frac{1}{m})^{p}\left(f(\tau)+g(\tau)\right)^{p}. \tag{24}$$

Now, multiplying both side of [4.7] by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $\tau \in (0,t)$, then integrating the resulting inequality with respect to τ, over $(0,t)$, we obtain

$$\left[J^{\alpha} g^{p}(t)\right]^{1/p} \le \frac{1}{m+1} \left[J^{\alpha}(f+g)^{p}(t)\right]^{1/p}. \tag{25}$$

Adding the inequality [4.6] and [4.8], we obtain the inequality

$$\left[J^{\alpha} f^{p}(t)\right]^{1/p} + \left[J^{\alpha} g^{p}(t)\right]^{1/p} \le \frac{1+M(m+2)}{(m+1)(M+1)} \left[J^{\alpha}(f+g)^{p}(t)\right]^{1/p}.$$

5. HERMIT-HADAMARD'S INEQUALITY

We recall Hermit-Hadamard inequality in [10] as follows:

Hermit-Hadamard Inequality: Let $f : I \subset R \to R$ be convex function on interval I of real numbers and $a, b \in I$ with $a < b$. The following Hermit-Hadamard inequality for convex function holds:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_{a}^{b} f(t) dt \le \frac{f(a)+f(b)}{2}. \tag{26}$$

If the function f is concave, the inequality [5.1] can be written as follows:

$$\frac{f(a)+f(b)}{2} \le \frac{1}{b-a}\int_a^b f(t)dt \le f\left(\frac{a+b}{2}\right). \tag{27}$$

In [4] author develop some new result related to Hermit-Hadamard Integral Inequality by using Riemann-Liouville fractional integral as follows

Lemma 5.1: *Let h be a concave function on $[a,b]$. Then we have*

$$h(a)-h(b) \le h(b+a-t)+h(t) \le 2h\left(\frac{a+b}{2}\right). \tag{28}$$

Theorem 5.1: *Let $\alpha>0, \beta>0, p>1, q>1$ and let f, g be two positive function on $[0,\infty[$. If f^p, g^q are concave function on $[0,\infty[$, then we have*

$$\begin{aligned}&2^{2-p-q}\big(f(0)+f(t)\big)^p\big(g(0)+g(t)\big)^q\left[J^\alpha(t^{\beta-1})\right]^2\\ &\le\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta(t^{\alpha-1}f^p(t))+J^\alpha(t^{\beta-1}f^p(t))\right]\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta(t^{\alpha-1}g^q(t))+J^\alpha(t^{\beta-1}g^q(t))\right].\end{aligned} \tag{29}$$

Proof: Since f^p and g^q are concave function on $[0,\infty[$, then by Lemma[5.1] for any $t>0$, we have

$$f^p(0)+f^p(t) \le f^p(t-\tau))+f^p(\tau) \le 2f^p(\frac{t}{2}). \tag{30}$$

$$g^q(0)+g^q(t) \le g^q(t-\tau))+g^q(\tau) \le 2g^q(\frac{t}{2}). \tag{31}$$

Multiplying both side on [5.5] and [5.6] by $\frac{(t-\tau)^{\alpha-1}J^{\beta-1}}{\Gamma(\alpha)}$, $\tau\in(0,1)$, then integrating the resulting inequality w.r.t τ over $(0,t)$, we obtain

$$\begin{aligned}\frac{f^p(0)+f^p(t)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau &\le \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}f^p(t-\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}f^p(\tau)d\tau\\ &\le\frac{2f^p(\frac{t}{2})}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau\end{aligned} \tag{32}$$

and

$$\begin{aligned}\frac{g^q(0)+g^q(t)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau &\le \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}g^q(t-\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}g^q(\tau)d\tau\\ &\le\frac{2g^q(\frac{t}{2})}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau.\end{aligned} \tag{33}$$

Using change of variable $t-\tau=y$, we obtain,

$$\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}f^p(t-\tau)d\tau=\frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta(t^{\alpha-1}f^p(t)). \tag{34}$$

and

$$\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}g^q(t-\tau)d\tau=\frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta(t^{\alpha-1}g^q(t)). \tag{35}$$

By the relation [5.7] and [5.9], we can state that

$$\left(f^p(0)+f^p(t)\right)\left(J^\alpha(t^{\beta-1})\right) \le \frac{\Gamma(\beta)}{\Gamma(\alpha)} J^\beta(t^{\alpha-1} f^p(t)) + J^\alpha(t^{\beta-1} f^p(t)) \le 2 f^p(\frac{t}{2})(J^\alpha(t^{\beta-1})). \tag{36}$$

and [5.8] and [5.10], we can write

$$\left(g^q(0)+g^q(t)\right)\left(J^\alpha(t^{\beta-1})\right) \le \frac{\Gamma(\beta)}{\Gamma(\alpha)} J^\beta(t^{\alpha-1} g^q(t)) + J^\alpha(t^{\beta-1} g^q(t)) \le 2 g^q(\frac{t}{2})(J^\alpha(t^{\beta-1})). \tag{37}$$

The inequality [5.11] and [5.12] implies that

$$\begin{aligned}&\left(f^p(0)+f^p(t)\right)\left(g^q(0)+g^q(t)\right)\left[J^\alpha(t^{\beta-1})\right]^2 \\ &\le \left[\frac{\Gamma(\beta)}{\Gamma(\alpha)} J^\beta(t^{\alpha-1} f^p(t)) + J^\alpha(t^{\beta-1} f^p(t))\right]\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)} J^\beta(t^{\alpha-1} g^q(t)) + J^\alpha(t^{\beta-1} g^q(t))\right].\end{aligned} \tag{38}$$

As before, since f and g are positive function, then for any $t>0$, $p \ge 1$, $q \ge 1$ we have

$$\frac{(f^p(0)+f^p(t)}{2} J^\alpha(t^{\beta-1}) \ge 2^{-p}(f(0)+f(t))^p J^\alpha(t^{\beta-1}), \tag{39}$$

and

$$\frac{(g^q(0)+g^q(t)}{2} J^\alpha(t^{\beta-1}) \ge 2^{-q}(q(0)+q(t))^q J^\alpha(t^{\beta-1}). \tag{40}$$

The inequalities [5.14] and [5.15] implies that

$$\frac{(f^p(0)+f^p(t))(g^q(0)+g^q(t))}{4}\left[J^\alpha(t^{\beta-1})\right]^2 \ge 2^{-p-q}(f(0)+f(t))^p(q(0)+q(t))^q\left[J^\alpha(t^{\beta-1})\right]^2. \tag{41}$$

Combining [5.13]and [5.16], we obtain required inequality [5.4].

6. OPIAL-TYPE INEQUALITIES FOR FRACTIONAL DERIVATIVE

We know that the original Opial-Type inequality see [1. Chapter 1 Theorem 1.1].

Theorem 6.1: *Let* $f \in C^1([0,h])$ *such that*

$$f(0)=f(h)=0,\ f(t)>0 \text{ on } (0,h).$$

Then

$$\int_0^h |f(t)f'(t)|\,dt \le \frac{h}{4}\int_0^h f'^2(t)dt, \tag{42}$$

where $\frac{h}{4}$ the best constant.

In [1. Chapter 2] author established Opial-Type Inequality For Fractional Derivative as:

Theorem 6.2: *Let* $f \in C^\upsilon_{t_0}([a,b])$, $\upsilon \ge 0$ *and* $f^i(t_0)=0, i=0,1,2\ldots n-1, n:=[\upsilon]$. *Here* $t,t_0 \in [a,b]; t \ge t_0$, *Let* $p,q>1$ *such that* $1/p+1/q=1$. *Then*

$$\int_{t_0}^t |f(w)||D^\upsilon_{t_0} f(w)|\,dw \le \left(\frac{2^{-1/p}(t-t_0)^{p\upsilon-p+2/p}}{\Gamma(\upsilon)\left((p\upsilon-p+1)(p\upsilon-p+2)\right)^{1/p}}\right)\times\left(\int_{t_0}^t |(D_{t_0} f)^\upsilon(w)|^q\,dw\right)^{2/q} \tag{43}$$

Proof: Let $f \in C_{t_0}^{\upsilon}([a,b])$ $\upsilon \geq 0$ and $f^{i}(t_0) = 0, i = 0,1,2...n-1, n := [\upsilon]$. Then by fractional Taylor formula [1. Chapter 2 Theorem 2.1] as:

$$f(t) = \frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} (t-\tau)^{\upsilon-1} (D_{t_0}^{\upsilon} f)(\tau) d\tau. \tag{44}$$

hence from [6.3] and Holder's inequity we get $(t \geq t_0)$.

$$\begin{aligned} |f(t)| &\leq \frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} (t-\tau)^{\upsilon-1} |(D_{t_0}^{\upsilon} f)(\tau)| d\tau \\ &\leq \frac{1}{\Gamma(\upsilon)} \left(\int_{t_0}^{t} ((t-\tau)^{\upsilon-1})^{p} dt \right)^{1/p} \left(\int_{t_0}^{t} (|D_{t_0}^{\upsilon} f|(\tau))^{q} dt \right)^{1/q} \\ &= \frac{(t-t_0)^{p\upsilon-p+1/p}}{\Gamma(\upsilon)(p\upsilon-p+1)^{1/p}} \left(\int_{t_0}^{t} (|D_{t_0}^{\upsilon} f|(\tau))^{q} d\tau \right)^{1/q}. \end{aligned} \tag{45}$$

Set $z(t) := \int_{t_0}^{t} (|D_{t_0}^{\upsilon} f|(t))^{q} dt$, $(z(t_0) = 0)$.

Then $(z^{'}(t)) = (|D_{t_0}^{\upsilon} f|(t))^{1/q}$ and $|D_{t_0}^{\upsilon} f| = (z^{'}(t))^{1/q}$, $\forall$ $t_0 \leq t \leq b$.

Therefore [6.4], we have

$$|f(w)||D_{t_0}^{\upsilon} f|(w) \leq \frac{(w-t_0)^{p\upsilon-p+1}}{\Gamma(\upsilon)(p\upsilon-p+1)^{1/p}} \left\{ \int_{t_0}^{w} (|D_{t_0}^{\upsilon} f|(\tau))^{q} d\tau \, z^{'}(w) \right\}^{1/q}. \tag{46}$$

all $t_0 \leq w \leq t$. Next we integrate [6.5] over $[t_0, t]$

$$\begin{aligned} \int_{t_0}^{t} |f(w)||D_{t_0}^{\upsilon} f|(w) dw &\leq \frac{1}{\Gamma(\upsilon)(p\upsilon-p+1)^{1/p}} \int_{t_0}^{t} (w-t_0)^{p\upsilon-p+1/p} (z(w) z^{'}(w))^{1/q} dw \\ &\leq \frac{1}{\Gamma(\upsilon)(p\upsilon-p+1)^{1/p}} \int_{t_0}^{t} \left((w-t_0)^{p\upsilon-p+1} dw \right)^{1/p} \left(z(w)(z^{'}(w)) dw \right)^{1/q} \\ &= \frac{(t-t_0)^{p\upsilon-p+2/p}}{\Gamma(\upsilon)(p\upsilon-p+1)^{1/p}(p\upsilon-p+2)^{1/p}} \frac{[z(t)]^{2/q}}{2^{1/q}} \\ &= \frac{2^{-1/q}(t-t_0)^{p\upsilon-p+2/p}}{\Gamma(\upsilon)\left((p\upsilon-p+1)(p\upsilon-p+2)^{1/p}\right)^{1/p}} \left(\int_{t_0}^{t} (|D_{t_0}^{\upsilon} f(w)|)^{q} dw \right)^{2/q}. \end{aligned}$$

Thus the proof.

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