

ON TOTAL DOMINATION SETS AND POLYNOMIALS OF CYCLES

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ABSTRACT

Let $G = (V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a total dominating set of G , if every vertex $u \in V$ is adjacent to an element of S . Let $\mathcal{D}_i(C_n)$ be the family of total dominating sets of a cycle C_n with cardinality i . Let $d_i(C_n)$ be the number of total dominating sets in $\mathcal{D}_i(C_n)$. In this paper, we study the concept of total domination polynomial for any cycle C_n . The total domination polynomial for any cycle C_n is the polynomial $D_t(C_n, x) = \sum_{i=1+n/2}^n d_i(C_n, i) x^i$, if $n \equiv 2 \pmod{4}$ and $D_t(C_n, x) = \sum_{i=\lceil n/2 \rceil}^n d_i(C_n, i) x^i$ if $n \not\equiv 2 \pmod{4}$. We obtain some properties of $D_t(C_n, x)$ and its coefficients. Also, we calculate the reduction formula to derive the total domination polynomial of cycles.

Keywords: cycles, total dominating set, total domination number, total domination polynomial.

Mathematics subject classification: 05C69, 11B83.

1. INTRODUCTION

Let $G = (V, E)$ be a graph. For any vertex $u \in V$, we define the open neighborhood of u as the set $N(u)$ defined by $N(u) = \{ v \mid uv \in E \}$ and the closed neighborhood of u as the set $N[u]$ defined by $N[u] = N(u) \cup \{u\}$. For a subset S of V , the open neighborhood of S is $N(S)$ which is defined as the union of $N(u)$ for all $u \in S$ and the closed neighborhood of S is defined as $N(S) \cup S$. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A set S of vertices in a graph G is said to be a dominating set if every vertex $u \in V$ is either an element of S or is adjacent to an element of S . A set of vertices in a graph G is said to be a total dominating set if every vertex $u \in V$ is adjacent to an element of S . The domination number of a graph, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G . The total domination number of a graph G , denoted by $\gamma_t(G)$, is the minimum cardinality of the total dominating sets in G .

We use the notation $\lceil x \rceil$ for the smallest integer greater than or equal to x . Also, we denote the set $\{1, 2, 3, \dots, n\}$ by $[n]$, throughout this paper.

2. TOTAL DOMINATING SETS OF CYCLES

In this section, we are going to investigate the total domination sets of cycles and some of its properties.

Definition 2.1: Let G be a graph of order n with no isolated vertices. Let $\mathcal{D}_i(G)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_i(G)|$. Then the total domination polynomial $D_t(G, x)$ of G is defined as $D_t(G, x) = \sum_{i=\gamma_t(G)}^n d_t(G, i) x^i$, where $\gamma_t(G)$ is the total domination number of G .

Let C_n , $n \geq 3$ be the cycle with n vertices. Let $V(C_n) = \{1, 2, \dots, n\}$ and $E(C_n) = \{(1,2), (2,3), \dots, (n-1,n), (n,1)\}$. Let $\mathcal{D}_i(C_n)$ be the collection of total domination sets in C_n with cardinality i . We shall investigate the total domination sets of cycles.

Lemma 2.2: [4] For $n \geq 3$, the total domination number of the cycle, C_n is given by

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$$\gamma_t(C_n) = \begin{cases} 1 + \frac{n}{2} & \text{if } n \equiv 2 \pmod{4} \\ \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \pmod{4} \end{cases}$$

Lemma 2.3: Let $C_n, n \geq 3$ be the cycle with $|V(C_n)| = n$. Then $d_t(C_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n$ and $d_t(C_n, i) > 0$ if $\lceil \frac{n}{2} \rceil < i \leq n$.

Proof: If $n \equiv 2 \pmod{4}$, then the total domination number of the cycle C_n is $\gamma_t(C_n) = 1 + \frac{n}{2}$. Therefore, $d_t(C_n, i) = 0$, when $i < 1 + \frac{n}{2}$ or $i > n$. And $d_t(C_n, i) > 0$, when $1 + \frac{n}{2} \leq i \leq n$. When $n \not\equiv 2 \pmod{4}$, then the total domination number of C_n is $\gamma_t(C_n) = \lceil \frac{n}{2} \rceil$. Therefore, $d_t(C_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n$. Also, $d_t(C_n, i) > 0$ when $\lceil \frac{n}{2} \rceil \leq i \leq n$. Hence, in general, we have $d_t(C_n, i) = 0$ when $i < \lceil \frac{n}{2} \rceil$ or $i > n$ and $d_t(C_n, i) > 0$ when $\lceil \frac{n}{2} \rceil < i \leq n$.

Lemma 2.4: Let $C_n, n \geq 3$ be the cycle with $|V(C_n)| = n$. Then

- (i) $\mathcal{D}_t(C_n, i) = \phi$ if $i < \gamma_t(C_n)$ or $i > n$.
- (ii) $\mathcal{D}_t(C_n, x)$ has no constant term and first degree terms.
- (iii) $\mathcal{D}_t(C_n, x)$ is a strictly increasing function on $[0, \infty)$.

Proof is obvious.

Lemma 2.5: Let $C_n, n \geq 3$ be the cycle with $|V(C_n)| = n$.

- (i) If $\mathcal{D}_t(C_{n-1}, i-1) = \mathcal{D}_t(C_{n-3}, i-1) = \phi$, then, $\mathcal{D}_t(C_{n-2}, i-1) = \phi$.
- (ii) If $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-3}, i-1) \neq \phi$ then $\mathcal{D}_t(C_{n-2}, i-1) \neq \phi$.
- (iii) If $\mathcal{D}_t(C_{n-1}, i-1) = \mathcal{D}_t(C_{n-3}, i-1) = \mathcal{D}_t(C_{n-2}, i-1) = \phi$ then $\mathcal{D}_t(C_n, i) = \phi$.

The proof of the lemma follows from lemma 2.3.

Lemma 2.6: Let $C_n, n \geq 3$ be the cycle with $|V(C_n)| = n$. Suppose that $\mathcal{D}_t(C_n, i) \neq \phi$, then we have

- (i) $\mathcal{D}_t(C_{n-2}, i-1) = \phi$ and $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$ if and only if $i = n$.
- (ii) $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi, \mathcal{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-3}, i-1) = \phi$ if and only if $i = n-1$.
- (iii) $\mathcal{D}_t(C_{n-1}, i-1) = \phi$ and $\mathcal{D}_t(C_{n-2}, i-1) = \phi$ if and only if $n = 4k$, and $i = 2k$ for some positive integer k .
- (iv) $\mathcal{D}_t(C_{n-1}, i-1) = \phi, \mathcal{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-3}, i-1) \neq \phi$ if and only if $n = 4k-1$, and $i = 2k$ for some k .
- (v) $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi, \mathcal{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-3}, i-1) \neq \phi$ if and only if $\lceil \frac{n-1}{2} \rceil + 1 < i \leq n-2$.

Proof: The proof of the lemma is similar to the proof of lemma [2.4] in [7]

Theorem 2.7 For every $n \geq 5$ and $i > \lceil \frac{n}{2} \rceil + 1$,

- (i) $\mathcal{D}_t(C_{4k}, 2k) = \{ \{1, 2, 5, 6, \dots, 4k-3, 4k-2\}, \{2,3,6,7, \dots, 4k-2, 4k-1\}, \{3,4,7,8, \dots, 4k-1, 4k\}, \{1,4,5,8, \dots, 4k-3, 4k\} \}$, where $k \geq 1$.
- (ii) If $\mathcal{D}_t(C_{n-2}, i-1) = \phi$ and $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$ then $\mathcal{D}_t(C_n, i) = \mathcal{D}_t(C_n, n) = \{ \{1, 2, 3, \dots, n\} \}$.
- (iii) If $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi, \mathcal{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-3}, i-1) = \phi$ then $\mathcal{D}_t(C_n, i) = \mathcal{D}_t(C_n, n-1) = \{ [n] - \{x\} / x \in [n] \}$.
- (iv) If $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-2}, i-1) \neq \phi$ then $\mathcal{D}_t(C_n, i) =$

$$\left\{ \begin{aligned} & \{X \cup \{n\}\} \cup \\ & \{Y \cup \{n\} \text{ if } 1 \in Y\} \cup \\ & \{Y \cup \{n-1\} \text{ if } n-2 \in Y\} \cup \\ & \{Y \cup \{1\} \text{ if } n-2 \in Y \text{ and } 1 \notin Y\} \cup \\ & \{Y \cup \{n-2\} \text{ if } n-3, n-4 \in Y \text{ and } n-2 \notin Y\} \cup \\ & \{(Y - \{1\}) \cup \{n, n-1\} \text{ if } 1, 2, 3, n-3 \in Y \text{ and } n-2 \notin Y\} \cup \\ & \{(Z - \{n-2\}) \cup \{n, n-1\} \text{ if } 1, 2 \notin Z \text{ and } n-4 \in Z\} \cup \\ & \{Z \cup \{n\}\} \text{ if } 1, n-2, n-3 \in Z \text{ and } 2 \notin Z\} \cup \\ & \{Z \cup \{n-1\}\} \text{ if } 1, 2, n-2 \in Z \text{ and } n-3 \notin Z\} \cup \\ & \{(Z - \{1\}) \cup \{n-1, n\} \text{ if } n-3, n-2 \notin Z \text{ and } 3 \in Z\} \cup \\ & \{(Z - \{n-2\}) \cup \{n-3, n\} \text{ if } 1, n-2 \in Z \text{ and } 2, n-3 \notin Z\} \cup \\ & \{(Z - \{n-3\}) \cup \{n-2, n-1\} \text{ if } n-3, n-4, n-5 \in Z \text{ and } 1 \notin Z\} \cup \\ & \{(Z - \{n-3\}) \cup \{n, n-1\} \text{ if } n-3, n-4, n-5 \in Z \text{ and } n-2 \notin Z\} \cup \\ & \{(Z - \{n-2, n-3\}) \cup \{n-4, n-3, n\} \text{ if } 1, n-6, n-5 \in Z \text{ and } n-4, n-3 \notin Z\} \cup \\ & \{(Z - \{n-3, n-4\}) \cup \{n-5, n-2, n-1\} \text{ if } 2, n-3 \in Z \text{ and } 1, n-2 \notin Z\} \cup \\ & \{(Z - \{n-4\}) \cup \{n-3, n-2\} \text{ if } n-6, n-5, n-4 \in Z \text{ and } n-2, n-3 \notin Z\} \cup \\ & \{(Z - \{n-2, n-3\}) \cup \{n-4, n-1, n\} \text{ if } n-5, n-3, n-2 \in Z \text{ and } 1, 2, n-4 \notin Z\} \end{aligned} \right.$$

where $X \in \mathcal{D}_t(C_{n-1}, i-1) - \mathcal{D}_t(C_{n-2}, i-1)$, $Y \in \mathcal{D}_t(C_{n-1}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1)$ and $Z \in \mathcal{D}_t(C_{n-1}, i-1) - (\mathcal{D}_t(C_{n-2}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1))$.

Proof:

(i) For any $k \geq 1$, split the vertices of C_{4k} in to k number of sets of the form $\{1,2,3,4\}$, $\{5,6,7,8\}$ and $\{4k-3, 4k-2, 4k-1, 4k\}$. The four total dominating sets of cardinality $2k$ are constructed by choosing the first two numbers in each set or second and third or third and fourth or first and fourth from each set. Hence, $\mathcal{D}_t(C_{4k}, 2k)$ has the only four total dominating sets, such as, $\{1, 2, 5, 6, \dots, 4k-3, 4k-2\}$, $\{2,3,6,7, \dots, 4k-2,4k-1\}$, $\{3,4,7,8, \dots, 4k-1,4k\}$, $\{1,4,5,8, \dots, 4k-3,4k\}$.

(ii) Since $\mathcal{D}_t(C_{n-2}, i-1) = \phi$ and $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$, by lemma 2.6, $i = n$. Therefore, $\mathcal{D}_t(C_n, i) = \mathcal{D}_t(C_n, n) = \{[n]\}$.

(iii) If $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$, $\mathcal{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-3}, i-1) = \phi$, then by lemma 2.6, $i = n-1$ then $\mathcal{D}_t(C_n, i) = \mathcal{D}_t(C_n, n-1) = \{[n] - \{x\} / x \in [n]\}$.

(iv) First, we consider the collection of total domination sets $\mathcal{D}_t(C_{n-1}, i-1) - \mathcal{D}_t(C_{n-2}, i-1)$. Each member of the above collection contains 1 and $n-1$ or $n-1$ and $n-2$ or 1 and 2. In particular, the total dominating sets contain 1 or $n-1$. So, we easily adjoin n to each of the member of $\mathcal{D}_t(C_{n-1}, i-1) - \mathcal{D}_t(C_{n-2}, i-1)$. Let $X \in \mathcal{D}_t(C_{n-1}, i-1) - \mathcal{D}_t(C_{n-2}, i-1)$. Let $X_1 = X \cup \{n\}$. Therefore, $X_1 \in \mathcal{D}_t(C_n, i)$.

Next, we consider $\mathcal{D}_t(C_{n-1}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1)$. The members of the intersection contain 1 or 2 or $n-2$ or $n-3$ or different combinations among themselves. In particular, 1 and $n-2$ play a very important role in the construction of new total dominating sets. Let Y belongs to the intersection. When $1 \in Y$, adjoin n with Y or when $n-2 \in Y$, adjoin $n-1$ with Y or when $n-2 \notin Y$ and $n-3, n-4 \in Y$, adjoin $n-2$ with Y or when $1 \notin Y$ and $n-2 \in Y$, adjoin 1 with Y or when $n-2 \notin Y$ and $1, 2, 3, n-3 \in Y$ then remove 1 from Y and adjoin n and $n-1$. Hence, in this collection we have the double the number of members of their intersection and the elements of the intersection give rise to double number of distinct elements in $\mathcal{D}_t(C_n, i)$. Therefore, in each case, the new element Y_1 (generated by Y) belongs to $\mathcal{D}_t(C_n, i)$.

Finally, we consider the set, $\mathcal{D}_t(C_{n-2}, i-1) - (\mathcal{D}_t(C_{n-1}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1))$. Let $Z \in \mathcal{D}_t(C_{n-2}, i-1) - (\mathcal{D}_t(C_{n-1}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1))$. When $1, n-2, n-3 \in Z$ and $2 \notin Z$, adjoin n with Z or when $1,2,n-2 \in Z$ and $n-3 \notin Z$ adjoin $n-1$ to Z or when $1,2 \notin Z$ and $n-4 \in Z$, remove $n-2$ from Z and adjoin n and $n-1$ or when $n-2, n-3 \notin Z$ and $3 \in Z$, remove 1 from Z and adjoin n and $n-1$ or when $n-3, n-4, n-5 \in Z$ and $1 \notin Z$ then remove $n-3$ and adjoin $n-1$ and $n-2$ or when $1, n-2 \in Z$ and $2, n-3 \notin Z$ then remove $n-2$ and adjoin n and $n-3$. Also, when $n-3, n-4, n-5 \in Z$ and $n-2 \notin Z$, remove $n-3$ and adjoin n and $n-1$ or when $n-6, n-5, 1 \in Z$ and $n-3, n-4 \notin Z$, remove $n-2$ and $n-3$ and adjoin $n-4, n-3, n$ or when $2, n-3 \in Z$ and $1, n-2 \notin Z$, remove $n-3$ and $n-4$ and adjoin $n-5, n-2, n-1$ or when $n-4, n-5, n-6 \in Z$ and $n-2, n-3 \notin Z$ remove $n-4$ and adjoin $n-2$ and $n-3$ or when $n-2, n-3, n-5 \in Z$ and $1, 2, n-4 \notin Z$ remove $n-2, n-3$ from Z and adjoin $n, n-1, n-4$. Hence the new total dominating set Z_1 (generated by Z) belongs to $\mathcal{D}_t(C_n, i)$. Therefore, we proved that $X_1, Y_1, Z_1 \in \mathcal{D}_t(C_n, i)$.

Conversely, Suppose that $K \in \mathcal{D}_t(C_n, i)$. The total dominating set K contains 1 or 2 or n or $n-1$. By the same argument as above, remove any one of the vertex from the above four vertices, we have, $K = M \cup \{x\}$, M is an element of $\mathcal{D}_t(C_{n-1}, i-1)$ or $\mathcal{D}_t(C_{n-2}, i-1)$ or both. Hence the statement.

Theorem 2.8: If $\mathcal{D}_t(C_n, i)$ is the family of the dominating sets of C_n with cardinality i , where $i > \lceil \frac{n}{2} \rceil + 1$, then,

$$d_t(C_n, i) = d_t(C_{n-1}, i-1) + d_t(C_{n-2}, i-1).$$

Proof: From theorem 2.6, we consider all the three cases as given below, where $i > \lceil \frac{n}{2} \rceil + 1$,

- (i) If $\mathcal{D}_t(C_{n-1}, i-1) = \mathcal{D}_t(C_{n-2}, i-1) = \phi$, then, $\mathcal{D}_t(C_n, i) = \phi$
- (ii) If $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-2}, i-1) = \phi$ then $\mathcal{D}_t(C_n, i) = \{\{n\} \cup X / X \in \mathcal{D}_t(C_{n-1}, i-1)\}$
- (iii) If $\mathcal{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_t(C_{n-2}, i-1) \neq \phi$, then $\mathcal{D}_t(C_n, i) =$

$$\left\{ \begin{aligned} & \{X \cup \{n\}\} \cup \\ & \{Y \cup \{n\} \text{ if } 1 \in Y\} \cup \\ & \{Y \cup \{n-1\} \text{ if } n-2 \in Y\} \cup \\ & \{Y \cup \{1\} \text{ if } n-2 \in Y \text{ and } 1 \notin Y\} \cup \\ & \{Y \cup \{n-2\} \text{ if } n-3, n-4 \in Y \text{ and } n-2 \notin Y\} \cup \\ & \{(Y - \{1\}) \cup \{n, n-1\} \text{ if } 1, 2, 3, n-3 \in Y \text{ and } n-2 \notin Y\} \cup \\ & \{(Z - \{n-2\}) \cup \{n, n-1\} \text{ if } 1, 2 \notin Z \text{ and } n-4 \in Z\} \cup \\ & \{Z \cup \{n\}\} \text{ if } 1, n-2, n-3 \in Z \text{ and } 2 \notin Z\} \cup \\ & \{Z \cup \{n-1\}\} \text{ if } 1, 2, n-2 \in Z \text{ and } n-3 \notin Z\} \cup \\ & \{(Z - \{1\}) \cup \{n-1, n\} \text{ if } n-3, n-2, \notin Z \text{ and } 3 \in Z\} \cup \\ & \{(Z - \{n-2\}) \cup \{n-3, n\} \text{ if } 1, n-2 \in Z \text{ and } 2, n-3 \notin Z\} \cup \\ & \{(Z - \{n-3\}) \cup \{n-2, n-1\} \text{ if } n-3, n-4, n-5 \in Z \text{ and } 1 \notin Z\} \cup \\ & \{(Z - \{n-3\}) \cup \{n, n-1\} \text{ if } n-3, n-4, n-5 \in Z \text{ and } n-2 \notin Z\} \cup \\ & \{(Z - \{n-2, n-3\}) \cup \{n-4, n-3, n\} \text{ if } 1, n-6, n-5 \in Z \text{ and } n-4, n-3 \notin Z\} \cup \\ & \{(Z - \{n-3, n-4\}) \cup \{n-5, n-2, n-1\} \text{ if } 2, n-3 \in Z \text{ and } 1, n-2 \notin Z\} \cup \\ & \{(Z - \{n-4\}) \cup \{n-3, n-2\} \text{ if } n-6, n-5, n-4 \in Z \text{ and } n-2, n-3 \notin Z\} \cup \\ & \{(Z - \{n-2, n-3\}) \cup \{n-4, n-1, n\} \text{ if } n-5, n-3, n-2 \in Z \text{ and } 1, 2, n-4 \notin Z\} \end{aligned} \right.$$

Where $X \in \mathcal{D}_t(C_{n-1}, i-1) - \mathcal{D}_t(C_{n-2}, i-1)$, $Y \in \mathcal{D}_t(C_{n-1}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1)$ and $Z \in \mathcal{D}_t(C_{n-1}, i-1) - (\mathcal{D}_t(C_{n-2}, i-1) \cap \mathcal{D}_t(C_{n-2}, i-1))$.

From the above construction in each case, we obtain that,

$$d_t(C_n, i) = d_t(C_{n-1}, i-1) + d_t(C_{n-2}, i-1).$$

Theorem 2.9: Let $C_n, n \geq 3$ be the cycle with $|V(C_n)| = n$. Then, the following properties hold:

- (i) For $n \geq 3, d_t(C_n, n) = 1$
- (ii) For $n \geq 3, d_t(C_n, n-1) = n$
- (iii) For $n \geq 5, d_t(C_n, n-2) = \frac{1}{2}n(n-3)$
- (iv) For $n \geq 7, d_t(C_n, n-3) = \frac{1}{6}[n(n^2 - 9n + 20)]$
- (v) For $k \geq 1, d_t(C_{4k}, 2k) = 4$
- (vi) For $k \geq 1, d_t(C_{2k+1}, k+1) = 2k+1$
- (vii) For $k \geq 1, d_t(C_{4k}, 2k+1) = 4k^2$
- (viii) For $k \geq 1, d_t(C_{4k+2}, 2k+2) = (2k+1)^2$

Proof:

- (i) For any graph G with n vertices, we have $d_t(G, n) = 1$. Hence, $d_t(C_n, n) = 1$.
- (ii) For any graph G with n vertices and $\delta(G) \geq 2$, then, we have $d_t(G, n-1) = n$. Hence $d_t(C_n, n-1) = n$.
- (iii) Proof by induction on n . First, suppose that $n = 5$, then $d_t(C_5, 3) = 5$. Now suppose that the result is true for all natural numbers less than n . From theorem 2.8,

$$\begin{aligned} \text{(iv)} \quad d_t(C_n, n-2) &= d_t(C_{n-1}, n-3) + d_t(C_{n-2}, n-3), \quad n \geq 6 \\ &= \frac{1}{2}(n-1)(n-4) + n-2 \\ &= \frac{1}{2}n(n-3). \end{aligned}$$

- (v) Proof by induction on n . First, suppose that $n = 7$, then $d_t(C_7, 7-3) = 7$. Now suppose that the result is true for all the natural numbers less than n . Therefore,

$$d_t(C_m, m-3) = m(m^2 - 9m + 20), \quad 7 \leq m \leq n-1.$$

From theorem 2.8, $d_t(C_n, n-3) = d_t(C_{n-1}, n-4) + d_t(C_{n-2}, n-4)$

$$\begin{aligned} &= \frac{1}{6}(n-1)((n-1)^2 - 9(n-1) + 20) + \frac{1}{2}(n-2)(n-5) \\ &= \frac{1}{6}(n-1)(n^2 - 11n + 30) + \frac{1}{2}(n^2 - 7n + 10) \\ &= \frac{1}{6}[n^3 - 9n^2 + 20n] \\ &= \frac{1}{6}n(n^2 - 9n + 20). \end{aligned}$$

Hence the result is true for all n .

(vi) $\mathcal{D}_t(C_{4k}, 2k)$ has the only four total dominating sets, such as, $\{1, 2, 5, 6, \dots, 4k-3, 4k-2\}$, $\{2,3,6,7, \dots, 4k-2, 4k-1\}$, $\{3,4,7,8, \dots, 4k-1, 4k\}$, $\{1,4,5,8, \dots, 4k-3, 4k\}$.

Hence $d_t(C_{4k}, 2k) = 4$.

(vii) Consider a cycle C_{2k+1} . Then it has $2k+1$ vertices, The total dominating sets of C_{2k+1} of cardinality $k+1$ are $\{1,2,5,6,9,10, \dots, 2k-3, 2k-2, 2k+1\}$, $\{2,3,6,7,10,11, \dots, 2k-2, 2k-1, 1\}$, $\{3,4,7,8,11,12, \dots, 2k-1, 2k, 2\}, \dots, \{2k+1, 1,4,5, \dots, 2k-4, 2k-3, 2k\}$. Therefore, we have $2k+1$ total dominating sets C_{2k+1} cardinality $k+1$. Hence $d_t(C_{2k+1}, k+1) = 2k+1$.

By observation, easily we can see (vii) and (viii).

3. TOTAL DOMINATION POLYNOMIAL OF CYCLE

Definition 3.1: Let $\mathcal{D}_t(C_n, i)$ be the family of total dominating sets of C_n with cardinality i , and let $d_t(C_n, i) = |\mathcal{D}_t(C_n, i)|$. Then the total dominating polynomial $D_t(C_n, x)$ of C_n is defined as

$$D_t(C_n, x) = \sum_{i=\gamma t(C_n)}^n d_t(C_n, i) x^i.$$

Particularly, the total domination polynomial of C_n is defined by

$$D_t(C_n, x) = \sum_{i=1+n/2}^n d_t(C_n, i) x^i, \text{ if } n \equiv 2 \pmod{4}.$$

$$D_t(C_n, x) = \sum_{i=\lceil n/2 \rceil}^n d_t(C_n, i) x^i, \text{ if } n \not\equiv 2 \pmod{4}.$$

Theorem 3.2: Let $C_n, n \geq 3$ be a cycle with $|V(C_n)| = n$. Then, for any $k \geq 1$

- (i) $D_t(C_{4k}, x) = 4x^{2k} + x [D_t(C_{4k-1}, x) + D_t(C_{4k-2}, x)]$
- (ii) $D_t(C_{4k+1}, x) = - 2x^{2k+1} + x [D_t(C_{4k}, x) + D_t(C_{4k-1}, x)]$
- (iii) $D_t(C_{4k+2}, x) = - 4x^{2k+1} + x [D_t(C_{4k+1}, x) + D_t(C_{4k}, x)]$
- (iv) $D_t(C_{4k+3}, x) = 2x^{2k+2} + x [D_t(C_{4k+2}, x) + D_t(C_{4k+1}, x)]$.

Proof: Proof of (i).

From theorem 2.7 and 2.8, $d_t(C_{4k}, 2k) = 4, k \geq 1$

$$d_t(C_{4k}, 2k+m) = d_t(C_{4k-1}, 2k+m-1) + d_t(C_{4k-2}, 2k+m-1), \quad 2 \leq m \leq 2k.$$

Summing all the equalities, we get,

$$\begin{aligned} D_t(C_{4k}, x) &= d_t(C_{4k}, 2k) x^{2k} + \sum_{m=1}^{2k} d_t(C_{4k}, 2k+m) x^{2k+m} \\ &= 4 x^{2k} + \sum_{m=1}^{2k} [d_t(C_{4k-1}, 2k+m-1) + d_t(C_{4k-2}, 2k+m-1)] x^{2k+m} \\ &= 4 x^{2k} + \sum_{m=1}^{2k} d_t(C_{4k-1}, 2k+m-1) x^{2k+m} + \sum_{m=2}^{2k} d_t(C_{4k-2}, 2k+m-1) x^{2k+m} \\ &= 4 x^{2k} + x \sum_{m=1}^{2k} d_t(C_{4k-1}, 2k+m-1) x^{2k+m-1} + x \sum_{m=1}^{2k} d_t(C_{4k-2}, 2k+m-1) x^{2k+m-1} \\ &= 4 x^{2k} + x [D_t(C_{4k-1}, x) + D_t(C_{4k-2}, x)]. \end{aligned}$$

Hence, $D_t(C_{4k}, x) = 4 x^{2k} + x [D_t(C_{4k-1}, x) + D_t(C_{4k-2}, x)]$

Proof of (ii).

From theorem 2.7 and 2.8, we have $D_t(C_{4k+1}, 2k+1) = 4k+1$ and

$$d_t(C_{4k+1}, 2k+m) = d_t(C_{4k}, 2k+m-1) + d_t(C_{4k-1}, 2k+m-1), \quad 2 \leq m \leq 2k+1.$$

$$\begin{aligned} \text{Now, } D_t(C_{4k+1}, x) &= d_t(C_{4k+1}, 2k+1) x^{2k+1} + \sum_{m=2}^{2k+1} d_t(C_{4k+1}, 2k+m) x^{2k+m} \\ &= d_t(C_{4k+1}, 2k+1) x^{2k+1} + \sum_{m=2}^{2k+1} [d_t(C_{4k}, 2k+m-1) + d_t(C_{4k-1}, 2k+m-1)] x^{2k+m} \\ &= d_t(C_{4k+1}, 2k+1) x^{2k+1} + \sum_{m=2}^{2k+1} d_t(C_{4k}, 2k+m-1) x^{2k+m} + \sum_{m=2}^{2k+1} d_t(C_{4k-1}, 2k+m-1) x^{2k+m} \\ &= (4k+1) x^{2k+1} + x [\sum_{m=1}^{2k+1} d_t(C_{4k}, 2k+m-1) x^{2k+m-1} - d_t(C_{4k}, 2k) x^{2k}] + x [\sum_{m=1}^{2k+1} d_t(C_{4k-1}, 2k+m-1) x^{2k+m-1} - d_t(C_{4k-1}, 2k) x^{2k}] \\ &= (4k+1) x^{2k+1} + x [D_t(C_{4k}, x) - 4 x^{2k}] + x [D_t(C_{4k-1}, x) - (4k-1) x^{2k}] \\ &= [(4k+1) - 4 - (4k-1)] x^{2k+1} + x D_t(C_{4k}, x) + x D_t(C_{4k-1}, x) \end{aligned}$$

$$D_t(C_{4k+1}, x) = - 2x^{2k+1} + x [D_t(C_{4k}, x) + D_t(C_{4k-1}, x)].$$

Proof of (iii):

From theorem 2.7 and 2.8, $d_t(C_{4k+2}, 2k+2) = (2k+1)^2$, $1 \leq k$ and

$$d_t(C_{4k+2}, 2k+m) = d_t(C_{4k+1}, 2k+m-1) + d_t(C_{4k}, 2k+m-1), \quad 3 \leq m \leq 2k+2$$

$$\begin{aligned} \text{Now } D_t(C_{4k+2}, x) &= d_t(C_{4k+2}, 2k+2) x^{2k+2} + \sum_{m=3}^{2k+2} d_t(C_{4k+2}, 2k+m) x^{2k+m} \\ &= d_t(C_{4k+2}, 2k+2) x^{2k+2} + \sum_{m=3}^{2k+2} [d_t(C_{4k+1}, 2k+m-1) + d_t(C_{4k}, 2k+m-1)] x^{2k+m} \\ &= (2k+1)^2 x^{2k+2} + \sum_{m=3}^{2k+2} d_t(C_{4k+1}, 2k+m-1) x^{2k+m} + \sum_{m=3}^{2k+2} d_t(C_{4k}, 2k+m-1) x^{2k+m} \\ &= (2k+1)^2 x^{2k+2} + x [\sum_{m=2}^{2k+2} d_t(C_{4k+1}, 2k+m-1) x^{2k+m-1} - d_t(C_{4k+1}, 2k+1) x^{2k+1}] \\ &\quad + x [\sum_{m=1}^{2k+2} d_t(C_{4k}, 2k+m-1) x^{2k+m-1} - d_t(C_{4k}, 2k) x^{2k} - d_t(C_{4k}, 2k+1) x^{2k+1}] \\ &= (2k+1)^2 x^{2k+2} - (4k+1) x^{2k+2} - 4 x^{2k+1} - 4k^2 x^{2k+2} + x D_t(C_{4k+1}, x) + x D_t(C_{4k}, x) \\ &= -4x^{2k+1} + (4k^2 + 4k + 1 - 4k - 1 - 4k^2) x^{2k+2} + x [D_t(C_{4k+1}, x) + D_t(C_{4k}, x)] \\ D_t(C_{4k+2}, x) &= -4x^{2k+1} + x [D_t(C_{4k+1}, x) + D_t(C_{4k}, x)]. \end{aligned}$$

Proof of (iv)

From theorem 2.7 and 2.8, $d_t(C_{4k+3}, 2k+2) = 4k+3$ and

$$d_t(C_{4k+3}, 2k+m) = d_t(C_{4k+2}, 2k+m-1) + d_t(C_{4k+1}, 2k+m-1), \quad 3 \leq m \leq 2k+3$$

$$\begin{aligned} \text{Now } D_t(C_{4k+3}, x) &= d_t(C_{4k+3}, 2k+2) x^{2k+2} + \sum_{m=3}^{2k+3} d_t(C_{4k+3}, 2k+m) x^{2k+m} \\ &= d_t(C_{4k+3}, 2k+2) x^{2k+2} + \sum_{m=3}^{2k+3} [d_t(C_{4k+2}, 2k+m-1) + d_t(C_{4k+1}, 2k+m-1)] x^{2k+m} \\ &= d_t(C_{4k+3}, 2k+2) x^{2k+2} + \sum_{m=3}^{2k+3} d_t(C_{4k+2}, 2k+m-1) x^{2k+m} + \sum_{m=3}^{2k+3} d_t(C_{4k+1}, 2k+m-1) x^{2k+m} \\ &= (4k+3) x^{2k+2} + x [\sum_{m=2}^{2k+3} d_t(C_{4k+2}, 2k+m-1) x^{2k+m-1}] \\ &\quad + x [\sum_{m=1}^{2k+3} d_t(C_{4k+1}, 2k+m-1) x^{2k+m-1} - d_t(C_{4k+1}, 2k+1) x^{2k+1}] \\ &= (4k+3) x^{2k+2} + x [\sum_{m=2}^{2k+3} d_t(C_{4k+2}, 2k+m-1) x^{2k+m-1}] \\ &\quad + x [\sum_{m=1}^{2k+3} d_t(C_{4k+1}, 2k+m-1) x^{2k+m-1} - (4k+1) x^{2k+1}] \\ &= [(4k+3) - (4k+1)] x^{2k+2} + x D_t(C_{4k+2}, x) + x D_t(C_{4k+1}, x) \end{aligned}$$

$$D_t(C_{4k+3}, x) = 2x^{2k+2} + x [D_t(C_{4k+2}, x) + D_t(C_{4k+1}, x)].$$

Using all the above theorems and lemmas, we obtain the coefficients of $D_t(C_k, i)$ for $2 \leq n \leq 18$ in Table 1.

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
C_3	3	1															
C_4	4	4	1														
C_5	0	5	5	1													
C_6	0	0	9	6	1												
C_7	0	0	7	14	7	1											
C_8	0	0	4	16	20	8	1										
C_9	0	0	0	9	30	27	9	1									
C_{10}	0	0	0	0	25	50	35	10	1								
C_{11}	0	0	0	0	11	55	77	44	11	1							
C_{12}	0	0	0	0	4	36	105	112	54	12	1						
C_{13}	0	0	0	0	0	13	91	182	156	65	13	1					
C_{14}	0	0	0	0	0	0	49	196	294	210	77	14	1				
C_{15}	0	0	0	0	0	0	15	140	378	450	275	90	15	1			
C_{16}	0	0	0	0	0	0	4	64	336	672	660	352	104	16	1		
C_{17}	0	0	0	0	0	0	0	17	204	714	1122	935	442	119	17	1	
C_{18}	0	0	0	0	0	0	0	0	81	540	1386	1782	1287	546	135	18	1

Table 1: $d_t(C_n, i)$, the number of total dominating sets of C_n with cardinality i .

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