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CHARACTERIZATION OF CLASS OF ATOMS IN LATTICE SIGMA ALGEBRAS

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ABSTRACT

This paper describes a class of null sets, lattice measure of an atom and lattice semi-finite measure. It has been established a result that the lattice measure of any two atoms are either disjoint or identical. In fact it has been proved that the class of all atoms in lattice sigma algebra is countable. Finally it has been observed various elementary characteristics of atoms in lattice sigma algebra.

Key words: *Lattice σ – algebra, measure of an atom, lattice measure, semi-finite measure.*

ASM Classification numbers: *03G10, 28A05, 28A12.*

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1. INTRODUCTION:

The concept of measure of a lattice has been attempted by Szasz (1963)[6]. The fundamentals of measure theoretical concepts were firstly described by Halmos (1974) [4]. Further a detailed attempt has been made on this concept by Royden (1981) [5]. Hahn decomposition theorem for signed lattice measure is originated by Tanaka (2009) [7].The concept of Radon – Nikodym theorem for signed lattice measure was obtained by Anil kumar etrl (2011) [2].

In section 2, based on Tanaka (2009) [7], the fundamentals of lattice sigma algebra, lattice measure on a lattice sigma algebra were described. Further based on Anil kumar etrl (2011) [2] the concepts of lattice measurable set, lattice measure space and lattice σ – finite measure were defined. Here some elementary properties of lattice measurable sets were derived.

In section 3, a class of null sets, atom, lattice measure of an atom and lattice semi-finite measure were introduced. Here it has been derived a result that the lattice measure of any two atoms are either disjoint or identical. Also proved that the class of all atoms in a lattice sigma algebra is countable. It has been obtained a theorem that if lattice sigma algebra is atomless, then it contains countable number of disjoint non-empty lattice measurable sets. Finally it has been observed that some elementary characteristics of atoms in a lattice sigma algebra.

2. PRELIMINARIES

This section briefly reviews the well-known facts of Birkhoff's [1967][3] lattice theory.
The system $(L, \wedge, \vee)$, where L is a non empty set, $\wedge$ and $\vee$ are two binary operations on L, is called a lattice if $\wedge$ and $\vee$ satisfies, for any elements x, y, z, in L:

(L1) commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
(L2) associative law: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.
(L3) absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$. Hereafter, the lattice $(L, \wedge, \vee)$ will often be

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written as L for simplicity. A lattice (L, $\wedge$, $\vee$) is called distributive if, for any x, y, z, in L.
(L4) distributive law holds: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee$ A and the infimum $\wedge$ A. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O respectively.

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^C such that

$x \vee x^C = 1$ (L5) the law of excluded middle

$x \wedge x^C = 0$ (L6) the law of non-contradiction

Let L be a lattice and C: L $\rightarrow$ L be an operator. Then C is called a lattice complement in L if the following conditions are satisfied.

(L5) and (L6); $\forall x \in L, x \vee x^C = 1$ and $x \wedge x^C = 0$,

(L7) the law of contrapositive; $\forall x, y \in L, x \leq y$ implies $x^C \geq y^C$,

(L8) the law of double negation; $\forall x \in L, (x^C)^C = x$.

Throughout this paper, we consider lattices as complete lattices which obey (L1) - (L8) except for (L6) the law of non-contradiction. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X.

Definition 2.1: If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra;

(1) $\forall h \in L, h^C \in L$

(2) if $h_n \in L$ for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in L$.

We denote σ (L) = ß, as the lattice σ-Algebra generated by L.

Example 2.1: [Halmos (1974)**][4].**
1. $\{\phi, X\}$ is a lattice σ-Algebra.
2. P(X) power set of X is a lattice σ-Algebra.

Example 2.2: Let X = $\Re$ and L = {measurable subsets of $\Re$ } with usual ordering ($\leq$).

Here L is a lattice and σ (L) = ß is a lattice σ - algebra generated by L.

Example2.3. Let X be any non-empty set, L = {All topologies on X}. Here L is a complete lattice but not σ - algebra.

Example 2.4: [Halmos (1974)**][4].** Let X = $\Re$ and L = {E < $\Re$ / E is finite or E^c is finite}.

Here L is lattice algebra but not lattice σ - algebra.

Definitition 2.2: The ordered pair (X, ß) is said to be lattice measurable space.

Example 2.5: Let X = $\Re$ and L = {All Lebesgue measurable sub sets of $\Re$ }. Then it can be verified that ($\Re$, ß) is a lattice measurable space.

Definitition 2.3: If the mapping μ: ß $\rightarrow$ R $\cup$ {∞} satisfies the following properties, then μ is called a lattice measure on the lattice σ-Algebra σ (L).

(1) $\mu(\phi) = \mu(0) = 0$.
(2) For all h, g $\in$ ß, such that μ(h), μ(g) $\geq$ 0 and $h \leq g \Rightarrow \mu(h) \leq \mu(g)$.
(3) For all h, g $\in$ ß, $\mu(h \vee g) + \mu(h \wedge g) = \mu(h) + \mu(g)$.
(4) If $h_n \subset$ ß, n $\in$ N such that $h_1 \leq h_2 \leq ... \leq h_n \leq ....$, then $\mu(\bigvee_{n=1}^{\infty} h_n) = \lim \mu(h_n)$.

Note 2.1: Let μ_1 and μ_2 be lattice measures defined on the same lattice σ-Algebra ß. If one of them is finite, then the set function $\mu(E) = \mu_1(E) - \mu_2(E)$, E $\in$ ß is well defined and is countably additive on ß.

Example 2.6: [Royden (1981)**][5]:** Let X be any set and ß = P(X) be the class of all sub sets of X. Define for any $A \in$ ß, μ(A) = $+\infty$ if A is infinite = |A| if A is finite, where |A| is the number of elements in A.

Then μ is a countable additive set function defined on ß and hence μ is a lattice measure on ß.

Definition 2.4: A set A is said to be lattice measurable set or lattice measurable if A belongs to ß.

Example 2.7: [Anilkumar etrl 2011][1] The interval (a, ∞) is a lattice measurable under usual ordering.

Example 2.8: [Anilkumar etrl 2011][1] [0, 1] < $\Re$ is lattice measurable under usual ordering.
Let X= $\Re$, L= {lebesgue measurable subsets of $\Re$} with usual ordering ($\leq$) clearly σ (L) is a lattice σ-algebra generated by L. Here [0, 1] is a member of σ(L). Hence it is a Lattice measurable set.

Example 2.9: [Anilkumar etrl 2011][1] Every Borel lattice is a lattice measurable.

Definition 2.5: The lattice measurable space (X, ß) together with a lattice measure μ is called a lattice measure space and it is denoted by (X, ß , μ).

Example 2.10: $\Re$ is a set of real numbers, μ is the lattice Lebesgue measure on $\Re$ and ß is the family of all Lebesgue measurable subsets of real numbers. Then ($\Re$, ß, μ) is a lattice measure space.

Example 2.11: $\Re$ be the set of real numbers and ß is the class of all Borel lattices, μ be a lattice Lebesgue measure on $\Re$ then ($\Re$, ß, μ) is a lattice measure space.

Definition 2.6: Let (X, ß, μ) be a lattice measure space. If μ(X) is finite then μ is called lattice finite measure.

Example 2.12: The lattice Lebesgue measure on the closed interval [0, 1] is a lattice finite measure.

Example 2.13: When a coin is tossed, either head or tail comes when the coin falls. Let us assume that these are the only possibilities. Let X = {H, T}, H for head and T for tail. Let ß = {ϕ, {H}, {T}, X}. Define the mapping P: ß $\to$[0, 1] by P (ϕ) = 0, P ({H}) = P ({T}) = ½, P (X) = 1. Then P is a lattice finite measure on the lattice measurable space (X, ß).

Definition 2.7: If μ is a lattice finite measure, then (X, ß, μ) is called a lattice finite measure space.

Example 2.14: Let ß be the class of all Lebesgue measurable sets of [0, 1] and μ be a lattice Lebesgue measure on [0, 1]. Then ([0, 1], ß, μ) is a lattice finite measure space.

Definition 2.8: Let (X, ß, μ) be a lattice measure space. If there exists a sequence of lattices measurable sets {x_n} such that

(i) $X = \overset{\infty}{\underset{n=1}{\vee}} x_n$ and (ii) $\mu(x_n)$ is finite then μ is called a lattice σ – finite measure.

Example2.15: The lattice Lebesgue measure on ($\Re$, μ) is a lattice σ – finite measure since $\Re = \overset{\infty}{\underset{n=1}{\vee}}$ (-n, n) and μ((-n, n)) = 2n is finite for every n.

Definition2.9: If μ be a lattice σ – finite measure, then (X, ß, μ) is called lattice σ – finite measure space.

Example2.16: Let ß be the class of all Lebesgue measurable sets on $\Re = \overset{\infty}{\underset{n=1}{\vee}}$ (-n, n) and μ be a lattice Lebesgue measure on $\Re$, then ($\Re$, ß, μ) is a lattice σ – finite measure space.

Theorem 2.1: Let {E_i} be an infinite decreasing sequence of lattice measurable sets. That is, a sequence with $E_{i+1} < E_i$ for each $i \in N$. Let μ (E_i) < ∞ for at least one $i \in$ N. Then

$$\mu\left(\overset{\infty}{\underset{i=1}{\wedge}} E_i\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof: Let p be the least integer such that $\mu(E_p) < \infty$. Then $\mu(E_i) < \infty$, for all $i \geq p$.

Let $E = \bigwedge_{i=1}^{\infty} E_i$ and $F_i = E_i - E_{i+1}$.

Then the sets F_i's are lattice measurable and pair wise disjoint, clearly

$E_p - E = \bigvee_{i=p}^{\infty} F_i$. Therefore, $\mu(Ep - E) = \sum_{i=p}^{\infty} \mu(F_i) = \sum_{i=p}^{\infty} \mu(E_i - E_{i+1})$

But $\mu(Ep) = \mu(E) + \mu(E_p - E)$ and $\mu(E_i) = \mu(E_{i+1}) + \mu(E_i - E_{i+1})$

For all $i \geq p$ since $E < E_p$ and $E_{i+1} < E_i$, further, using the fact that $\mu(E_i) < \infty$, for all $i \geq p$,

it follow that $\mu(Ep - E) = \mu(Ep) - \mu(E)$ and $\mu(E_i - E_{i+1}) = \mu(E_i) - \mu(E_{i+1})$ for all $i \geq p$.

Hence $\mu(Ep) - \mu(E) = \sum_{i=p}^{\infty} \mu(E_i) - \mu(E_{i+1})) = \operatorname{Lim}_{n\to\infty} \sum_{i=p}^{n} (\mu(E_i) - \mu(E_{i+1})) = \operatorname{Lim}_{n\to\infty} (\mu(E_p) - \mu(E_n))$

$= \mu(E_p) - \operatorname{Lim}_{n\to\infty} \mu(E_n)$. Since $\mu(E_p) < \infty$, it gives $\mu(E) = \operatorname{Lim}_{n\to\infty} \mu(E_n)$.

Theorem 2.2: Let $\{E_i\}$ be an infinite increasing sequence of lattice measurable sets. That is, a sequence with $E_{i+1} > E_i$ for each $i \in N$. Let $\mu(E_i) < \infty$ for at least one $i \in N$.

Then $\mu\left(\bigvee_{i=1}^{\infty} E_i\right) = \operatorname{Lim}_{n\to\infty} \mu(E_n)$.

Proof: If $\mu(E_p) = \infty$ for some $p \in N$, then the result is trivially true, since $\mu\left(\bigvee_{i=1}^{\infty} E_i\right) \geq \mu(E_p) = \infty$,

And $\mu(E_n) = \infty$, for each $n \geq P$. Let $\mu(E_i) < \infty$, for each $i \in N$.

Now $E = \bigvee_{i=1}^{\infty} E_i$, evidently $F_i = E_i - E_{i+1}$.

Then the sets F_i's are lattice measurable and pair wise disjoint, clearly $E - E_i = \bigvee_{i=p}^{\infty} F_i$

$\mu(E - E_i) = \mu(\bigvee_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i) = \sum_{i=1}^{\infty} \mu(E_{i+1} - E_i) = \mu(E) - \mu(E_i) = \sum_{i=1}^{\infty} (\mu(E_{i+1}) - \mu(E_i))$

$= \operatorname{Lim}_{n\to\infty} \sum_{i=1}^{n} (\mu(E_{i+1}) - \mu(E_i)) = \operatorname{Lim}_{n\to\infty} (\mu(E_{i+1}) - \mu(E_i))$

It gives $\mu(E) = \operatorname{Lim}_{n\to\infty} \mu(E_n)$.

3. CHARACTERIZATION OF CLASS OF ATOMS IN LATTICE SIGMA ALGEBRAS

Definition3.1: Let (X, ß) be a lattice measurable space. A nonempty class N of sets, where N is contained in ß is called a class of null sets of ß

(1) If $E \in N$ and $F \in$ ß, then $E \wedge F \in N$, and

(2) If $E_n \in N$, n=1, 2, 3...., then $\bigvee_{n=1}^{\infty} E_n \in N$.

Definition3.2: Let (X, ß, μ) be a lattice measure space. A set E in ß is called a μ-atom if

(1) μ (E) > 0 and

(2) If F ∈ ß such that F is contained in E, then either μ (E-F) = 0 or μ (F) = 0.

Definition3.3: Let ß be a lattice σ – algebra of a subsets of X. A lattice measurable set E is said to be an atom of ß if
(1) $E \neq \phi$ and
(2) F in ß, F is contained in E implies $F = \phi$ or F = E.

Example 3.1: Let ß = {all Lebesgue measurable subsets of a real line $\Re$ } here ß is a lattice σ – algebra. Clearly {1} is a member of ß, put {1}= E it can be easily verified that E is an atom of ß.
($\{1\} \neq \phi$ that is $E \neq \phi$ also $F \in$ ß, F < E, then $F = \phi$ or F = E).

Note 3.1: A lattice σ – algebra ß of X is said to be atomless if there are no atoms of ß.

Definition3.4: Lattice semi-finite measure: A lattice measure μ on a lattice σ – algebra ß of X is said to be semi finite if $F \in$ ß, $\mu(F) = \infty$ implies there exists $E \in$ ß such that E is contained in F and $0 < \mu(E) < \infty$.

Definition3.5: A partially ordered set X is said to satisfy the countable chain condition, or to be ccc, if every strong antichain in X is countable. In other words no two elements have a common lower bound.

Example 3.2: The partially ordered set of non-empty open sub lattices of X satisfies the countable chain condition that is every pairwise disjoint collection of non-empty open sublattices of X is countable.

Result 3.1: Let (X, ß, μ) be a lattice measure space. If E_1 and E_2 are atoms, then either $\mu(E_1 \Delta E_2) = 0$ or $\mu(E_1 \wedge E_2) = 0$ or (the lattice measure of any two atoms are either disjoint or identical)

Proof: Let $E_1$and E_2 are atoms. Since $E_1$is an atom by definition3.2, $E_2 \in$ ß such that E_2 is contained in E_1 implies $\mu(E_1\text{-}E_2) = 0$ or $\mu(E_2) = 0$.

Since E_2 is an atom $\mu(E_2) \neq 0$ implies $\mu(E_1\text{-}E_2) = 0$.

By similar argument we have that $\mu(E_2\text{-}E_1) = 0$.

Now consider $E_1 \Delta E_2 = (E_1\text{-}E_2) \vee (E_2\text{-}E_1)$ implies $\mu(E_1 \Delta E_2) = \mu(E_1\text{-}E_2) + \mu(E_2\text{-}E_1)$.

Which implies $\mu(E_1 \Delta E_2) = 0$. Also evidently $(E_1 \vee E_2) = (E_1 \wedge E_2) \vee (E_1 \Delta E_2)$.

This implies $\mu(E_1 \vee E_2) = \mu(E_1 \wedge E_2) + \mu(E_1 \Delta E_2)$.

Which leads to $\mu(E_1 \vee E_2) = \mu(E_1 \wedge E_2)$ (since $\mu(E_1 \Delta E_2) = 0$).

Again if $\mu(E_1\text{-}E_2) \neq 0$, then $\mu(E_2) = 0$.

Now $E_1 \wedge E_2 \leq E_2$ implies $\mu(E_1 \wedge E_2) \leq \mu(E_2)$.

Which implies $\mu(E_1 \wedge E_2) \leq 0$.

But $\mu(E_1 \wedge E_2) \geq 0$ (by definition 2.3).

Therefore $\mu(E_1 \wedge E_2) = 0$.

If $E_2 - E_1 \neq 0$ similarly we get $\mu(E_1 \wedge E_2) = 0$.

Result 3.2: Let (X, ß, μ) be a lattice measure space and μ is lattice σ – finite measure. Then the class A of all atoms in a lattice σ-algebra ß is countable.

Proof: Let $E_1, E_2 \in A$ be any two sets by result 3.1.

We have either $\mu(E_1 \Delta E_2) = 0$ or $\mu(E_1 \wedge E_2) = 0$.

If $\mu(E_1 \Delta E_2) = 0$, then the set $(E_1 \wedge E_2)$ represents an atom or if $\mu(E_1 \wedge E_2) = 0$ then $(E_1\text{-}E_2)$ and $(E_2\text{-}E_1)$ represents two disjoint atoms.

This implies two disjoint sets in ß – N.

Continuing this process for E_1, E_2 ……, we get a countable collection of disjoint sets in ß – N. Which leads to ß – N is countable.

Theorem 3.1: Let μ be a lattice semi-finite measure on a lattice σ – algebra ß of X. Let N denotes the collection of sets of μ - measure zero. Then ß – N satisfies countable chain condition (ccc) if and only if μ is lattice σ – finite measure.

Proof: If μ is lattice σ – finite measure, it is obvious that ß – N satisfies countable chain condition (ccc) (by result 3.2).

Conversely, if $\mu(X) < \infty$, then there is nothing to prove.

If $\mu(X) = \infty$, choose E_1 in ß such that $0 < \mu(E_1) < \infty$.

Choose E_2 in ß such that E_2 is contained in $X - E_1$ and $0 < \mu(E_2) < \infty$.

Continuing this process we get a sequence of disjoint sets $E_1, E_2, \ldots,$ in ß such that E_i in ß – N and $\mu(E_i) < \infty$.

If $\mu(X - \bigvee_{i=1}^{\infty} E_i) < \infty$, then we have a decomposition of X.

Which implies that μ is σ – finite.

Hence $\mu(X - \bigvee_{i=1}^{\infty} E_i) = \infty$. Choose E_α in ß such that E_α is contained in $X - \bigvee_{i=1}^{\infty} E_i$ and $0 < \mu(E_\alpha) < \infty$, where α is the first countable ordinal.

Proceeding inductively, since ß – N satisfies countable chain condition (ccc), there exists a countable ordinal β such that $\mu(X - \bigvee_{\alpha<\beta} A_\alpha) < \infty$.

This implies that μ is lattice σ – finite measure.

Theorem 3.2: Let ß be a lattice σ – algebra of a set X. Then ß is atomless if and only if every non empty set in ß contains countable number of disjoint non empty sets in ß.

Proof: Let E in ß be non empty set. Fix $x \in E$. We can choose E_1 in E such that $x \notin E_1$.

Now E_1 is non empty and E_1 is contained in E.

Choose E_2 in E such that $x \notin E_2$.

Now E_2 is non empty and E_2 is contained in $E - E_1$.

Choose E_3 in E such that $x \notin E_3$.

Continuing this process we get a family $\{E_\alpha / \alpha < \beta\}$ of non empty disjoint sets contained in ß where β is the first uncountable ordinal.

The converse part is trivial.

Theorem 3.3: Let ß be a lattice σ – algebra of a set X. Then it satisfies countable chain condition (ccc) if and only if ß is isomorphic to the power set.

Proof: We can prove this theorem by using theorem 3.1 and theorem 3.2.

If ß satisfies countable chain condition (ccc), then the number of atoms of ß is countable.

From X remove all atoms of ß.

In the view of above theorem 3.2, the remaining part is empty.

Hence it is isomorphic.

Example 3.3: Take the numbers 0, 1 and the fractions $\frac{m}{n}$, $0 < \frac{m}{n} < 1$

That is

0, 1, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$,................ order as follows

$0 < \frac{m}{n} < 1$ for all $\frac{m}{n}$; $\frac{m}{n} \le \frac{r}{s}$ only if max(m, r) = r ; $\frac{m}{n}, \frac{r}{s}$ in comparable if $n \neq s$.

Clearly the fractions from 0 to 1 have a countable infinity of atoms and of dual of atoms.

CONCLUSION

This work illustrates a class of null sets, lattice measure of an atom and lattice semi-finite measure. A crucial result obtained that the lattice measure of any two atoms are either disjoint or identical. Observed scrupulously that the class of all atoms in lattice sigma algebra is countable. Various elementary characteristics of atoms in a lattice sigma algebra have been identified.

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