International Journal of Mathematical Archive-3(4), 2012, Page: 1411-1422
Available online through www.ijma.info ISSN 2229 – 5046

APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR SOLVING SYSTEM OF LINEAR AND NON-LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

V. G. Gupta[1] & Pramod Kumar[2*]

[1]Department of Mathematics, University of Rajasthan, Jaipur-302004, India
E-mail: guptavguor@rediffmail.com

[2]Department of Mathematics, Jaipur National University, Jaipur-302025, India
E-mail: pramodgupta472@gmail.com

(Received on: 23-03-12; Accepted on: 09-04-12)

ABSTRACT

This paper present, a reliable algorithm for solving linear and non-linear system of fractional order ordinary differential equations. The fractional derivatives are considered in Caputo sense. The homotopy analysis method is applied to construct the numerical solutions. The proposed algorithm avoids the complexity provided by other numerical approaches. The method is applied to solve four systems of linear and non-linear fractional order ordinary differential equations. Numerical results shows that HAM is easy to implement and accurate when applied to solve system of equations.

Key words: *System of linear and non-linear fractional differential equations, Homotopy analysis method, Homotopy perturbation method, Caputo derivatives.*

INTRODUCTION

Fractional order ordinary differential equations, as generalization of classical ordinary differential equations, are increasingly used to model problems in fluid flow, continuum and statistical mechanics, physics and engineering, economics, biology and other applications. Half order derivatives and integrals proved to be more useful for the formulation of certain electro-chemical problems than the classical models. Fractional differential and fractional integrals provide more accurate models of system under consideration. The solutions of fractional order ordinary differential equations are much involved. One of the most recent works on the subject of fractional calculus i.e. on the theory of derivatives and integrals of fractional order, is the book of Podlubny [17] which deals principally with fractional differential equations and today there are many works on fractional calculus. For most of fractional differential equations there exist no method that yield on exact solution of fractional differential equations, so approximation and numerical techniques must be used such as homotopy perturbation method [7,14,16,19,20], Adomian's decomposition method [1,5,8,12,13] and variation iteration method [4,12,15,16]. However the region of convergence of the corresponding result is rather small as shown in this paper.

Recently, Liao [9] proposed a powerful analysis method, namely the homotopy analysis method [HAM], for solving linear and non-linear differential and integral equations. Different from perturbation techniques, the HAM does not depend upon any small or large parameters. A systematic and clear exposition on HAM is given in [10]. The HAM was successfully applied to solve many non-linear problems such as non-linear Riccati differential equation with fractional order [3], non-linear Vakhnenko equation [21], the Glauert-jet problem [2], fractional KdV-Bergers Kuramoto equation [18] and so on. In this paper we extend the application of HAM to solve linear and non-linear system of fractional ordinary differential equations. Besides we note that Adomian decomposition method and homotopy perturbation method are special cases of the HAM when $h = -1$.

The organization of this paper is as follows. A brief review of the fractional calculus is given in the next section. We use the homotopy analysis method to construct the numerical solutions for system of fractional order ordinary linear and non-linear equations in section 3, some examples are given to show the efficiency and simplicity in section 4, the numerical values with varying order α and $h = -1$ in tabular form is given in section 5.

Corresponding author: Pramod Kumar[2*], *E-mail: pramodgupta472@gmail.com

2. BASIC DEFINITIONS

1. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$ of a function $F(x) \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha F(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} F(t)\,dt \qquad ;\ \alpha > 0, x > 0$$

$$J^0 F(x) = F(x)$$

$\Gamma\alpha$ is well known Gamma function. Some of the basic properties of the operator J^α which

We will need here are as follows:

For $F \in C_\mu$, $\mu \geq -1$, $\quad \alpha, \beta \geq 0,\ \gamma > -1,\ x > 0$

(i) $\quad J^\alpha J^\beta F(x) = J^{\alpha+\beta} F(x)$

(ii) $\quad J^\alpha J^\beta F(x) = J^\beta J^\alpha F(x)$

(iii) $\quad J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

2. The fractional derivative (D_*^α) of F(x) in the Caputo' sense is defined as

$$D_*^\alpha F(x) = \begin{cases} J^{n-\alpha} F^{(n)}(x)\,] & ,\ n-1 < \alpha < n, \quad n \in N, \\ \dfrac{d^n F(x)}{dx^n} & ,\quad \alpha = n \end{cases}$$

For $F \in C_{-1}^n,\ n \in N \cup \{0\}$.

The following are the basic properties of the Caputo's fractional derivative:

(i) $\quad J^\alpha D^\alpha F(x) = F(x) - \sum_{k=0}^{n-1} F^{(k)}(0^+)\dfrac{x^k}{k!},\ x > 0, n-1 < \alpha \leq n$

(ii) $\quad D^\beta J^\alpha F(x) = \begin{cases} J^{\alpha-\beta} F(x) & ;\ \alpha > \beta \\ F(x) & ;\ \alpha = \beta \\ D^{\beta-\alpha} F(x) & ;\ \alpha < \beta \end{cases}$

(iii) $\quad D^\alpha D^m F(x) = D^{\alpha+m} F(x),\ m = 0,1,2,\ldots, n-1 < \alpha \leq n$

The Caputo fractional derivative [6] is considered here because it allows fractional initial and boundary conditions to be included in the formulation of the problem.

3. THE HOMOTOPY ANALYSIS METHOD

In this section the basic idea of the homotopy analysis method is introduced. Here a description of the method is given to handle the system of fractional differential equations.

$$D_*^{\alpha_i} x_i(t) = F_i(t, x_1, \ldots, x_n)\ ;\ i = 1,2,3,\ldots,n, \quad 0 < \alpha_i \leq 1 \tag{1}$$

Subject to the following initial conditions:

$$x_i(0) = c_i\ ;\ i = 1,2,3,\ldots,n \tag{2}$$

Zeroth-order deformation equation

Liao [10, 11], construct the so-called zeroth order deformation equations:

$$(1-q)\,L_i[\phi_i(t,q)-x_{i0}(t)] = q\,h_i\,H_i(t)\,N_i[\phi_i(t,q)] \quad ,i=1,2,...,n \tag{3}$$

Subject to the initial conditions:

$$\phi_i(0,q)=c_i \quad ,i=1,2,3,...,n,$$

Where $q \in [0, 1]$ is the embedding parameter, $h_i \neq 0$ are non-zero auxiliary parameters $H_i(t) \neq 0$ is an auxiliary functions, L_i are auxiliary linear operators. Satisfying $L_i(c) = 0$, x_{i0} (t) are initial guesses satisfying the given initial conditions and ϕ_i (t, q) are unknown functions. It is important, that one has great freedom to choose auxiliary parameters in HAM. Obviously, when q = 0 and q = 1, it holds

$$\phi_i(t,0)=x_{i0}(t)$$

$$\phi_i(t,1)=x_i(t) \quad , i = 1,2,3\ldots,n \tag{4}$$

Respectively. Thus as q increases from 0 to 1, the solution $\phi_i(t,q)$ varies from the initial guess x_{i0} (t) to the solution $x_i(t)$. Expanding $\phi_i(t,q)$ in Taylor's series with respect to the embedding parameter q, one has

$$\phi_i(t,q)=x_{i0}(t)+\sum_{m=1}^{\infty} x_{im}(t)\,q^m \;;\; i=1,2,...,n \tag{5}$$

Where $$x_{im}(t)=\frac{1}{m!}\left.\frac{\partial^m \phi_i(t,q)}{\partial q^m}\right|_{q=0} \;;\; i=1,2,...,n \tag{6}$$

If the auxiliary linear operator L_i, the initial guess $x_{i0}(t)$, the auxiliary parameter h_i and the auxiliary functions $H_i(t)$ are so properly chosen, the series (5) converges at q = 1. Then at q = 1 and by (4) the series (5) becomes

$$x_i(t)=x_{i0}(t)+\sum_{m=1}^{\infty} x_{im}(t), \quad i=1,2,...,n \tag{7}$$

The m-th order deformation equation

Define the vector

$$\vec{x}_i=\{x_{i0}(t), x_{i1}(t), x_{i2}(t),..., x_{i\ell}(t)\} \quad ; i=1,2,...,\ell \tag{8}$$

Differentiating equations (3) m times with respect of the embedding parameter q, then setting q = 0 and dividing by m! And using (6), we obtain the m^{th} order deformation equations for i = 1, 2, 3…., n

$$L_i[x_{im}(t)-\chi_m\,x_{im-1}(t)]=h_i\,H_i(t)\,R_{im}[\vec{x}_{im-1}(t)] \quad , i=1,2,...,n \tag{9}$$

Subject to the initial conditions:

$$x_{im}(0)=0, \; i=1,2,...,n \tag{10}$$

Where

$$R_{im}[\vec{x}_{im-1}(t)]=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1} N_i(\phi_i(t,q))}{\partial q^{m-1}}\right|_{q=0} \quad , i = 1, 2,\ldots, n \tag{11}$$

And

$$\chi_m=\begin{cases} 0 & ; m \le 1 \\ 1 & ; m>1 \end{cases} \tag{12}$$

If we choose

$$L_i = D_*^{\alpha_i} \qquad ; i = 1,2,...,n \tag{13}$$

and applying the Riemann-Liouville integral operator J^{α_i} on both sides of (9) and using initial conditions (10) we have

$$x_{im}(t) = \chi_m \, x_{m-1}(t) + h_i \, J^{\alpha_i}[H_i(t) \, R_{im}(x_{im-1}(t)] \ , \ i = 1,2,3,...,n \tag{14}$$

Subject to the initial conditions

$$x_{im}(0) = 0, \ i = 1,2,...,n \tag{15}$$

4. APPLICATIONS

We will apply the homotopy analysis method to the system of linear and non-linear fractional differential equations.

Example 1: Consider the following system of non-linear FDEs:

$$\begin{aligned} &D^{\alpha_1} x = 2\,y^2 \\ &D^{\alpha_2} y = t\,x \\ &D^{\alpha_2} z = yz \end{aligned} \tag{16}$$

With initial conditions as

$$x(0) = 0, \ y(0) = 1, \ z(0) = 1 \tag{17}$$

In view of the HAM presented above, if we select the auxiliary functions $H_1(t) = H_2(t) = H_3(t) = 1$ and the initial guesses $x_0(t) = 0, y_0(t) = 1, z_0(t) = 1,$ we can construct the homotopy

$$\begin{aligned} &R_{1m}(\vec{x}_{m-1}(t)) = D^{\alpha_1} x_{m-1} - 2\sum_{j=0}^{m-1} y_j \, y_{m-1-j} \\ &R_{2m}(\vec{y}_{m-1}(t)) = D^{\alpha_2} \, y_{m-1} - t\,x_{m-1} \\ &R_{3m}(\vec{z}_{m-1}(t)) = D^{\alpha_3} \, z_{m-1} - \sum_{j=0}^{m-1} y_j \, z_{m-1-j} \end{aligned} \tag{18}$$

By HAM the m-th order deformation equations are given by

$$\begin{aligned} &L_1[x_m(t) - \chi_m \, x_{m-1}(t)] = h_1 \, R_{1m}[\vec{x}_{m-1}(t)] \\ &L_2[y_m(t) - \chi_m y_{m-1}(t)] = h_2 \, R_{2m}[\vec{y}_{m-1}(t)] \\ &L_3[z_m(t) - \chi_m \, z_{m-1}(t)] = h_3 \, R_{3m}[\vec{z}_{m-1}(t)] \end{aligned} \tag{19}$$

Using equation (18) in (19) and applying the Riemann-Liouville integral operator J^{α_i} on both sides, one has

$$x_m(t) = (\chi_m + h)[x_{m-1}(t) - x_{m-1}(0)] + h_1 \, J^{\alpha_1}\left[-2\sum_{j=0}^{m-1} y_j y_{m-1-j}\right]$$

$$y_m(t) = (\chi_m + h_2)[y_{m-1}(t) - y_{m-1}(0)] + h_2 \, J^{\alpha_2}[-t\,x_{m-1}]$$

$$z_m(t) = (\chi_m + h_3)\ [z_{m-1}(t) - z_{m-1}(0)] + h_3\ J^{\alpha_3}\left[-\sum_{j=0}^{m-1} y_j z_{m-1-j}\right] \tag{20}$$

Then we obtain the following:

$$x_0(t) = 0$$

$$y_0(t) = 1$$

$$z_0(t) = 1$$

$$x_1(t) = -2h_1 \frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}$$

$$y_1(t) = 0$$

$$z_1(t) = -h_3 \frac{t^{\alpha_3}}{\Gamma(1+\alpha_1)}$$

$$x_2(t) = -2h_1(1+h_1)\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}$$

$$y_2(t) = 2h_1h_2(1+\alpha_1)\frac{t^{1+\alpha_1+\alpha_2}}{\Gamma(2+\alpha_1+\alpha_2)}$$

$$z_2(t) = h_3^2 \frac{1}{\Gamma(1+2\alpha_3)} t^{2\alpha_3} - h_3(1+h_3)\frac{1}{\Gamma(1+\alpha_3)} t^{\alpha_3}$$

$$x_3(t) = -2h_1(1+h_1)^2 \frac{1}{\Gamma(1+\alpha_1)} t^{\alpha_1} - 8h_1^2 h_2 \frac{(1+\alpha_1)}{\Gamma(2+2\alpha_1+\alpha_2)} t^{1+2\alpha_1+\alpha_2}$$

$$y_3(t) = 2h_1h_2(1+h_1)\frac{(1+\alpha_1)}{\Gamma(2+\alpha_1+\alpha_2)} t^{1+\alpha_1+\alpha_2} + 2h_1h_2(1+h_2)\frac{(1+\alpha_1)}{\Gamma(2+\alpha_1+\alpha_2)} t^{1+\alpha_1+\alpha_2}$$

$$z_3(t) = (1+h_3)\left[h_3^2 \frac{1}{\Gamma(1+2\alpha_3)} t^{2\alpha_3} - h_3(1+h_3)\frac{1}{\Gamma(1+\alpha_3)} t^{\alpha_3}\right]$$
$$-h_3\left[h_3^2 \frac{1}{\Gamma(1+3\alpha_3)} t^{2\alpha_3} - \frac{h_3(1+h_3)}{\Gamma(1+2\alpha_3)} t^{\alpha_3} + 2h_1h_2 \frac{(1+\alpha_1)}{\Gamma(2+\alpha_1+\alpha_2+\alpha_3)} t^{1+\alpha_1+\alpha_2}\right] t^{\alpha_3}$$

$$x(t) = -2h_1[1+(1+h_1)+(1+h_1)^2+\ldots]\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)} - 8[h_1^2h_2+\ldots]\frac{(1+\alpha_1)}{\Gamma(2+2\alpha_1+\alpha_2)} t^{1+2\alpha_1+\alpha_2}$$

$$y(t) = 1 + 2h_1h_2[1+(1+h_1)+(1+h_2)+\ldots]\frac{(1+\alpha_1)}{\Gamma(2+\alpha_1+\alpha_2)} t^{1+\alpha_1+\alpha_2} \tag{21}$$

$$z(t) = 1 - h_3[1+(1+h_3)+(1+h_3)^2+\ldots]\frac{t^{\alpha_3}}{\Gamma(1+\alpha_3)} + h_3^2[1+2(1+h_3)+\ldots]\frac{t^{2\alpha_3}}{\Gamma(1+2\alpha_3)}$$

$$-h_3^3[1+...]\frac{t^{3\alpha_3}}{\Gamma(1+3\alpha_3)}-2h_1h_2h_3\frac{t^{1+\alpha_1+\alpha_2+\alpha_3}}{\Gamma(2+\alpha_1+\alpha_2+\alpha_3)}$$

Example 2

$$D^{\alpha}x=y$$
$$D^{\alpha}y=z$$
$$D^{\alpha}z=-x-y-z+x^2 \tag{22}$$

Subject to the initial conditions

$$x(0)=0.2,\ y(0)=-0.3,\ z(0)=0.1 \tag{23}$$

Here we have assumed the initial guesses $x_0(t)=0.2, y_0(t)=-0.3 \text{and } z_0(t)=0.1.$

Using the equation (9) and (14) and taking $H_i(t) = 1$, we can construct the following

$$x_m(t)=(\chi_m+h_1)[x_{m-1}(t)-x_{m-1}(0)]+h_1\,J^{\alpha}[-y_{m-1}(t)]$$
$$y_m(t)=(\chi_m+h_2)[y_{m-1}(t)-y_{m-1}(0)]+h_2\,J^{\alpha}[-z_{m-1}(t)]$$
$$z_m(t)\ =\ (\chi_m+h_3)[z_{m-1}(t)-z_{m-1}(0)]+h_3\,J^{\alpha}\left[x_{m-1}(t)+y_{m-1}(t)+z_{m-1}(t)\text{-}\sum_{j=0}^{m-1}x_j\,x_{m-1-j}\right]$$

Then we obtain the following:

$$x_0(t)=0.2$$
$$y_0(t)=\text{-}0.3$$
$$z_0(t)=0.1$$
$$x_1(t)=0.3\,h_1\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$y_1(t)=-0.1h_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$z_1(t)=-0.04\,h_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$x_2(t)=0.3\,h_1(1+h_1)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+0.1h_1h_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$y_2(t)=-0.1h_2(1+h_2)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+0.4h_2\,h_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$z_2(t)=-0.04h_3(1+h_3)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+h_3[0.18h_1-0.01h_2-0.04h_3]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$x=0.2+0.3\,h_1[1+(1+h_1)+...]\frac{t^{\alpha}}{\Gamma(1+\alpha)}+0.1h_1[h_2+...]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$y = -0.3 - 0.1h_2[1+(1+h_2)+...]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 0.04h_2[h_3+...]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{24}$$

$$z = 0.1 - 0.04h_3[1+(1+h_3)+...]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + h_3[(0.18h_1 - 0.01h_2 - 0.04h_3)+...]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

Example 3

$$x^{\alpha_1} = \lambda(Tx - y) - \lambda x^3 \tag{25}$$

$$y^{\alpha_2} = x \qquad\qquad 0<\alpha_1,\alpha_2 \le 1$$

Subject to the initial conditions

$$x(0) = c_1 \text{ and } y(0) = c_2 \tag{26}$$

Here we have assumed the initial guesses $x_0(t) = c_1$ and $y_0(t) = c_2$.

Using the equation (9) and (14) and taking $H_i(t) = 1,$ we can construct the following:

$$x_m(t) = (\chi_m + h_1)[x_{m-1}(t) - x_{m-1}(0)]$$
$$+h_1 J^{\alpha_1}\left[-\lambda(Tx_{m-1} - y_{m-1}) + \lambda\sum_{j=0}^{m-1} x_j \sum_{i=0}^{m-1-j} x_i x_{m-1-i-j}\right]$$

$$y_m(t) = (\chi_m + h_2)[y_{m-1}(t) - y_{m-1}(0)] + h_2 J^{\alpha_2}[-x_{m-1}(t)] \tag{27}$$

Then we obtain the following:

$$x_0(t) = c_1$$

$$y_0(t) = c_2$$

$$x_1(t) = h_1\lambda(-Tc_1 + c_1^3 + c_2)\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}$$

$$y_1(t) = c_1 h_2 \frac{t^{\alpha_2}}{\Gamma(1+\alpha_1)}$$

$$x_2(t) = h_1(1+h_1)\lambda(-T c_1 + c_1^3 + c_2)\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}$$
$$-t^{\alpha_1}\left[t^{\alpha_1}\lambda(T - 3c_1^2)\frac{(-Tc_1 + c_1^3 + c_2)h_1}{\Gamma(1+2\alpha_1)} + \frac{c_1 h_2 t^{\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}\right]\lambda h_1$$

$$y_2(t) = -h_1 h_2\lambda(-Tc_1 + c_1^3 + c_2)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} - c_1 h_2(1+h_2)\frac{t^{\alpha_2}}{\Gamma(1+\alpha_1)}$$

$$x(t) = c_1 + \lambda_1 h_1(-Tc_1 + c_1^3 + c_2)[1+(1+h_1)+...]\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}$$

$$+\lambda^2 h_1^2(3c_1^2 - T)\frac{(-Tc_1 + c_1^3 + c_2)}{\Gamma(1+2\alpha_1)}t^{2\alpha_1} - \lambda c_1 h_1 h_2 \frac{1}{\Gamma(1+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2} \tag{28}$$

$$y(t) = c_2 - c_1 h_2[1 + (1+h_2) + \ldots]\frac{t^{\alpha_2}}{\Gamma(1+\alpha_2)} - h_1 h_2 \lambda(-Tc_1 + c_1^3 + c_2)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}$$

Now by setting the values of $c_1 = 0.45, c_2 = -0.02025, \lambda = 40, T = 0.1575$ than the numerical result are given in table (3) for $h_1 = h_2 = -1$ and for different values of α_1 and α_2.

Example 4

$$D^\alpha x = x - y$$

$$D^\alpha y = -x + y \tag{29}$$

Subject to the initial conditions

$$x(0) = 1,\ y(0) = 3 \tag{30}$$

Here we assume the initial guesses $x_0(t) = 1$ and $y_0(t) = 3.$

Using the equations (9) and (14) and taking $H_i(t) = 1$, we can construct the following

$$x_m(t) = (\chi_m + h_1)[x_{m-1}(t) - x_{m-1}(0)] + h_1 J^\alpha[-x_{m-1} + y_{m-1}]$$

$$y_m(t) = (\chi_m + h_2)[y_{m-1}(t) - y_{m-1}(0)] + h_2 J^\alpha[x_{m-1} - y_{m-1}] \tag{31}$$

Then we obtain the following:

$$x_0(t) = 1$$

$$y_0(t) = 3$$

$$x_1 = 2h_1 \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$y_1 = -2h_2 \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$x_2 = 2h_1(1+h_1)\frac{t^\alpha}{\Gamma(\alpha+1)} - 2h_1(h_1+h_2)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}$$

$$y_2 = -2h_2(1+h_2)\frac{t^\alpha}{\Gamma(\alpha+1)} + 2h_2(h_1+h_2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

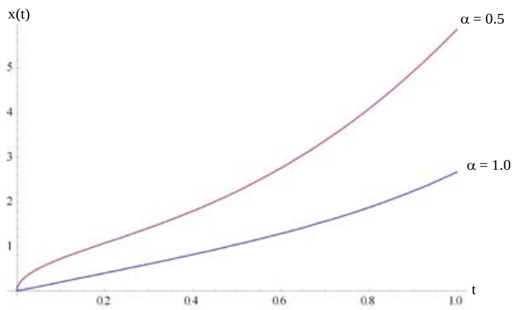
$$x_3(t) = 2h_1(1+h_1)\frac{t^\alpha}{\Gamma(1+\alpha)} - 2h_1(1+h_2)[2(h_1+h_2) + (h_1^2 + h_2^2)]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$+2h_1(h_1+h_2)^2 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

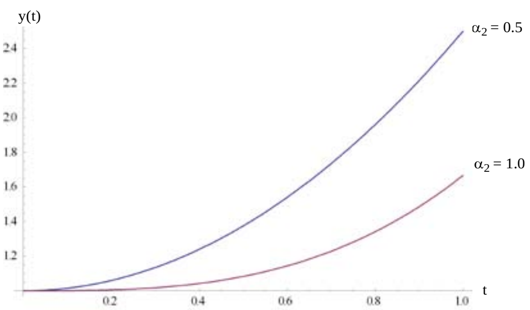
$$y_3(t) = -2h_2(1+h_2^2)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 2h_2[2(h_1+h_2)+(h_1^2+h_2^2)]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$-2h_2(h_1+h_2)^2\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$x(t) = 1 + 2h_1[1+(1+h_1)+(1+h_1)^2+\ldots]\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$-2h_1[(h_1+h_2)+(h_1+h_2)^2+[h_1(1+h_1)+h_2(1+h_2)]+\ldots]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$+2h_1[(h_1+h_2)^2+\ldots]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

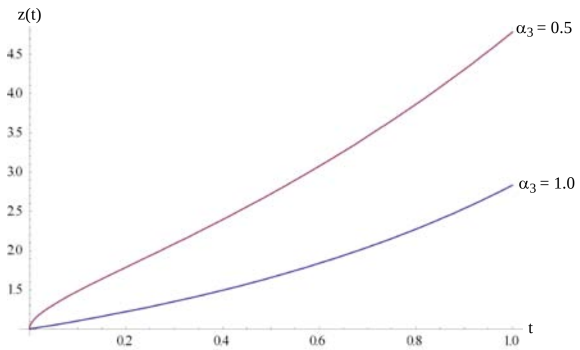
$$y(t) = 3 - 2h_2[1+(1+h_2)+(1+h_2)^2+\ldots]\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$+2h_2[(h_1+h_2)+(1+h_2)(h_1+h_2)+[h_1(1+h_1)+h_2(1+h_2)+\ldots]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$-2h_2[(h_1+h_2)^2+\ldots]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \tag{32}$$



1(a)

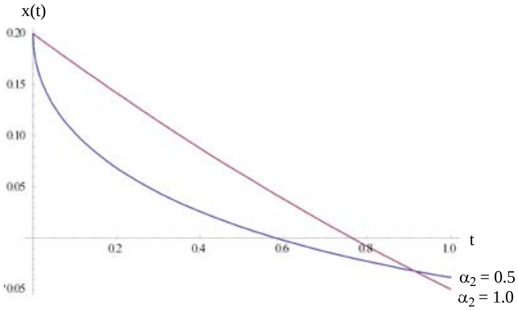


1(b)

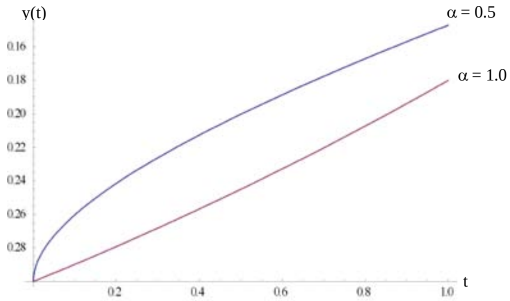


1(c)

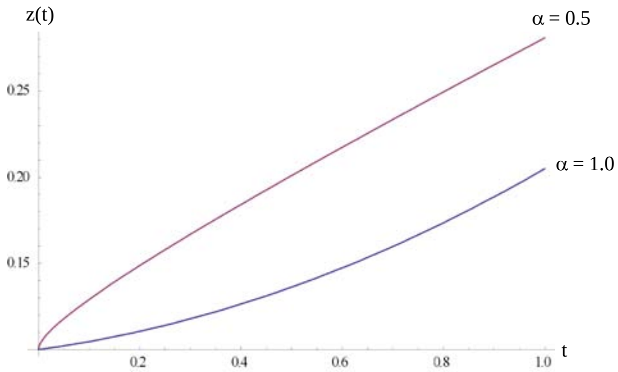
Figure 1: *Plot of the system (16) when $h_1 = h_2 = h_3 = -1$*



2(a)

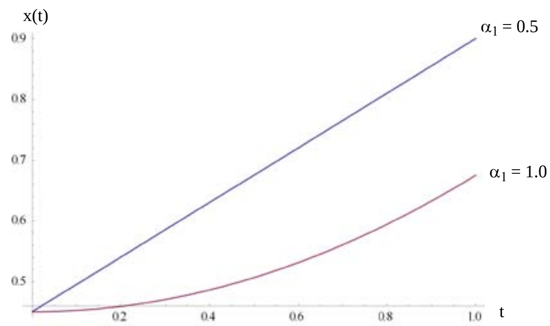


2(b)

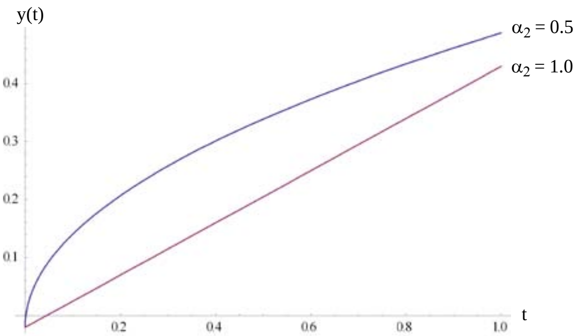


2(c)

Figure 2. *Plot of the system (22) when $h_1 = h_2 = h_3 = -1$*

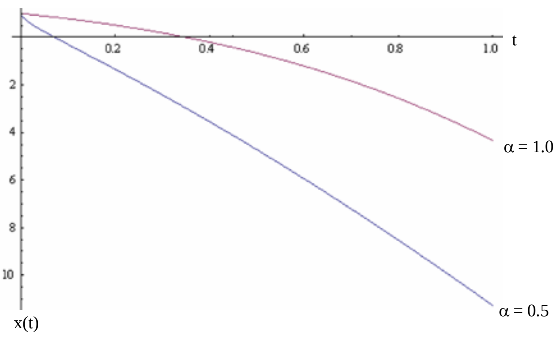


3(a)

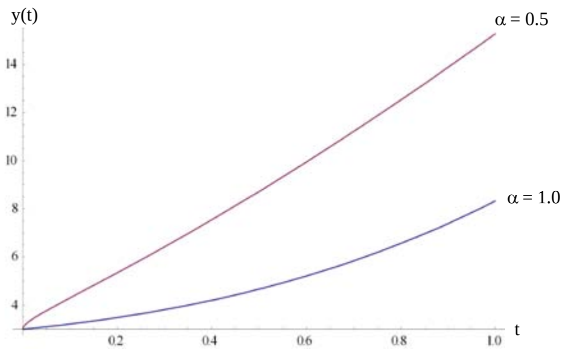


3(b)

Figure 3: *Plot of the system (25) when $h_1 = h_2 = -1$*



4(a)



4(b)

Figure 4: *Plot of the system (29) when $h_1 = h_2 = -1$*

5. Numerical values with varying order α_i and $h_i = -1$

Table 1: ***For Example 1***

t	$\alpha_1=\alpha_2=\alpha_3=0.5$			$\alpha_1=\alpha_2=\alpha_3=1$		
	X	y	z	x	y	z
0.0	0	1	1	0	1	1
0.2	1.07385	0.06	1.78806	0.401067	1.00533	1.2216
0.4	1.79269	1.24	2.3953	0.817067	1.04267	1.49493
0.6	2.75497	1.54	3.07538	1.2864	1.144	1.8376
0.8	4.08546	1.96	3.86426	1.87707	1.34133	2.2736
1.0	5.86757	2.5	4.78884	2.66667	1.66667	2.83333

Table 2: ***For Example 2***

t	$\alpha=0.5$			$\alpha=1$		
	X	y	z	x	y	z
0.0	0.2	-0.3	0.1	0.2	-0.3	0.1
0.2	0.68612	-0.241537	0.148776	0.142	-0.2792	0.1106
0.4	0.0259051	-0.212635	0.184211	0.088	-0.2568	0.1264
0.6	-0.00221162	-0.88596	0.21745	0.038	-0.2320	0.1474
0.8	-0.0227759	-0.167075	0.249553	-0.008	-0.2072	0.1736
1.0	-0.0385138	-0.147142	0.28093	-0.05	-0.18	0.2050

Table 3:*For Example 3*

t	$\alpha_1=\alpha_2=0.5$		$\alpha_1=\alpha_2=1.0$	
	X	y	x	y
0.0	0.45	-0.02025	0.45	-0.020205
0.2	0.54	0.206632	0.459	0.06975
0.4	0.63	0.300892	0.486	0.15975
0.6	0.72	0.373067	0.531	0.24975
0.8	0.81	0.433914	0.594	0.33975
1.0	0.90	0.487521	0.675	0.42975

Table 4: ***For Example 4***

t	$\alpha= 0.5$		$\alpha = 1.0$	
	X	y	x	y
0.0	1	3	1	3
0.2	-1.34752	5.39752	0.509333	3.49067
0.4	-3.54975	7.54975	0.205333	4.20533
0.6	-5.945	9.945	-1.208	5.208
0.8	-8.52465	12.5241	-2.56267	0.50267
1.0	-11.2748	15.2748	-4.33333	8.3333

CONCLUSION

This paper has focused on the successful employment of the powerful mathematical tool homotopy analysis method to find the solution of a system of linear and non-linear system of equations with fractional order. The method provides in a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. Thus it may be concluded that HAM is simple and accurate and represent a very powerful analytical approach for handling linear and non-linear system of differential equations with fractional order. Finally, generally speaking, we proposed method can be further implemented to save other problems in fractional calculus field.

Mathematica has been used for computation in this paper

REFERENCES

1. Arora, H.L., Abdelwahid, F.I. Solution of non-integer order differential equations via the Adomian decompositing method, Appl. Math. Lett. 6, (1993), 21-23.

2. Bourermel, Y. Explicit Series Solution for the Glauert-jet problem by means of the homotopy analysis method, Int. J. Non-linear Sci. Numer. Simulat. 12 (5) (2007), 714-724.

3. Cang, J., Tan, Y., Xu, H. Liao, S. Series solution of non-linear Riccati differential equations with fractional order, Chaos Soliton. Fract. 40 (1) (2007), 1-9.

4. Daftardar-Gejji, V., Jafari, H. An iterative method solving non-linear fractional equations. J. Math. Anal. Appl. 316 (2006), 753-763.

5. Deftardar-Giejji, V., Jafari, H. Solving a multi order fractional differential equations using a Adomain decomposition. J. Math. Anal. Appl. 189 (2007), 541-548.

6. Gorenflow, R., Mainardi, F. Fractional Calculus : integral and differential equations of fractional order, in : A Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York (1997), 223-276.

7. He, J.H. Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178 (1999), 257-262.
8. Jafari, H., Daftardar-Gejii, V. Solving linear and non-linear fractional diffusion and wave equations by Adomian-decomposition. Appl. Math. Comput. 180 (2006), 488-497.

9. Liao, S.J. The proposed homotopy analysis technique for the solution of non-linear problems. Ph.D. Thesis, Shanghai, Jiqo Tong University (1992).

10. Liao, S.J. Beyond Perturbation: Introduction to homotopy analysis method. CRC Press/Champan & Hall, BocaRaton (2003).

11. Liao, S.J. On the homotopy analysis method for non-linear problems. Appl. Math. Comput. 147 (2004), 499-513.

12. Momani, S. Analytic approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method. Appl. Math. Compute. 165 (2005), 459-472.

13. Momani, S., Al-Khaled, A. Numerical solutions for system of fractional differential equations by decomposition method. Appl. Math. Comput (196) (2005), 644-665.

14. Momani, S., Odibat, Z. Homotopy perturbation method for non-linear partial differential equations of fractional order. Phys. Lett. A 365 (5-6), (2007), 345-350.

15. Odibat, Z., Momani, S. Application of variation iteration method to non-liner differential equations of fractional order. Int. J. non-linear Science. Number. Simulat., 1(7) (2006), 15-27.

16. Odibat, Z., Momani, S. Numerical comparison of methods for solving linear differential equations of fractional order. Chaos. Soliton. Fract. 31 (5) (2007), 1248-1255.

17. Podlubny, I. Fractional differential equations. Sam Diego: Academic Press (1999).

18. Song, L., Zhang, H. Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation, Phys. Lett. A 367 (1-2) (2007), 88-94.

19. Wang, Q. Homotopy perturbation method for fractional KdV equation. Appl. Math. Comput. 190 (2007) 1795-1802.

20. Wang, Q. Homotopy perturbation method for fractional KdV-Burgers equation. Chaos. Solitons fractals. 35 (2008), 843-850.

21. Wu, Y.Y., Liao, S.J. Solving the one loop soliton solution of the Vakhnenko equation by means of the homotopy analysis method. Chaos. Soliton. Fract. 23(5) (2004), 1733-1740.

****

