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Certain Transformations of Truncated Basic Bilateral q-Series

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ABSTRACT

In this paper we have established certain transformations of truncated basic bilateral hypergeometric series with one and more than one base. These results, in turn, lead to very interesting transformations of truncated bi-basic and poly-basic bilateral q-series. A few of the results which are representative of the many results obtained are presented in this article.

Key Words & Phrases: *Basic bilateral hypergeometric series, basic hypergeometric series poly-basic, q-series and truncated series.*

Subject Classification: *33A30, Secondary 10A45, 05A17.*

1. INTRODUCTION:

The transformation theory of basic hypergeometric series is very powerful in dealing with various functions motivated by q-series. A new direction to this theory can be given by developing the transformation theory of truncated hypergeometric series. With this intension we set out to develop it.

In this paper, we make use of the following series identity

$$\sum_{k=-m}^{n} a_{k+m} \sum_{j=0}^{n-k} A_j = \sum_{k=-m}^{n} A_{k+m} \sum_{j=0}^{n-k} a_j \tag{1.1}$$

to establish transformation of truncated basic bilateral q-series, which may be deduce from the following known identity (cf. Gasper Rahman [6])

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} a_j \tag{1.2}$$

We shall use the following known summations of truncated series to establish our results:

$${}_2\phi_1\begin{bmatrix} a, y; q; q \\ ayq \end{bmatrix}_N = \frac{[aq, yq; q]_N}{[q, ayq; q]_N} \tag{1.3}$$

[1; App. II (8)].

$${}_4\phi_3\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, e; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e \end{bmatrix}_N = \frac{[aq, eq; q]_N}{[q, aq/e; q]_N e^N} \qquad \text{[1; App. II (23)].} \tag{1.4}$$

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$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix}\right]_N = \frac{[aq, bq, cq, dq; q]_N}{[q, aq/b, aq/c, aq/d; q]_N} \tag{1.5} $$

[1; App. II (25)].

$$ \sum_{k=0}^{n} \frac{(1-ap^k q^k)[a;p]_k [c;q]_k c^{-k}}{(1-a)[q;q]_k [ap/c;p]_k} = \frac{[ap;p]_n [cq;q]_n c^{-n}}{[q;q]_n [ap/c;p]_n} \tag{1.6} $$

[6; App. II (34)].

$$ \sum_{k=0}^{n} \frac{(1-adp^k q^k)\left(1-\frac{b}{d}p^k q^{-k}\right)[a,b;p]_k [c, ad^2/bc;q]_k q^k}{(1-a)(1-b/d)[dq, adq/b;q]_k [adp/c, bcp/d;p]_k} $$
$$ = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} $$
$$ \times\left\{\frac{[ap, bp;p]_n [cq, ad^2 q/bc;q]_n}{[dq, adq/b;q]_n [adp/c, bcp/d;p]_n} - \frac{(b-ad)(c-ad)(d-bc)(1-d)}{d(1-a)(1-b)(1-c)(bc-ad^2)}\right\} \tag{1.7} $$

which is the case m=0 in [6; eq.12 p.83].

2. DEFINITIONS AND NOTATIONS:

A generalized bi-basic hypergeometric function of one variable is defined as

$$ \phi\left[\begin{matrix} (a);(b);q,q_1;z \\ (c);(d);q^i, q_1^j \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[(a);q]_n [(b);q_1]_n z^n q^{i\binom{n}{2}} q_1^{j\binom{n}{2}}}{[q;q]_n [(c);q]_n [(d);q_1]_n} \tag{2.1} $$

where (a) stands for the sequence of A-parameters $a_1, a_2, \ldots\ldots, a_A$.

Also,

$$ [(a);q]_n = [a_1, a_2, \ldots\ldots, a_A;q]_n = [a_1;q]_n [a_2;q]_n \ldots\ldots [a_A;q]_n $$

and

$$ [a;q]_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\ldots\ldots(1-aq^{n-1}) & , n > 0 \\ 1 & , n = 0 \end{cases} $$

The symbol $\binom{n}{2}$ shall mean $\frac{n(n-1)}{2}$. The series on the right of (2.1) converges for |q|,|q_1|< 1 and |z|<∞ when i,j >0 and max.(|q|,|q_1|,|z|) < 1 when i = j = 0.

We also define a poly-basic hypergeometric series of one variable as

$$ \phi\left[\begin{matrix} a_1, a_2, \ldots, a_r; c_{1,1}, \ldots, c_{1,r_1}; \ldots; c_{m,1}, \ldots, c_{m,r_m}; q, q_1, \ldots, q_m; z \\ b_1, b_2, \ldots, b_s; d_{1,1}, \ldots, d_{1,s_1}; \ldots; d_{m,1}, \ldots, d_{m,s_m} \end{matrix}\right] $$
$$ = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r;q]_n z^n}{[q, b_1, b_2, \ldots, b_s;q]_n} \prod_{j=1}^{m} \frac{[c_{j,1}, \ldots, c_{j,r_j};q_j]_n}{[d_{j,1}, \ldots, d_{j,s_j};q_j]_n} \tag{2.2} $$

The series (2.2) converges for |q|,|q_1|,……,|q_m|,|z|< 1.

A series of the type

$$\sum_{r=-\infty}^{\infty} u_r$$

is a bilateral series, convergent under appropriate condition, which may terminate on either or both sides.

A truncated poly-basic bilateral series is defined as

$$\psi\left[\begin{matrix} a_1, a_2, \ldots\ldots, a_r; c_{1,1}, \ldots\ldots, c_{1,r_1}; \ldots\ldots; c_{s,1}, \ldots\ldots, c_{s,r_s}; q, q_1, \ldots\ldots, q_s; z \\ b_1, b_2, \ldots\ldots, b_r; d_{1,1}, \ldots\ldots, d_{1,r_1}; \ldots\ldots; d_{s,1}, \ldots\ldots, d_{s,r_s} \end{matrix}\right]_m^n$$
$$= \sum_{j=-m}^{n} \frac{[a_1, a_2, \ldots\ldots, a_r; q]_j z^j}{[b_1, b_2, \ldots\ldots, b_r; q]_j} \prod_{i=1}^{s} \frac{[c_{i,1}, \ldots\ldots, c_{i,r_i}; q_i]_j}{[d_{i,1}, \ldots\ldots, d_{i,r_i}; q_i]_j} \tag{2.3}$$

The other notations that follow will carry their usual meaning.

3. MAIN RESULTS:

Here we shall establish our transformations of truncated basic bilateral hypergeometric series.

If we take $A_j = z^j$, the identity (1.1) takes form,

$$\sum_{k=-m}^{n} a_{k+m} - z^{1+n} \sum_{k=-m}^{n} a_{k+m} z^{-k} = z^m (1-z) \sum_{k=-m}^{n} z^k \sum_{j=0}^{n-k} a_j \tag{3.1}$$

(I) If we take

$$a_j = \frac{[a, y; q]_j q^j}{[q, ayq; q]_j}$$

in (3.1) and make use of (1.3), we get(2)

$${}_2\psi_2\left[\begin{matrix} aq^m, yq^m; q; q \\ q^{1+m}, ayq^{1+m} \end{matrix}\right]_m^n - z^{1+n}\, {}_2\psi_2\left[\begin{matrix} aq^m, yq^m; q; \dfrac{q}{z} \\ q^{1+m}, ayq^{1+m} \end{matrix}\right]_m^n$$
$$= (1-z)\left(\frac{z}{q}\right)^m \frac{[q, ayq; q]_m [aq, yq; q]_n}{[a, y; q]_m [q, ayq]_n} \times {}_2\psi_2\left[\begin{matrix} q^{-n}, q^{-n}/ay; q; z \\ q^{-n}/a, q^{-n}/y \end{matrix}\right]_m^n \tag{3.2}$$

(II) Again, setting

$$a_j = \frac{q^j [adpq; pq]_j \left[\dfrac{b}{d}\dfrac{p}{q}; \dfrac{p}{q}\right]_j [a, b; p]_j [c, ad^2/bc; q]_j}{[ad; pq]_j \left[\dfrac{b}{d}; \dfrac{p}{q}\right]_j [dq, adq/b; q]_j [adp/c, bcp/d; p]_j}$$

in (3.1) and using (1.7), we get

$$
{}_6\psi_6\left[\begin{matrix} cq^m,\dfrac{ad^2}{bc}q^m;ap^m,bp^m;adp^{1+m}q^{1+m};\dfrac{b}{d}\dfrac{p^{1+m}}{q^{1+m}};q,p,pq,\dfrac{p}{q};q \\ dq^{1+m},\dfrac{ad}{b}q^{1+m};\dfrac{ad}{c}p^{1+m},\dfrac{bc}{d}p^{1+m};adp^mq^m;\dfrac{b}{d}\dfrac{p^m}{q^m} \end{matrix}\right]_m^n
$$

$$
-z^{1+n}\,{}_6\psi_6\left[\begin{matrix} cq^m,\dfrac{ad^2}{bc}q^m;ap^m,bp^m;adp^{1+m}q^{1+m};\dfrac{b}{d}\dfrac{p^{1+m}}{q^{1+m}};q,p,pq,\dfrac{p}{q};\dfrac{q}{z} \\ dq^{1+m},\dfrac{ad}{b}q^{1+m};\dfrac{ad}{c}p^{1+m},\dfrac{bc}{d}p^{1+m};adp^mq^m;\dfrac{b}{d}\dfrac{p^m}{q^m} \end{matrix}\right]_m^n
$$

$$
=\left(\frac{z}{q}\right)^m(1-z)\frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times
$$

$$
\times\frac{[ap,bp;p]_n[cq,ad^2/bc;q]_n}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}\frac{[ad;pq]_m\left[\dfrac{b}{d};\dfrac{p}{q}\right]_m[dq,adq/b;q]_m}{[adpq;pq]_m\left[\dfrac{b}{d}\dfrac{p}{q};\dfrac{p}{q}\right]_m[a,b;p]_m}\times
$$

$$
\times\frac{[adp/c,bcp/d;p]_m}{[c,ad^2/bc;q]_m}\times{}_4\psi_4\left[\begin{matrix} q^{-n}/d,bq^{-n}/ad;cp^{-n}/ad,dp^{-n}/bc;q,p;z \\ q^{-n}/c,bcq^{-n}/ad^2;p^{-n}/a,p^{-n}/b \end{matrix}\right]_m^n-
$$

$$
-(1-z^{1+m+n})\frac{(b-ad)(c-ad)(d-bc)(1-d)}{bcd^2(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times
$$

$$
\times\frac{[ad;pq]_m\left[\dfrac{b}{d};\dfrac{p}{q}\right]_m[dq,adq/b;q]_m[adp/c,bcp/d;p]_m}{[adpq;pq]_m\left[\dfrac{b}{d}\dfrac{p}{q};\dfrac{p}{q}\right]_m[a,b;p]_m[c,ad^2/bc;q]_m} \tag{3.3}
$$

If we let $d\to 1$ in (3.3), we get

$$
{}_6\psi_6\left[\begin{matrix} cq^m,\dfrac{a}{bc}q^m;ap^m,bp^m;ap^{1+m}q^{1+m};b\dfrac{p^{1+m}}{q^{1+m}};q,p,pq,\dfrac{p}{q};q \\ q^{1+m},\dfrac{a}{b}q^{1+m};\dfrac{a}{c}p^{1+m},bcp^{1+m};ap^mq^m;b\dfrac{p^m}{q^m} \end{matrix}\right]_m^n
$$

$$
-z^{1+n}\,{}_6\psi_6\left[\begin{matrix} cq^m,\dfrac{a}{bc}q^m;ap^m,bp^m;ap^{1+m}q^{1+m};b\dfrac{p^{1+m}}{q^{1+m}};q,p,pq,\dfrac{p}{q};\dfrac{q}{z} \\ q^{1+m},\dfrac{a}{b}q^{1+m};\dfrac{a}{c}p^{1+m},bcp^{1+m};ap^mq^m;b\dfrac{p^m}{q^m} \end{matrix}\right]_m^n
$$

$$
=\left(\frac{z}{q}\right)^m(1-z)\times\frac{[ap,bp;p]_n[cq,a/bc;q]_n}{[q,aq/b;q]_n[ap/c,bcp;p]_n}\frac{[a;pq]_m\left[b;\dfrac{p}{q}\right]_m[q,aq/b;q]_m}{[apq;pq]_m\left[b\dfrac{p}{q};\dfrac{p}{q}\right]_m[a,b;p]_m}
$$

$$
\times\frac{[ap/c,bcp;p]_m}{[c,a/bc;q]_m}\times{}_4\psi_4\left[\begin{matrix} q^{-n},bq^{-n}/a;cp^{-n}/a,p^{-n}/bc;q,p;z \\ q^{-n}/c,bcq^{-n}/a;p^{-n}/a,p^{-n}/b \end{matrix}\right]_m^n \tag{3.4}
$$

(III) Further, taking

$$a_j = \frac{[apq;pq]_j[a;p]_j[c;q]_j c^{-j}}{[a;pq]_j[q;q]_j[ap/c;p]_j}$$

in (3.1) and using (1.6), we get

$$
{}_3\psi_3\left[\begin{matrix} cq^m; ap^m; ap^{1+m}q^{1+m}; q, p, pq; \dfrac{1}{c} \\ q^{1+m}; \dfrac{a}{c}p^{1+m}; ap^m q^m \end{matrix}\right]_m^n - z^{1+n}\,{}_3\psi_3\left[\begin{matrix} cq^m; ap^m; ap^{1+m}q^{1+m}; q, p, pq; \dfrac{1}{zc} \\ q^{1+m}; \dfrac{a}{c}p^{1+m}; ap^m q^m \end{matrix}\right]_m^n
$$

$$
= (zc)^m(1-z)\frac{[ap;p]_n[cq;q]_n[a;pq]_m[q;q]_m[ap/c;p]_m c^{-n}}{[q;q]_n[ap/c;p]_n[apq;pq]_m[a;p]_m[c;q]_m} \times \times_2\psi_2\left[\begin{matrix} q^{-n}; \dfrac{c}{a}p^{-n}; q, p; \dfrac{z}{c} \\ q^{-n}/c; p^{-n}/a \end{matrix}\right]_m^n \qquad (3.5)
$$

(IV) Next, taking

$$a_j = \frac{\left[a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q\right]_j q^j}{\left[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d; q\right]_j}$$

with a=bcd in (3.1) and using (1.5), we get

$$
{}_6\psi_6\left[\begin{matrix} aq^m, q^{m+1}\sqrt{a}, -q^{m+1}\sqrt{a}, bq^m, cq^m, dq^m; q; q \\ q^{1+m}, q^m\sqrt{a}, -q^m\sqrt{a}, aq^{1+m}/b, aq^{1+m}/c, aq^{1+m}/d \end{matrix}\right]_m^n -
$$

$$
- z^{1+n}\,{}_6\psi_6\left[\begin{matrix} aq^m, q^{m+1}\sqrt{a}, -q^{m+1}\sqrt{a}, bq^m, cq^m, dq^m; q; q/z \\ q^{1+m}, q^m\sqrt{a}, -q^m\sqrt{a}, aq^{1+m}/b, aq^{1+m}/c, aq^{1+m}/d \end{matrix}\right]_m^n
$$

$$
= \left(\frac{z}{q}\right)^m(1-z)\frac{[aq, bq, cq, dq; q]_n\left[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq, d; q\right]_m}{[q, aq/b, aq/c, aq/d; q]_n\left[a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q\right]_m} \times
$$

$$
\times_4\psi_4\left[\begin{matrix} q^{-n}, bq^{-n}/a, cq^{-n}/a, dq^{-n}/a; q; z \\ q^{-n}/a, q^{-n}/b, q^{-n}/c, q^{-n}/d \end{matrix}\right]_m^n \qquad (3.6)
$$

(V) Finally, taking

$$a_j = \frac{\left[a, q\sqrt{a}, -q\sqrt{a}, e; q\right]_j}{\left[q, \sqrt{a}, -\sqrt{a}, aq/e; q\right]_j e^j}$$

in (3.1) and using (1.4), we get

$$
{}_4\psi_4\left[\begin{matrix} aq^m, q^{m+1}\sqrt{a}, -q^{m+1}\sqrt{a}, eq^m; q; 1/e \\ q^{1+m}, q^m\sqrt{a}, -q^m\sqrt{a}, aq^{1+m}/e \end{matrix}\right]_m^n - z^{1+n}\,{}_4\psi_4\left[\begin{matrix} aq^m, q^{m+1}\sqrt{a}, -q^{m+1}\sqrt{a}eq^m; q; 1/e \\ q^{m+1}, q^m\sqrt{a}, -q^m\sqrt{a}aq^{m+1}/e \end{matrix}\right]_m^n
$$

$$
= \left(\frac{z}{q}\right)^m(1-z)\frac{[aq, eq; q]_n\left[q, \sqrt{a}, -\sqrt{a}, aq/e; q\right]_m}{[q, aq/e; q]_n\left[a, q\sqrt{a}, -q\sqrt{a}, e; q\right]_m}\times_2\psi_2\left[\begin{matrix} q^{-n}, eq^{-n}/a; q; z/e \\ q^{-n}/a, q^{-n}/e \end{matrix}\right]_m^n \qquad (3.7)
$$

To conclude, in this short article we have shown that starting from a modified Gasper and Rahman [6] identity, it is possible to establish transformation of truncated basic bilateral hypergeometric series in terms of a similar series. Only a few examples have been shown here to illustrate our methodology.

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