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ON KAEHLERIAN CONHARMONIC* RECURRENT AND SYMMETRIC MANIFOLD

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ABSTRACT

Tachibana (1967) has studied on the Bochner curvature tensor. Sinha (1973) has studied H-curvature tensors in Kaehler manifold. In the present paper, the authors have defined Kaehlerian conharmonic recurrent and symmetric manifold and several theorems have been investigated.*

Key Words: *Kaehlerian, Holomorphically Bochner curvature tensor, Conharmonic*, Recurrent, Symmetric, Manifold.*

Classification Number: *32C15, 46A13, 46M40, 53B35, 53C55.*

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1. INTRODUCTION

An n (=2m) dimensional Kaehlerian manifold K^n is a Riemannian space which admits a structure tensor field F^h_i satisfying the relations,

(1.1) $F^i_j F^h_i = -\delta^h_j$,

(1.2) $F_{ij} = -F_{ji}$, $(F_{ij} = F^a_i g_{aj})$ and

(1.3) $F^h_{i,j} = 0$,

Where the comma (,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, which we denote by R^h_{ijk}, is given by

$$R^h_{ijk} = \partial_i \{_j{}^h{}_k\} - \partial_j \{_i{}^h{}_k\} + \{_i{}^h{}_l\} \{_j{}^l{}_k\} - \{_j{}^h{}_l\} \{_i{}^l{}_k\},$$

Where as the Ricci-tensor and the scalar curvature are respectively given by

$$R_{ij} = R^a{}_{aij} \quad \text{and} \quad R = R_{ij}\, g^{ij}$$

It is well known that these tensors satisfy the identities (Tachibana 1967)

(1.4) $F^a_i R^j_a = R^a_i F^j_a$ and

(1.5) $F^a_i R_{aj} = - R_{ia} F^a_j$

In view of (1.1), the relation (1.4) gives

(1.6) $F^a_i R^b_a F^j_b = - R^j_i$.

Also, multiplying (1.5) by g^{ij}, we obtain

$$F^a_i R^i_a = - R^j_a F^a_j$$

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which implies,

(1.7) $F^a_i R^i_a = 0.$

If we define a tensor S_{ij} by

(1.8) $S_{ij} = - F^a_i R_{aj}$,

We have

(1.9) $S_{ij} = - S_{ji}.$

The holomorphically conharmonic* curvature tensor $*T^h_{ijk}$ given by(Ram Behari and L.R. Ahuja)

(1.10) $$*T^h_{ijk} \overset{def}{==} R^h_{ijk} + \frac{1}{n-2}(g_{ik} R^h_j - g_{jk} R^h_i).$$

whereas the holomorphically Bochner curvature tensor K^h_{ijk} are given by (Sinha 1973).

(1.11) $$K^h_{ijk} = R^h_{ijk} + \frac{1}{n+4}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik} R^h_j - g_{jk} R^h_i + S_{ik} F^h_j - S_{jk} F^h_i + F_{ik} S^h_j - F_{jk} S^h_i + 2S_{ij} F^h_k + 2F_{ij} S^h_k)$$
$$- \frac{R}{(n+2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k).$$

The equation (1.11), in view of (1.10), may be expressed as

(1.12) $$K^h_{ijk} = *T^h_{ijk} - \frac{6}{(n-2)(n+4)}(g_{ik} R^h_j - g_{jk} R^h_i) + \frac{1}{(n+4)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i - S_{jk} F^h_i + F_{ik} S^h_j - F_{jk} S^h_i + 2S_{ij} F^h_k + 2F_{ij} S^h_k)$$
$$- \frac{R}{(n+2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k).$$

We shall use the following:

Definition (1.1): A Kaehlerian manifold K^n satisfying ([4])

(1.13) $R^h_{ijk,ab} = \lambda_a R^h_{ijk}$

For a non-zero recurrence tensor λ_a, will be called a Kaehlerian recurrent manifold.

The space K^n is called Kaehlerian Ricci-recurrent if it satisfying the relation.

(1.14) $R_{ij,a} = \lambda_a R_{ij}$,

Then, multiplying the above equation by g^{ij} and using the fact that $g^{ij}{}_{,a}$ =o, we get

(1.15) $R{,a} = \lambda_a R.$

Remark (1.1): From (1.13), it follows that every Kaehlerian recurrent manifold is Kaehlerian Ricci-Recurrent, but the converse is not necessarily true.

Definition (1.2): A Kaehler manifold is called Kaehlerian symmetric in the sense of Cartan if it satisfies ([4])

(1.16) $R^h_{ijk,a} = 0$, or equivalently $R_{ijkl,a} = 0$

Obviously a Kaehlerian symmetric manifold is Kaehlerin Ricci-symmetric, i.e.

(1.17) $R_{ij,a} = 0.$

Definition (1.3): A Kaehlerian manifold in which the Bochner curvature Tensor $K^h_{ijk,}$ satisfies the relation

(1.18) $K^h_{ijk,a} = \lambda_a K^h_{ijk}$,

For some non-zero vector λ_a, will be called a Kaehlerian manifold with recurrent Bochner curvature tensor, or Kaehlerian Bochner recurrent manifold ([4])

2. KAEHLERIAN CONHARMONIC* RECURRENT MANIFOLD.

Definition (2.1): A Kaehlerian manifold satisfying the relation

(2.1) $*T^h_{ijk,a} = \lambda_a *T^h_{ijk}$,

For some non-zero recurrence vector $\lambda_{a,}$ will be called a **Kaehlerian conharmonic* recurrent manifold.**

We have the following:

Theorem (2.1): Every Kaehlerian recurrent manifold is Kaehlerian conharmonic* recurrent.

Proof: A Kaehlerian recurrent manifold is characterized by the equation (1.13), which yields (1.14).By differentiating (1.10) covariently with respect to x^a and using equation(1.14), we get after some simplification.

$$*T^h_{ijk,a} = \lambda_a {}^*T^h_{ijk}.$$

Which shows that the space is Kaehlerian conharmonic* recurrent.

Theorem (2.2): A Kaehlerian conharmonic* recurrent manifold will be Kaehlerian recurrent provided that it is Kaehlerian Ricci-recurrent.

Proof : Differentiating (1.10) coveriantly with respect to $x^{a,}$ we obtain

(2.2) $$*T^h_{ijk,a} = R^h_{ijk,a} + \frac{1}{n-2} (g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a})$$

Multiplying (1.10) by λ_a and subtracting the result thus obtained from (2.2), we have

(2.3) $$*T^h_{ijk,a} -\lambda_a *T^h_{ijk,} = R^h_{jk,a} -\lambda_a R^h_{ijk} + \frac{1}{n-2} [g_{ik}(R^h_{j,,a} -\lambda_a R^h_{,l}) - g_{jk} (R^h_{i,a} - \lambda_a R^{\ h}_{i})]$$

Let the space be Kaehlerian Ricci-recurrent. Then (2.3) yields

(2.4) $*T^h_{ijk,a} - \lambda_a *T^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}$,

Which shows that the Kaehlerian conharomnic* recurrent manifold is Kaehlerian recurrent.

Theorem (2.3): The necessary and sufficient condition that a Kaehlerian manifold is Kaehlerian Ricci-recurrent, is that

$$*T^h_{ijk,a} - \lambda_a *T^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk} ,$$

Proof: Let the space be Kaehlerian Ricci-recurrent, then the (1.14) is satisfied and so the (2.3) reduces to

$$*T^h_{ijk,a} - \lambda_a *T^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk} ,$$

Conversely, if in a Kaehler space the above equation is satisfied, then (2.3) yields

(2.5) $g_{ik} (R^h_{j,a} -\lambda_a R^h_{j}) -g_{jk} (R^h_{i,a} - \lambda_a R^{h}_{\ i}) = 0,$

Which yields

$$R_{ij,\ a} - \lambda_a R_{ij} =0,$$

i.e. the manifold is Kaehlerian Ricci-recurrent, which completes the Proof.

Theorem (2.4): Every Kaelherian conhermonic* recurrent manifold is a Kaehler manifold with recurrent Bochner curvature tensor.

Proof: Let the space be Kaehlerian conharmonic* recurrent (2.1), in view of (1.10) gives

$$(2.6) \qquad R^h_{ijk,a} + \frac{1}{n-2}(g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a}] = \lambda_a [R^h_{ijk} + \frac{1}{n-2}(g_{ik} R^h_j - g_{jk} R^h_i)]$$

Which gives

$$(2.7) \qquad R_{,a} - \lambda_a R = 0$$

Differentiating (1.12) covariantly with respect to x^a, we obtain

$$(2.8) \qquad K^h_{ijk,a} = {}^*T^h_{ijk,a} - \frac{6}{(n-2)(n+4)}(g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a}) + \frac{1}{n+4}(R_{ik,a} \delta^h_j -$$

$$R_{jk,a} \delta^h_i + S_{ik,a} F^h_j - S_{jk,a} F^h_i + F_{ik} S^h_{j,a} - F_{jk} S^h_{i,ab} + 2S_{ij,a} F^h_k$$

$$+ 2F_{ij} S^h_{k,a}) - \frac{R_{,a}}{(n+2)(n+4)} (g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k)$$

Multiplying (1.12) by λ_a and subtracting from (2.8), we have

$$(2.9) \qquad K^h_{ijk,a} - \lambda_a K^h_{ijk} = {}^*T^h_{ijk,a} - \lambda_a {}^*T^h_{ijk} - \frac{6}{(n-2)(n+4)}[g_{ik}(R^h_{j,a} - \lambda_a R^h_j) -$$

$$-g_{jk} (R^h_{i,a} - \lambda_a R^h_i)] + \frac{1}{n+4}[\delta^h_j(R_{ik,a} - \lambda_a R_{ik}) - \delta^h_i(R_{jk,a}$$

$$- \lambda_a R_{jk}) + F^h_j (S_{ik,a} - \lambda_a S_{ik}) - F^h_i (S_{jk,a} - \lambda_a S_{jk}) + F_{ik} (S^h_{j,a}$$

$$- \lambda_a S^h_j) - F_{jk} (S^h_{i,a} - \lambda_a S^h_i) + 2F^h_k (S_{ij,a} - \lambda_a S_{ij}) + 2F_{ij}$$

$$(S^h_{k,a} - \lambda_a S^h_k)] - \frac{(R_{,a} - \lambda_a R)}{(n+2)(n+4)} [g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k]$$

Making use of equations (1.7),(1.14),(2.1) and (2.7) in (2.9), we get

$$K^h_{ijk,a} - \lambda_a K^h_{ijk} = 0$$

Which shows that space is a Kaehler manifold with recurrent Bochner curvature tensor.

Theorem (2.5): The necessary and sufficient condition for a Kaehler manifold to be Kaehlerian conharmonic* recurrent are that the space is a Kaehlerian Ricci-recurrent and a Kaehlerian-Bochner recurrent both.

Proof: The necessary part has been proved in theorem (2.4) for the sufficient part, let us suppose that the space be both Kaehlerian Ricc-recurrent and Kaehlerian Bochner recurrent. Then equations (1.14), (1.15) and (1.18), are satisfied.

Equation (1.12) yields (2.9), which in view of (1.14), (1.15) and (1.18), reduces to

$${}^*T^h_{ijk,a} - \lambda_a {}^*T^h_{ijk} = 0$$

This shows that the space is Kaehlerian conharmonic * recurrent. Hence the sufficient part is proved.

This completes the proof.

3. KAEHLERIAN CONHARMONIC* SYMMETRIC MANIFOLD.

Definition (3.1): A Kaehler manifold satisfying the relation

(3.1) $\quad {}^{*}T^{h}_{ijk,a} = 0$ or equivalently ${}^{*}T_{ijkl,a} = 0$,

Will be called a Kaehlerian conharmonic* symmetric manifold.

Theorem (3.1) : Every Kaehlerian symmetric manifold is a Kaeherian conharmoic* symmetric.

Proof: If the manifold is Kaehlerian symmetric, then the relations (1.16) and (1.17) are satisfied.

Differentisting (1.10) covariantly with respect to x^a and using (1.16) and (1.17), we get

$$ {}^{*}T^{h}_{ijk,a} = 0, $$

Which shows that the manifold is Kaehlerian conharmonic* symmetric.

Theorem (3.2): The necessary ad sufficient condition that a Kaehlerian conharmonic* symmetric manifold be Kaehlerian Ricci-recurrent, is that

$$ R^{h}_{ijk,a} + \lambda_a ({}^{*}T^{h}_{ijk} - R^{h}_{ijk,}) = 0. $$

Proof: Since the manifold is Kaehlerian conharmonic* symmetric (3.1) is satisfied and (2.3) takes the form

$$ (3.2)\,) \qquad R^{h}_{ijk,a} - \lambda_a R^{h}_{ijk} + \lambda_a\, {}^{*}T^{h}_{ijk} + \frac{1}{n-2}\,[g_{ik}(R^{h}_{j,a} - \lambda_a R^{h}_{j}) - g_{jk}(R^{h}_{i,a} - \lambda_a R^{h}_{i})] = 0 $$

If the manifold is Kaehlerian Ricci-recurrent, then the above equation reduces to

$$ (3.3) \qquad R^{h}_{ijk,a} - \lambda_a R^{h}_{ijk} + \lambda_a\, {}^{*}T^{h}_{ijk} = 0 $$

Which is the necessary condition.

Conversely, if the given condition is satisfied, then (3.2) reduces to

$$ (3.4) \qquad g_{ik}(R^{h}_{j,a} - \lambda_a R^{h}_{j}) - g_{jk}(R^{h}_{i,a} - \lambda_a R^{h}_{i}) = 0 $$

After some simplification the above equation gives us

$$ R_{ij,a} - \lambda_a R_{ij} = 0 $$

Which shows that the manifold is Kaehlerian Ricci-recurrent.

This completes the Proof.

Theorem (3.3): In a Kaehlerian conharmonic* symmetric manifold, the scalar curvature is constant.

Proof: From equations (1.10) and (3.1),we obtain

$$ (3.5) \qquad R^{h}_{ijk,a} = -\frac{1}{n-2}(g_{ik} R^{h}_{j,a} - g_{jk} R^{h}_{i,a}) $$

Multiplying the above equation g^{jk} and using that every symmetric manifold is Ricci-symmetric, then after simplification, we have

$$ (3.6) \qquad R_{,a} = 0, $$

i.e. R is a constant.

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