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DOMINATION SUBDIVISION STABLE GRAPHS

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ABSTRACT

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set - abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by $\gamma(G)$. A graph G is said to be domination subdivision stable (DSS), if the γ- value of G does not change by subdividing any edge of G. In this paper, we have obtained necessary and sufficient condition for a graph G to be a DSS graph. We have discussed conditions under which a graph is DSS and not DSS. We have generated new DSS graphs from existing ones and proved that every graph G is an induced sub graph of DSS graph.

AMS Subject Classification (2010): *05C.*

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1. INTRODUCTION

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for G is called the domination number of *G* and it is denoted by $\gamma (G)$. A γ - Set denotes a dominating set for G with minimum cardinality.

The subgraph of G induced by the vertices in D is denoted by < D>. The open neighborhood of vertex $v \in V (G)$ is denoted by $N (v) = \{ u \in V (G) \mid uv \in E (G) \}$ while its closed neighborhood is the set $N [v] = N (v) \cup \{ v \}$. A vertex v is said to be a, down vertex if $\gamma (G – u) < \gamma (G)$, level vertex if $\gamma (G – u) = \gamma (G)$, up vertex if $\gamma (G – u) > \gamma (G)$. A vertex v is said to be selfish in the γ- set D, if v is needed only to dominate itself. A vertex v is said to be good if there is a γ – set of G containing v. If there is no γ – set of G containing v, then v is said to be a bad vertex. A vertex in V – D is k – dominated if it is dominated by at least k – vertices in D ie., $|N (v) \cap D| \geq k$. If every vertex in V – D is k – dominated then D is called k – dominating set.

For a pair of adjacent vertices u, v of G, we denote by G. uv the graph obtained by identifying u and v. Let uv denote the identified vertex. In [1], Tamara Burton and David. P. Sumner defined a graph to be domination dot critical (DDC) if $\gamma (G.uv) < \gamma (G)$, $\forall$ u, v $\in$ V (G). In [3], M. Yamuna and K. Karthika have introduced the concepts of domination dot stable graphs. A graph G is said to be to domination dot stable (DDS) if $\gamma (G.uv) = \gamma (G)$ $\forall$ u, v $\in$ V (G), such that $u \perp v$. They have obtained necessary and sufficient conditions for a graph G to be DDS and have discussed properties of DDS graphs.

A subdivision of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints {u, v} yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, {u, w} and {w, v}.

In this paper we define domination subdivision stable graphs and initiate a study on them.

2. DOMINATION SUBDIVISION STABLE GRAPHS

A graph G is said to be domination subdivision stable (DSS) if the γ - value of G does not change by subdividing any edge of G.

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We shall denote the graph obtained by subdividing any edge uv of a graph G, by G $_{sd}$ uv. Let w be a vertex introduced by subdividing uv. We shall denote this by G $_{sd}$ uv = w.

Examples of DSS graphs

1. P_n is DSS if and only if $\gamma (P_n) = \gamma (P_{n+1})$.
2. C_n is DSS if and only if $\gamma (C_n) = \gamma (C_{n+1})$.
3. Petersen's graph.
4. A Complete Bipartite graph $K_{m, n}$.

The graph *G* given in Fig. 1 is DSS.

Fig. 1

In Fig. 1, γ (G) = γ (G $_{sd}$ uv) = 2. This is true$\forall$ e = (a b)$\in$ E (G). Here G is a DSS graph.

Theorem 2.1: A graph G is DSS if and only if for every u, v $\in$ V (G), either $\exists$ a γ - set containing u and v or $\exists$ a γ - set D such that

1. PN [u, D] = {v}
2. v is 2 – dominated.

Proof: Let G be a DSS graph and let D be a γ - set for G. Let u, v $\in$ V (G) and D′ be a γ - set for G_{sd} uv.

Case 1: w $\in$ D′

In this case u, v $\notin$ D′. If u $\in$ D′, then D = D′ - {w} is a γ - set for G such that | D | < | D′ | [v is dominated by u] which is a contradiction for G is a DSS graph.

When w $\in$ D′, u, v $\notin$ D. D = D′ - {w}$\cup${u} is a γ set for G. v is 2 – dominated in G, if v $\notin$ PN [w, D′] else v $\in$ PN [u, D] if v $\in$ PN [w, D′].

Case 2: w $\notin$ D′

Subcase 1: u $\in$ D′, v $\notin$ D′

D′ is a γ - set for G such that v is 2- dominated.

Subcase 2: u, v $\in$ D′

D′ is a γ - set for G containing u and v.

Subcase 3: u $\notin$ D′, v $\in$ D′

D′ is a γ - set for G such that u is 2- dominated.

Conversely if $\exists$ a γ - set D′ containing u and v or D′ is a γ - set such that u $\in$ D′ , v is 2 – dominated, then D′ itself is a γ - set for G $_{sd}$ uv. If D′ is a γ - set for G such that PN [u, D′] = {v}, then D = D′ – {u} – {w} is a γ - set for G $_{sd}$ uv.

Hence G is DSS.

Theorem 2.2: For any graph G, γ (G $_{sd}$ uv) $\geq$ γ (G) $\forall$ e = (u v) $\in$ E (G).

Proof: Let G be a graph and D be its dominating set. Consider G $_{sd}$ uv, where e = (u v) $\in$ E (G). Let D$'$ be a γ - set for G $_{sd}$ uv. If possible let | D$'$ | < | D |.

Case 1: w $\in$ D$'$

In this case either u or v may belong to D$'$, but both u and v cannot be in D$'$.

If u, v $\notin$ D$'$, w $\in$ D$'$, then D$''$ = D$'$ – {w} $\cup$ {u} is a γ - set for G such that | D$''$ | < | D |.

If either u or v $\in D'$, then $D''' = D' - \{w\}$ is a γ - set for G such that such that $| D''' | < | D |$.

Case 2: $w \notin D'$
In this case D' itself is a γ - set for G such that $| D' | < | D |$.

In both cases, we get a contradiction. Hence $\gamma (G_{sd}\, uv) \geq \gamma (G)\ \forall\ e = (u\, v) \in E (G)$.

Theorem 2.3: If G is a graph such that every vertex is a down vertex, then G is DSS.

Proof: Let G be a graph and let $e = (u\, v) \in E (G)$. Let $G_{sd}\, uv = w$. Since every vertex is a down vertex, u is down vertex, ie., $\gamma (G - u) = \gamma (G) - 1$. Also $D' = \gamma (G - \{u\}) \cup \{u\}$ is a γ - set for G, where any $x \in N (u)$ is 2 – dominated, if $x \notin D'$.

If $v \in D'$, then D' is a γ - set for $G_{sd}\, uv$ also.

If $v \notin D'$, then by theorem [2.1], $G_{sd}\, uv$ is DSS. This is true $\forall\ e = (u\, v) \in E (G)$ ie., G is DSS.

Corollary 1: If G is a graph such that $\forall\ e = (u\, v) \in E (G)$, either u or v is critical, then G is DSS.

Proof: Let G be graph and $e = (u\, v) \in E (G)$. Either u or v is critical. Let us assume that u is critical $\gamma (G - u) = \gamma (G) - 1$. Also $\gamma (G - \{u\}) \cup \{u\}$ is a γ - set for G. By theorem [2.3] $\gamma (G_{sd}\, u\, v) = \gamma (G)\ \forall\ e = (u\, v) \in E (G)$. Hence G is DSS.

Corollary 2: If G is a graph that has at least one down vertex u, then G has a γ - set that contains at least one selfish vertex u. Also $\gamma (G_{sd}\, uv) = D = \gamma (G), \forall\ v \in N (u)$.

Proof: Let G be a graph that has a down vertex u. Then by theorem [2.3] $\exists$ a γ - set for G such that N (u) is 2 – dominated ie., u is a selfish vertex, since N (u) is 2 – dominated. By theorem [2.1], $\gamma (G_{sd}\, u\, v) = \gamma (G),\ \forall\ v \in N (u)$ ie., if u is a selfish vertex, then $\gamma (G_{sd}\, u\, v) = \gamma (G),\ \forall\ v \in N (u)$.

Corollary 3: Let G be a graph $\forall\ u \in V (G)$, $\exists$ a γ – set for G such that u is selfish. Then G is DSS.

Proof: Let $e = (u\, v) \in E (G)$. By the given conditions $\exists$ a γ – set for G such that u is selfish. By corollary [2] of theorem [2.3], $\gamma (G_{sd}\, uv) = \gamma (G)$. This is true $\forall\ e \in E (G)$. Hence G is DSS.

Corollary 4: Every DDC graph is DSS.

Proof: Let G be DDC graph. Let $u, v \in V (G)$ and D be a γ - set for G. In [1], it has been proved that, "Let $a, b \in V (G)$ for a graph G. Then $\gamma(G_{\bullet}\, ab) < \gamma (G)$ if and only if either there exists an MDS S of G such that $a, b \in S$ or atleast one of a or b is critical in G".

If $u, v \in D$, then D is γ - set for $G_{sd}\, uv$ also ie., G is DSS.

If either u or v is critical, G is DSS [By corollary 1 of theorem 2.3].

Hence every DDC graph is DSS.

Theorem 2.4: Let G be a DSS graph, then
1. Every support vertex has exactly one pendant vertex adjacent to it.
2. If v is a pendant vertex then $\exists$ at least one γ - set of G including v.
3. If the pendant vertex v is selfish then v is a down vertex.
4. If the pendant vertex v is not selfish then v is a level vertex.

Proof:
1. Let u be support vertex. Let $x, y \in V (G)$ where x, y are pendant vertices such that $x, y \perp u$. Then u is included in every γ - set. $\gamma (G_{sd}\, ux) = \gamma (G_{sd}\, uy) = \gamma (G) + 1$, which is contradiction as G is DSS.
2. Let G be DSS and let $v \in V (G)$ be a pendant vertex and u be the support vertex. Let $G_{sd}\, uv = w$. Let D be a γ - set for $G_{sd}\, uv$. In $G_{sd}\, uv$ either $v \in D$ or $v \notin D$.

If $v \in D$ then D is a γ - set for G also such that $v \in D$.

If $v \notin D$ then $w \in \gamma (G_{sd}\ uv)$ since v is pendant. $D - \{w\} \cup \{v\}$ is a γ - set for G ie., γ (G) contains v. Hence v belongs to some γ - set of G.

3. If v is a pendant vertex then $\exists$ a γ – set containing v. If v is selfish, then $\gamma (G - v) = \gamma (G) - 1$ ie., v is a down vertex.
4. Let PN [v, D] = u and $\exists$ no $w \in N(u)$, $w \neq v$ such that $w \in D$ [Since if such a vertex exist then v becomes selfish]. $\gamma (G - v) \leq \gamma (G)$. If $\gamma (G - v) < \gamma (G)$, then by corollary [2] of theorem [2.3], v is selfish which is not possible. Hence $\gamma (G - v) = \gamma (G)$ ie., v is a level vertex.

Theorem 2.5: Let G be a graph such that

1. G is DDS,
2. $N (u) \geq 2, \forall u \in V (G)$,
3. $v \in PN [u, D]$, for some $u \in D$, $\forall v \in V - D$, $\forall \gamma$ - set D of G,

then G is not DSS.

Proof: Let G be a graph satisfying assumptions 1, 2, and 3. Let us assume that G is DSS. Let $G_{sd}\ uv = w$. Let D' be γ - set for $G_{sd}\ uv$.

Case 1: $u \in D'$, $v, w \notin D'$

Since u does not dominate v, $\exists$ one x such that $x \in D'$ and $x \perp v$. D' is a γ -set for G where $u, x \in D'$ such that $u \perp v \perp x$ ie., v is 2 – dominated which is a contradiction.

Case 2: $w \in D'$, $u, v \notin D'$

Subcase 1: If w is selfish, then u and v are 2 – dominated vertices in $G_{sd}\ uv$. $D'' = D' - \{w\} \cup \{u\}$ is a γ - set for G where u is selfish which is contradiction as G is DDS.

Subcase 2: If either $u \in PN [w, D']$ or $v \in PN [w, D']$. Let us assume that $u \in PN [w, D']$. Then $\exists$ one x such that $x \in D$, $x \perp y \perp u$ where $y \in N (u)$. D'' is a γ - set for G such that $u \perp y \perp x$, where $u, x \in D''$ which is contradiction as y is 2 – dominated.

Subcase 3: If $u, v \in PN [w, D']$ then as in subcase 2, D'' is a γ - set for G such that $u \perp y \perp x$, which is a contradiction as y is 2 – dominated.

Subcase 4: If $u, v \notin PN [w, D']$. Let v be 2 – dominated, then $\exists$ one $x \in D'$ such that $v \perp x$. D'' is a γ - set for G such that $u \perp v \perp x$, where $u, x \in D'$ which is contradiction as v is 2 – dominated. Also $D''' = D' - \{w\} \cup \{v\}$ is a γ - set for G such that $v \perp x$, where $v, x \in D'''$ which is contradiction as G is DDS.

Case 3: $v \in D'$, $u, w \notin D'$

Since v does not dominates u, this case is similar to case 1, where u will be a 2 – dominated vertex for G with respect to D'.

Case 4: $u, w \in D'$, $v \notin D'$

Subcase 1: If $v \in PN [w, D']$, then $\exists$ one x such that $x \in D'$ and $x \perp y \perp v$, where $y \in N (v)$. $D''' = D' - \{w\} \cup \{v\}$ is a γ - set for G such that $v \perp y \perp x$, where $v, x \in D'$ which is contradiction as y is 2 – dominated. Also $u, v \in D'''$ such that $u \perp v$ which is contradiction as G is DDS.

Subcase 2: If $v \notin PN [w, D']$ ie., v is 2 – dominated say v is dominated by x, w, then $G - \{w\} \cup \{v\}$ is a γ - set for G such that $u \perp v \perp x$, where $u, v, x \in D'$ ie., $D' - \{v\}$ is a γ - set for G which is contradiction as we assume that G is DSS.

Case 5: $w, v \in D'$, $u \notin D'$

We get contradiction, similar to case 4.

Case 6: $u, v \in D'$, $w \notin D'$

D' is a γ - set for G such that $u \perp v$, which is contradiction as G is DDS.

In all cases we get a contradiction and hence G is not DSS.

Remark: If G is a DDS graph such that,

1. $N (u) < 2$, for some $u \in V (G)$,
2. $v \in PN [u, D]$, for some $u \in D$ and, $\forall v \in V - D$, $\forall \gamma$ - set D of G, then G may or may not be DSS.

Example:

Fig. 2

In Fig. 2, G is DDS, N (u) = N (v) = 1. v ∈ PN [u, D] where v ∈ V – D. G is also DSS.

3 CONSTRUCTIONS

Theorem 3.1: Every graph is an induced subgraph of DSS.

Proof: Let G be DSS graph with n – vertices say u_i, i = 1, 2, …, n. Let H = G o K_1. Label the pendant vertices as v_1, v_2, …, v_n . {u_1, u_2, …, u_n} or {v_1, v_2, …, v_n} are the possible γ – sets for H. Let {u_1, u_2, …, u_n} be the γ – set under consideration.

Consider $H_{sd}\ u_iv_i$. $\gamma (H_{sd}\ u_iv_i) = \gamma (H) - \{u_i\} \cup \{w_i\}$, where i = 1, 2, …, n ie., $\gamma (H_{sd}\ u_iv_i) = \gamma (H)$.

Consider $H_{sd}\ u_iu_j$. $\gamma (H_{sd}\ u_iu_j) = \gamma (H) - \{u_i\} - \{u_j\} \cup \{ u_iu_j \}$, where $i \neq j$, i, j = 1, 2, …, n.

Hence every graph is an induced subgraph of DSS graph.

Theorem 3.2: Let G_1 and G_2 be DSS graphs. Let D_1 and D_2 be γ – sets for G_1 and G_2 respectively. Let u ∈ V (G_1) such that u is both level and bad vertex in G_1 and v ∈ V (G_2) such that v is selfish. Obtain a graph H by adding an edge between u and v then H is DSS.

Proof: Let G_1 and G_2 be DSS graphs. Let D_1 and D_2 be γ – sets for G_1 and G_2 respectively. Let u ∈ V (G_1) such that u is both level and bad vertex in G_1 and v ∈ V (G_2) such that v is selfish. Obtain a graph H by adding an edge between u and v. $\gamma (H) = \gamma (G_1) + \gamma (G_2)$. [Since u is both level and bad vertex in G_1 and v is selfish, then γ – value does not change when we add an edge between u and v]. Consider H_{sd} uv. Let H_{sd} uv = w. $\gamma (H_{sd}\ uv) = \gamma (H)$, since u is 2 – dominated. Also $\gamma (H_{sd}\ uv) = \gamma (H), \forall$ u, v ∈ V (G_1) and ∀ u, v ∈ V (G_2). Hence H is DSS.

Theorem 3.3: Let G_1 and G_2 be DSS graphs. Let D_1 and D_2 be γ – sets for G_1 and G_2 respectively. Let u ∈ D_1 and v ∈ D_2 be selfish vertices in G_1 and G_2, then the graph H obtained by merging two vertices u and v is DSS.

Proof: Let G_1 and G_2 be DSS graphs. Let D_1 and D_2 be γ – sets for G_1 and G_2 respectively. Let u ∈ D_1 and v ∈ D_2 be selfish vertices in G_1 and G_2. H is obtained by merging vertices u and v. $\gamma (H) = \gamma (G_1) + \gamma (G_2) - \{u\} - \{v\} \cup \{uv\}$
ie., $\gamma (H) = \gamma (G_1) + \gamma (G_2) - 1$. Since G_1 and G_2 are DSS. Also $\gamma (H_{sd}\ uv) = \gamma (H), \forall$ u, v ∈ V (G_1) and ∀ u, v ∈V (G_2).

Hence H is DSS.

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