International Journal of Mathematical Archive-3(4), 2012, Page: 1439-1442
IJMA Available online through www.ijma.info ISSN 2229 – 5046

b-coloring in the context of some graph operations

Mrs. D.Vijayalakshmi[1*] & Dr. K. Thilagavathi[2]

[1]Assistant Professor & Head Department of Maths CA, Kongunadu Arts and Science College, Coimbatore – 641 029, India

[2]Associate Professor, Department of Mathematics, Kongunadu Arts and Science College, Coimbatore – 641 029, India
E-mail: vijikasc@gmail.com, ktmaths@yahoo.com

(Received on: 24-03-12; Accepted on: 09-04-12)

ABSTRACT

In this paper, we discuss about some graph operations like Corona product, Strong product and Cartesian product. We find the b-chromatic number of Corona product of Path, Cycle and Star graph with complete graph, the Strong product of Path with Cycle and Cartesian product of cycles.

***Keywords**: Cycle, Path, Star graph, Complete graph, Corona product, Strong product, Cartesian Product.*

***Subject classification:** 05C15.*

---

1. INTRODUCTION

Let G be a graph without loops [1] and multiple edges with vertex set V (G) and edge set E (G). A proper k-coloring of graph G is a function c defined on the V (G), onto a set of colors C = {1, 2. . . k} such that any two adjacent vertices have different colors. In fact for every i, $1 \leq i \leq k$, the set c^{-1} {i} is an independent set of vertices which is called a color class. The minimum cardinality k for which G has a proper k-coloring is the chromatic number $\chi(G)$ of G. A b-coloring of a graph G[2] is a proper vertex coloring of G such that each color class contains a vertex that has at least one neighbor in every other color class and b-chromatic number of a graph G is the largest integer $\varphi(G)$ for which G has a b-coloring with $\varphi(G)$ colors. A vertex of color i that has all other colors in its neighborhood is called color i dominating vertex. The invariant $\varphi(G)$ has the chromatic number $\chi(G)$ as a trivial lower bound, however the difference between both of them can be arbitrary large [3]. A trivial upper bound for $\varphi(G)$ is $\Delta(G) + 1$. Let $d(v_1) \geq d(v_2) \geq . . . \geq d(v_n)$ be the degree sequence of G. Then $m(G) = \max\{i \mid d(v_i) \geq i - 1\}$ is an improved upper bound for $\varphi(G)$. The concept of b-chromatic number was introduced by R. W. Irwing and D. F. Manlove in 1999.

2. PRELIMINARIES

In this section, we give the definition of Path, Cycle, Star graph, Corona product, Strong product and Cartesian product.

2.1Definition: Let P_n be a path graph with n vertices and n-1 edges.

2.2 Definition: Let Cn be a cycle with n vertices and n edges.

2.3 Definition: A star S_n is the complete bipartite graph $K_{1,n}$ is a tree with one internal node and n leaves.

2.4 Definition: Corona product or simply corona of graph G_1 and G_2 is a graph which is the disjoint union of one copy of G_1 and $|v_1|$ copies of G_2 ($|v_1|$ is number of vertices of G_1) in which each vertex copy of G_1 is connected to all vertices of separate copy of G_2 .

2.5 Definition: The strong product of two graphs G_1 and G_2 has the vertex set $v(G_1) \times v(G_2)$ and two distinct vertices (u, u') and (v, v') are connected if and only if they are adjacent or equal in each coordinates.

2.6 Definition: The cartesian product of two graphs G_1 and G_2 has the vertex set of $G_1 \times G_2$ is the Cartesian product $v(G_1) \times v(G_2)$ and two distinct vertices (u, u') and (v, v') are adjacent in $G_1 \times G_2$ if and only if either u = v and u' is adjacent with v' or u'=v' and u is adjacent with v.

***Corresponding author: Mrs. D.Vijayalakshmi[1*]**, ***E-mail:** vijikasc@gmail.com*

3. b-CHROMATIC NUMBEROF CORONA PRODUCT OF PATH GRAPH AND K_1

3.1 Theorem: For any $n\geq 3$, $\varphi\ [P_n \circ (n-1)\ K_1] = n$

Proof: Let P_n be a path graph of length n-1 ie $V(P_n) = \{v_1,v_2,v_3..v_n\}$ and $E(P_n) = \{e_1,e_2,e_3..e_{n-1}\}$.By the definition of corona product, attach (n-1) copies of K_1 to each vertex of P_n.

i.e $V[P_n \circ (n-1)\ K_1] = \{v_i/1\leq i\leq n\} \cup \{v_{ij}/1\leq i\leq n,\ 1\leq j\leq n-1\}$.

$E[P_n \circ (n-1)\ K_1] = \{e_i/1\leq i\leq n-1\} \cup \{e_{ij}/1\leq i\leq n, 1\leq j\leq n-1\}$.

Consider the colour class $C= \{c_1,c_2,c_3..c_n\}$. Assign a proper coloring to the vertices as follows. Give the colour c_i to vertex v_i for i=1,2,3…n and assign the colour c_{n+1} to v_{ij} for i=1,2…n and j=1,2,3..n-1. We see that each v_i is adjacent with v_{i-1} and v_{i+1} for i=2, 3,..n-1, v_1 is adjacent with v_2 and v_n is adjacent with $v_{n-1,}$ due to this non-adjacency condition v_i for i=1,2,3..n does not realizes its own colour, which does not produce a b-chromatic colouring. Hence to make the coloring as b-chromatic, assign the coloring to v_{ij}'s as follows.
For $1 \leq i\leq n$, assign the color c_i to v_i.

For i=1,2,3..n, j=1,2,3…n-1, assign the colour c_{i+j} to v_{ij} when $i+j\leq n$ and assign the colour c_{i+j-n} when $i+j>n$. Now the vertices v_i for i=1,2,3..n realizes its own colour which produces a b-chromatic coloring. Hence by coloring procedure the above said coloring is maximal and b-chromatic.

Hence the proof.

3.2 Properties:

- Number of vertices in $[Pn \circ (n-1)\ K_1] = n^2$
- Number of edges in $[Pn \circ (n-1)\ K_1] = n^2-1$
- Maximum degree = n+1
- Minimum degree =1

4. b-CHROMATIC NUMBER OF CORONA PRODUCT OF CYCLE AND K_1

4.1 Theorem: For any $n\geq 3$, $\varphi\ [C_n. (n-3)\ K_1] = n$

Proof: Let C_n be a cycle of length n ie $V (C_n) = \{v_1, v_2, v_3..v_n\}$ and $E(C_n) = \{e_1,e_2,e_3..e_n\}$. By the definition of Corona product, attach a (n-3) copies of K_1 to each vertex of C_n.

i.e $V[C_n . (n-3)\ K_1] = \{v_i/1\leq i\leq n\} \cup \{v_{ij}/1\leq i\leq n,\ 1\leq j\leq n-3\}$.

$E[C_n . (n-3)\ K_1] = \{e_i/1\leq i\leq n\} \cup \{e_{ij}/1\leq i\leq n,1\leq j\leq n-3\}$. Here $|V(G)| = n(n-2)$ and $|E(G)| = n(n-2)$

Consider the colour class $C= \{c_1,c_2,c_3..c_n\}$. Assign a proper coloring to the vertices as follows. Give the colour c_i to vertex v_i for i=1,2,3…n and assign the colour c_{n+1} to v_{ij} for i=1,2…n and j=1,2,3..n-3. Here v_i for i=1,2,3..n does not realize its own colour because each v_i is adjacent with v_{i-1} and v_{i+1} for i=2,3,..n-1, v_1 is adjacent with $v_{2,}v_n$ and v_n is adjacent with v_{n-1} and v_1. Hence to make the coloring as b-chromatic, assign the coloring to v_{ij}'s as follows.
For $1 \leq i\leq n$, assign the color c_i to v_i.

For $1 \leq i\leq n$ and $1 \leq j\leq n-3$ assign the colour c_{i+j+1} to v_{ij}'s when $i+j< n$ and assign $c_{i+j+1-n}$ to remaining v_{ij}'s when $i+j\geq n$.
Now here all the vertices v_i for i=1, 2, 3...n realizes its own colour which produces a b-coloring. Hence by coloring procedure the above said coloring is maximal and b-chromatic.

Hence the proof.

4.2 Properties:

- Number of vertices and edges in $[Cn \circ (n-3)\ K_1] = n(n-2)$
- Maximum degree = n-1
- Minimum degree =1

5. b-CHROMATIC NUMBER OF CORONA PRODUCT OF STAR GRAPH WITH K_1

5.1Theorem: For any $n\geq 2$, $\varphi\,[K_{1,n}.\ (n-1)\ K_1] = n+1$ where $n\geq 2$

Proof: Let $v_1, v_2, v_3 \ldots v_n$ be the pendant vertices of the graph $K_{1,n}$ with v as the root vertex. Here v is adjacent with v_i for i=1, 2, 3..n. ie $V[K_{1,n}] = \{v\}\ U\{v_i/1\leq i\leq n\}$.

Now consider $[\ K_{1,n} \circ (n-1)\ K_1]$ .ie$V[\ K_{1,n} \circ (n-1)\ K_1] = \{v\}\ U\{v_i/1\leq i\leq n\}U\{v_{ij};\ 1\leq i\leq n,\ 1\leq j\leq n-1\}U\{\ v_i'/1\leq i\leq n-1\}$

Consider the colour class $C = \{c_1, c_2, c_3..c_n, c_{n+1}\}$. Assign a proper coloring to these vertices as follows. Give the colour c_i to vertex v_i for i=1, 2, 3…n and assign the colour c_{n+1} to v. Here the root vertex realizes its own colour but the vertices v_i for i=1,2,3..n does not realize its own colour, which does not produce a b-chromatic colouring. Hence to make the coloring as b-chromatic, assign the coloring as follows.

For $1 \leq i\leq n$, assign the color c_i to v_i. Assign the color c_{n+1} to root vertex v.Next assign proper colouring to the v_{ij}'s as follows. For i=1,2,3..n, j=1,2,3…n-1, assign the color c_{i+j} to v_{ij} when $i+j\leq n$ and assign the colour c_{i+j-n} when i+j>n. For the remaining vertex v_i' assign any color c_i for $1\leq i\leq n-1$. Now all the vertices v_i for i=1, 2, 3...n and the root vertex v realizes its own colour, which produces a b- chromatic coloring. Hence by coloring procedure the above said coloring is maximal and b-chromatic.

5.2 Properties:

- Number of vertices in $[\ K_{1,n} \circ (n-1)K_1]= n(n+1)$
- Number of edges in $[K_{1,n} \circ K_{1,m}] = n^2+n-1$
- Maximum degree = 2n-1
- Minimum degree =1

6. b-CHROMATIC NUMBER OF STRONG PRODUCT OF P_2 WITH CYCLE C_n

6.1 Theorem: If P_m is a path graph on m vertices and C_n be a cycle on n vertices respectively. Then $\varphi[P_m\otimes C_n] = 6$ where $n\geq 3$ and m=2

Proof: By observation, we say that the strong product of $P_m\otimes C_n$ is a 5-regular graph. Therefore the b-chromatic number of $P_2\otimes Cn$ will be more than 5. Hence by coloring procedure we assign six colours to every $P_m\otimes C_n$ which produces b-chromatic coloring.Suppose if we assign more than sixcolors, it contradicts the definition of b-colouring. Hence the b-chromatic number of Strong product of $P_m\otimes C_n$ is six.

6.2 Properties:

1. Number of vertices in $P_2\otimes C_n$ is two times the number of vertices in cycle C_n
2. Number of edges in $P_2\otimes C_n$ is five times the number of edges in cycle C_n
3. Every $P_2\otimes C_n$ is a 5 regular graph.

6.3 Theorem: If P_n is a path graph on n vertices and k_m be a complete graph on 2 vertices respectively. Then

$$\varphi\ (Pn\otimes K_2) = \begin{cases} 4 \text{ for } n\leq 3,\ m = 2 \\ 6 \text{ for } n\geq 4 \end{cases}$$

6.4 Theorem: Let $P_n$and $P_m$be paths on n and m vertices respectively.

$$\varphi\ (P_n\otimes P_m) = \begin{cases} n+2 \text{ for } 2\leq n\leq 4,\ m=3 \\ 9 \text{ for } n\geq 5 \end{cases}$$

Proof: The proof of the theorem 6.3 and 6.4 is similar to the theorem 6.1.

7. b-Chromatic Number Of Cartesian Product Of $C_3 \times C_n$

7.1 Theorem: Let C_n and C_m be cycles on n and m vertices respectively.

$\varphi[C_m \times C_n] = 5$ for every $n \geq 6, m=3$

Proof: By observation, we say that the cartesian product of $C_m \times C_n$ is a 4-regular graph. Therefore, the b-chromatic number of $C_m \times C_n$ will be more than 4. Hence by coloring procedure we assign five colours to every $C_m \times C_n$, which produces b-chromatic coloring. Suppose if we assign more than five colors, it contradicts the definition of b-colouring. Hence the b-chromatic number of cartesian product of $C_m \times C_n$ is five.

7.2 Corollary:

$\varphi[C_m \times C_n] = n$ for every $n \leq 5$ and $m=3$

ACKNOWLEDGEMENT

We are much grateful for referees for their valuable suggestions which lead to improve this paper.

REFERENCES

[1].Raminjavadi, Behnazomoomi, On b-coloring of the Kneser graphs, Discrete mathematics (2009) 4399-4408.

[2] Marko jakov, Faculty of Natural Sciences and Mathematics, University of Maribor and IztokPeterin, On the b-chromatic number of some graph products, Mathematika 2011.

[3] R. W. Irving, D. F. Manlove. The b-chromatic number of a graph, Discrete Appl. Math.,91 (1999) 127-141.

[4] FlaviaBonomo et.al.,On the b-Coloring of Cographs and *P*4-Sparse Graphs, Graphs and Combinatorics (2009), 25:153–167

[5] Zhendong Shao, Sandi Klavzar, WaiCheeShiu, and David Zhang, Senior Member, IEEE Improved Bounds onthe L(2; 1)-Number of Direct and Strong Products of Graphs.

[6] M. Kouider, M. Mah´o, The b-chromatic number of the Cartesian product oftwo graphs, Studia Sci. Math. Hungar. 44 (2007) 49–55.

[7] W. Imrich, S. Klavar, Product Graphs: Structure and Recognition, John Wiley& Sons, New York, 2000.

[8] F. Harary, Graph theory, Addison-Wesley, Reading, Massachusetts, 1972.

****