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ON TOTAL DOMINATION SETS AND POLYNOMIALS OF CYCLES

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ABSTRACT

Let G = (V, E) be a graph without isolated vertices. A set $S \subseteq V$ is a total dominating set of G, if every vertex $u \in V$ is adjacent to an element of S. Let $\mathcal{D}_t(C_n, i)$ be the family of total dominating sets of a cycle C_n with cardinality i. Let $d_t(C_n, i)$ be the number of total dominating sets in $\mathcal{D}_t(C_n, i)$. In this paper, we study the concept of total domination polynomial for any cycle C_n. The total domination polynomial for any cycle C_n is the polynomial $D_t(C_n, x) = \sum_{i=1+n/2}^{n} d_t(C_n, i)\, x^i$, if $n \equiv 2(mod4)$ and $D_t(C_n, x) = \sum_{i=\lceil n/2\rceil}^{n} d_t(C_n, i)\, x^i$ if $n \not\equiv 2(mod4)$. We obtain some properties of $D_t(C_n, x)$ and its coefficients. Also, we calculate the reduction formula to derive the total domination polynomial of cycles.

Keywords: *cycles, total dominating set, total domination number, total domination polynomial.*

Mathematics subject classification: *05C69, 11B83.*

1. INTRODUCTION

Let G = (V, E) be a graph. For any vertex $u \in V$, we define the open neighborhood of u as the set N(u) defined by N(u) = { v / uv ∈ E } and the closed neighborhood of u as the set N[u] defined by N[u] = N(u) ∪ {u}. For a subset S of V, the open neighborhood of S is N(S) which is defined as the union of N(u) for all u ∈ S and the closed neighborhood of S is defined as N(S) ∪ S. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A set S of vertices in a graph G is said to be a dominating set if every vertex u ∈ V is either an element of S or is adjacent to an element of S. A set of vertices in a graph G is said to be a total dominating set if every vertex u ∈ V is adjacent to an element of S. The domination number of a graph, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of the total dominating sets in G.

We use the notation $\lceil x\rceil$ for the smallest integer grater than or equal to x. Also, we denote the set {1, 2, 3,….n} by [x], throughout this paper.

2. TOTAL DOMINATING SETS OF CYCLES

In this section, we are going to investigate the total domination sets of cycles and some of its properties.

Definition 2.1: Let G be a graph of order n with no isolated vertices. Let $\mathfrak{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathfrak{D}_t(G, i)|$. Then the total domination polynomial $D_t(G, x)$ of G is defined as $D_t(G, x) = \sum_{i=\gamma t(G)]}^{n} d_t(G, i)\, x^i$,where $\gamma_t(G)$ is the total domination number of G.

Let C_n, $n \geq 3$ be the cycle with n vertices. Let $V(C_n)$ = { 1,2,…..n} and $E(C_n)$ = { (1,2), (2,3) …….. ……..(n-1,n), (n, 1)}. Let $\mathfrak{D}_t(C_n, i)$ be the collection of total domination sets in C_n with cardinality i. We shall investigate the total domination sets of cycles.

Lemma 2.2: [4] For $n \geq 3$, the total domination number of the cycle, C_n is given by

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$$\gamma_t(C_n) = \begin{cases} 1+\frac{n}{2} & if\ n \equiv 2(mod4) \\ \lceil \frac{n}{2} \rceil & if\ n \not\equiv 2(mod4) \end{cases}$$

Lemma 2.3: Let C_n, $n \geq 3$ be the cycle with $|V(C_n)| = n$. Then $d_t(C_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n$ and $d_t(C_n, i) > 0$ if $\lceil \frac{n}{2} \rceil < i \leq n$.

Proof: If $n \equiv 2(mod4)$, then the total domination number of the cycle C_n is $\gamma_t(C_n) = 1+\frac{n}{2}$. Therefore, $d_t(C_n, i) = 0$, when $i < 1+\frac{n}{2}$ or $i > n$. And $d_t(C_n, i) > 0$, when $1+\frac{n}{2} \leq i \leq n$. When $n \not\equiv 2(mod4)$, then the total domination number of C_n is $\gamma_t(C_n) = \lceil \frac{n}{2} \rceil$. Therefore, $d_t(C_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n$. Also, $d_t(C_n, i) > 0$ when $\lceil \frac{n}{2} \rceil \leq i \leq n$. Hence, in general, we have $d_t(C_n, i) = 0$ when $i < \lceil \frac{n}{2} \rceil$ or $i > n$ and $d_t(C_n, i) > 0$ when $\lceil \frac{n}{2} \rceil < i \leq n$.

Lemma 2.4: Let C_n, $n \geq 3$ be the cycle with $|V(C_n)| = n$. Then
(i) $\mathfrak{D}_t(C_n, i) = \phi$ if $i < \gamma_t(C_n)$ or $i > n$.
(ii) $D_t(C_n, x)$ has no constant term and first degree terms.
(iii) $D_t(C_n, x)$ is a strictly increasing function on $[0, \infty)$.

Proof is obvious.

Lemma 2.5: Let C_n, $n \geq 3$ be the cycle with $|V(C_n)| = n$.
(i) If $\mathfrak{D}_t(C_{n-1}, i-1) = \mathfrak{D}_t(C_{n-3}, i-1) = \phi$, then, $\mathfrak{D}_t(C_{n-2}, i-1) = \phi$.
(ii) If $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-3}, i-1) \neq \phi$ then $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$.
(iii) If $\mathfrak{D}_t(C_{n-1}, i-1) = \mathfrak{D}_t(C_{n-3}, i-1) = \mathfrak{D}_t(C_{n-2}, i-1) = \phi$ then $\mathfrak{D}_t(C_n, i) = \phi$.

The proof of the lemma follows from lemma 2.3.

Lemma 2.6: Let C_n, $n \geq 3$ be the cycle with $|V(C_n)| = n$. Suppose that $\mathfrak{D}_t(C_n,i) \neq \phi$, then we have
(i) $\mathfrak{D}_t(C_{n-2}, i-1) = \phi$ and $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$ if and only if $i = n$.
(ii) $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$, $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-3}, i-1) = \phi$ if and only if $i = n-1$.
(iii) $\mathfrak{D}_t(C_{n-1}, i-1) = \phi$ and $\mathfrak{D}_t(C_{n-2}, i-1) = \phi$ if and only if $n = 4k$, and $i = 2k$ for some positive integer k.
(iv) $\mathfrak{D}_t(C_{n-1}, i-1) = \phi$, $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-3}, i-1) \neq \phi$ if and only if $n = 4k-1$, and $i = 2k$ for some k.
(v) $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$, $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-3}, i-1) \neq \phi$ if and only if $\lceil \frac{n-1}{2} \rceil + 1 < i \leq n-2$.

Proof: The proof of the lemma is similar to the proof of lemma [2.4] in [7]

Theorem 2.7 For every $n \geq 5$ and $i > \lceil \frac{n}{2} \rceil +1$,
(i) $\mathfrak{D}_t(C_{4k}, 2k) = \{\{1, 2, 5, 6, \ldots, 4k-3, 4k-2\}, \{2,3,6,7, \ldots, 4k-2, 4k-1\}, \{3,4,7,8, \ldots, 4k-1, 4k\}, \{1,4,5,8, \ldots, 4k-3, 4k\}\}$, where $k \geq 1$.
(ii) If $\mathfrak{D}_t(C_{n-2}, i-1) = \phi$ and $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$ then $\mathfrak{D}_t(C_n, i) = \mathfrak{D}_t(C_n, n) = \{\{1,2,3, \ldots n\}\}$.
(iii) If $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$, $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-3}, i-1) = \phi$ then $\mathfrak{D}_t(C_n, i) = \mathfrak{D}_t(C_n, n-1) = \{[n] - \{x\} / x \in [n]\}$.
(iv) If $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$ then $\mathfrak{D}_t(C_n, i) =$

$$\begin{cases} \{X \cup \{n\}\} \cup \\ \{Y \cup \{n\}\ \ if\ 1 \in Y\} \cup \\ \{Y \cup \{n-1\}\ if\ n-2 \in Y\} \cup \\ \{Y \cup \{1\}\ if\ n-2 \in Y\ and\ 1 \notin Y\} \cup \\ \{Y \cup \{n-2\}\ if\ n-3, n-4 \in Y\ and\ n-2 \notin Y\} \cup \\ \{(Y - \{1\}) \cup \{n, n-1\}\ if\ 1,2,3, n-3 \in Y\ and\ n-2 \notin Y\} \cup \\ \{(Z - \{n-2\}) \cup \{n, n-1\}\ \ if\ 1,2 \notin Z and\ n-4 \in Z\} \cup \\ \{Z \cup \{n\}\}\ if\ 1, n-2, n-3 \in Z\ and\ 2 \notin Z\} \cup \\ \{Z \cup \{n-1\}\}\ if\ 1,2, n-2 \in Z\ and\ n-3 \notin Z\} \cup \\ \{(Z - \{1\}) \cup \{n-1, n\}\ \ if\ n-3, n-2, \notin Z\ and\ 3 \in Z\} \cup \\ \{\{Z - \{n-2\}\} \cup \{n-3, n\}\ if\ 1, n-2 \in Z\ and\ 2, n-3 \notin Z\} \cup \\ \{(Z - \{n-3\}) \cup \{n-2, n-1\}\ if\ n-3, n-4, n-5 \in Z\ and\ 1 \notin Z\} \cup \\ \{(Z - \{n-3\}) \cup \{n, n-1\}\ if\ n-3, n-4, n-5 \in Z\ and\ n-2 \notin Z\} \cup \\ \{(Z - \{n-2, n-3\}) \cup \{n-4, n-3, n\}\ if\ 1, n-6, n-5 \in Z\ and\ n-4, n-3 \notin Z\} \cup \\ \{(Z - \{n-3, n-4\}) \cup \{n-5, n-2, n-1\}\ if\ 2, n-3 \in Z\ and\ 1, n-2 \notin Z\} \cup \\ \{(Z - \{n-4\}) \cup \{n-3, n-2\}\ if\ n-6, n-5, n-4 \in Z\ and\ n-2, n-3 \notin Z\} \cup \\ \{(Z - \{n-2, n-3\}) \cup \{n-4, n-1, n\}\ if\ n-5, n-3, n-2 \in Z\ and\ 1,2, n-4 \notin Z\} \end{cases}$$

where $X \in \mathfrak{D}_t(C_{n-1}, i-1) - \mathfrak{D}_t(C_{n-2}, i-1)$, $Y \in \mathfrak{D}_t(C_{n-1}, i-1) \cap \mathfrak{D}_t(C_{n-2}, i-1)$ and $Z \in \mathfrak{D}_t(C_{n-1}, i-1) - (\mathfrak{D}_t(C_{n-2}, i-1) \cap \mathfrak{D}_t(C_{n-2}, i-1))$.

Proof:

(i) For any $k \geq 1$, split the vertices of C_{4k} in to k number of sets of the form {1,2,3,4 }, {5,6,7,8 }………… and { 4k-3, 4k-2, 4k-1, 4k }. The four total dominating sets of cardinality 2k are constructed by choosing the first two numbers in each set or second and third or third and fourth or first and fourth from each set. Hence, $\mathfrak{D}_t(C_{4k}, 2k)$ has the only four total dominating sets, such as, {1, 2, 5, 6, ……,4k-3, 4k-2}, {2,3,6,7, ……… ,4k-2,4k-1}, {3,4,7,8,…… ,4k-1,4k}, {1,4,5,8, ……….., 4k-3,4k}.

(ii) Since $\mathfrak{D}_t(C_{n-2}, i-1) = \phi$ and $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$, by lemma 2.6, i = n. Therefore, $\mathfrak{D}_t(C_n, i) = \mathfrak{D}_t(C_n, n) = \{[n]\}$.

(iii) If $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$, $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-3}, i-1) = \phi$, then by lemma 2.6, i = n-1 then $\mathfrak{D}_t(C_n, i) = \mathfrak{D}_t(C_n, n-1) = \{ [n]- \{x\}/x \in [n] \}$.

(iv) First, we consider the collection of total domination sets $\mathfrak{D}_t(C_{n-1}, i-1) - \mathfrak{D}_t(C_{n-2}, i-1)$. Each member of the above collection contains 1 and n-1 or n-1 and n-2 or 1 and 2. In particular, the total dominating sets contain 1 or n-1. So, we easily adjoin n to each of the member of $\mathfrak{D}_t(C_{n-1}, i-1) -\ _t(C_{n-2}, i-1)$. Let $X \in \mathfrak{D}_t(C_{n-1}, i-1) - \mathfrak{D}_t(C_{n-2}, i-1)$. Let $X_1 = X \cup\{n\}$. Therefore, $X_1 \in \mathfrak{D}_t(C_n, i)$.

Next, we consider $\mathfrak{D}_t(C_{n-1}, i-1) \cap \mathfrak{D}_t(C_{n-2}, i-1)$. The members of the intersection contain 1 or 2 or n-2 or n-3 or different combinations among themselves. In particular, 1 and n-2 play a very important role in the construction of new total dominating sets. Let Y belongs to the intersection. When $1 \in Y$, adjoin n with Y or when $n-2 \in Y$, adjoin n-1 with Y or when $n-2 \notin Y$ and $n-3, n-4 \in Y$, adjoin n-2 with Y or when $1 \notin Y$ and $n-2 \in Y$, adjoin 1 with Y or when $n-2 \notin Y$ and $1, 2, 3, n-3 \in Y$ then remove 1 from Y and adjoin n and n-1. Hence, in this collection we have the double the number of members of their intersection and the elements of the intersection give rise to double number of distinct elements in $\mathfrak{D}_t(C_n, i)$. Therefore, in each case, the new element Y_1 (generated by Y) belongs to $\mathfrak{D}_t(C_n, i)$.

Finally, we consider the set, $\mathfrak{D}_t(C_{n-2}, i-1) - (\mathfrak{D}_t(C_{n-1}, i-1) \cap \mathfrak{D}_t(C_{n-2}, i-1))$. Let $Z \in \mathfrak{D}_t(C_{n-2}, i-1) - (\mathfrak{D}_t(C_{n-1}, i-1) \cap \mathfrak{D}_t(C_{n-2}, i-1)$. When $1, n-2, n-3 \in Z$ and $2 \notin Z$, adjoin n with Z or when $1,2,n-2 \in Z$ and $n-3 \notin Z$ adjoin n-1 to Z or when $1,2 \notin Z$ and $n-4 \in Z$, remove n-2 from Z and adjoin n and n-1 or when $n-2, n-3 \notin Z$ and $3 \in Z$, remove 1 from Z and adjoin n and n-1 or when $n-3, n-4, n-5 \in Z$ and $1 \notin Z$ then remove n-3 and adjoin n-1 and n-2 or when $1, n-2 \in Z$ and $2, n-3 \notin Z$ then remove n-2 and adjoin n and n-3. Also, when $n-3, n-4, n-5 \in Z$ and $n-2 \notin Z$, remove n-3 and adjoin n and n-1 or when $n-6, n-5, 1 \in Z$ and $n-3, n-4 \notin Z$, remove n-2 and n-3 and adjoin n-4, n-3, n or when $2, n-3 \in Z$ and $1, n-2 \notin Z$,remove n-3 and n-4 and adjoin n-5, n-2, n-1 or when $n-4, n-5, n-6 \in Z$ and $n-2, n-3 \notin Z$ remove n-4 and adjoin n-2 and n-3 or when $n-2, n-3, n-5 \in Z$ and $1, 2, n-4 \notin Z$ remove n-2, n-3 from Z and adjoin n, n-1, n-4. Hence the new total dominating set Z_1 (generated by Z) belongs to $\mathfrak{D}_t(C_n, i)$. Therefore, we proved that $X_1, Y_1, Z_1 \in \mathfrak{D}_t(C_n, i)$.

Conversely, Suppose that $K \in \mathfrak{D}_t(C_n, i)$. The total dominating set K contains 1 or 2 or n or n-1. By the same argument as above, remove any one of the vertex from the above four vertices, we have, $K = M \cup \{x\}$, M is an element of $\mathfrak{D}_t(C_{n-1}, i-1)$ or $\mathfrak{D}_t(C_{n-2}, i-1)$ or both. Hence the statement.

Theorem 2.8: If $\mathfrak{D}_t(C_n, i)$ is the family of the dominating sets of C_n with cardinality i, where $i > \lceil \frac{n}{2} \rceil + 1$, then,

$$d_t(C_n, i) = d_t(C_{n-1}, i-1) + d_t(C_{n-2}, i-1).$$

Proof: From theorem 2.6, we consider all the three cases as given below, where $i > \lceil \frac{n}{2} \rceil + 1$,

(i) If $\mathfrak{D}_t(C_{n-1}, i-1) = \mathfrak{D}_t(C_{n-2}, i-1) = \phi$, then, $\mathfrak{D}_t(C_n, i) = \phi$

(ii) If $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-2}, i-1) = \phi$ then $\mathfrak{D}_t(C_n, i) = \{\{n\} \cup X / X \in \mathfrak{D}_t(C_{n-1}, i-1)\}$

(iii) If $\mathfrak{D}_t(C_{n-1}, i-1) \neq \phi$ and $\mathfrak{D}_t(C_{n-2}, i-1) \neq \phi$, then $\mathfrak{D}_t(C_n, i) =$

$$
\left\{
\begin{array}{c}
\{X \cup \{n\}\} \cup \\
\{Y \cup \{n\} \;\; if\ 1 \in Y\} \cup \\
\{Y \cup \{n-1\}\ if\ n-2 \in Y\} \cup \\
\{Y \cup \{1\}\ if\ n-2 \in Y\ and\ 1 \notin Y\} \cup \\
\{Y \cup \{n-2\}\ if\ n-3, n-4 \in Y\ and\ n-2 \notin Y\} \cup \\
\{(Y - \{1\}) \cup \{n, n-1\}\ if\ 1,2,3, n-3 \in Y\ and\ n-2 \notin Y\} \cup \\
\{(Z - \{n-2\}) \cup \{n, n-1\} \;\; if\ 1,2 \notin Z and\ n-4 \in Z\} \cup \\
\{Z \cup \{n\}\} \;\; if\ 1, n-2, n-3 \in Z\ and\ 2 \notin Z\} \cup \\
\{Z \cup \{n-1\}\} \;\; if\ 1,2, n-2 \in Z\ and\ n-3 \notin Z\} \cup \\
\{(Z - \{1\}) \cup \{n-1, n\} \;\; if\ n-3, n-2, \notin Z\ and\ 3 \in Z\} \cup \\
\left\{\{Z - \{n-2\}\} \cup \{n-3, n\}\ if\ 1, n-2 \in Z\ and\ 2, n-3 \;\notin Z\right\} \cup \\
\{(Z - \{n-3\}) \cup \{n-2, n-1\}\ if\ n-3, n-4, n-5 \in Z\ and\ 1 \notin Z\} \cup \\
\{(Z - \{n-3\}) \cup \{n, n-1\}\ if\ n-3, n-4, n-5 \in Z\ and\ n-2 \notin Z\} \cup \\
\{(Z - \{n-2, n-3\}) \cup \{n-4, n-3, n\}\ if\ 1, n-6, n-5 \in Z\ and\ n-4, n-3 \notin Z\} \cup \\
\{(Z - \{n-3, n-4\}) \cup \{n-5, n-2, n-1\}\ if\ 2, n-3 \in Z\ and\ 1, n-2 \notin Z\} \cup \\
\{(Z - \{n-4\}) \cup \{n-3, n-2\}\ if\ n-6, n-5, n-4 \in Z\ and\ n-2, n-3 \notin Z\} \cup \\
\{(Z - \{n-2, n-3\}) \cup \{n-4, n-1, n\}\ if\ n-5, n-3, n-2 \in Z\ and\ 1,2, n-4 \notin Z\}
\end{array}
\right.
$$

Where $X \in \mathfrak{D}_t$ (C_{n-1}, i-1) — $\mathfrak{D}_t$ (C_{n-2}, i-1), $Y \in \mathfrak{D}_t$ (C_{n-1}, i-1) $\cap$ $\mathfrak{D}_t$ (C_{n-2}, i-1) and $Z \in \mathfrak{D}_t$ (C_{n-1}, i-1) — ($\mathfrak{D}_t$ (C_{n-2}, i-1) $\cap$ $\mathfrak{D}_t$ (C_{n-2}, i-1)).

From the above construction in each case, we obtain that,

$d_t(C_n, i) = d_t(C_{n-1}, i-1) + d_t(C_{n-2}, i-1)$.

Theorem 2.9: Let C_n, $n \geq 3$ be the cycle with $|V(C_n)| = n$. Then, the following properties hold:

(i) For $n \geq 3$, $d_t(C_n, n) = 1$

(ii) For $n \geq 3$, $d_t(C_n, n-1) = n$

(iii) For $n \geq 5$, $d_t(C_n, n-2) = \frac{1}{2} n(n-3)$

(iv) For $n \geq 7$, $d_t(C_n, n-3) = \frac{1}{6}[n(n^2 - 9n+20)]$

(v) For $k \geq 1$, $d_t(C_{4k}, 2k) = 4$

(vi) For $k \geq 1$, $d_t(C_{2k+1}, k+1) = 2k+1$

(vii) For $k \geq 1$, $d_t(C_{4k}, 2k+1) = 4k^2$

(viii) For $k \geq 1$, $d_t(C_{4k+2}, 2k+2) = (2k+1)^2$

Proof:

(i) For any graph G with n vertices, we have $d_t(G, n) = 1$. Hence, $d_t(C_n, n) = 1$.

(ii) For any graph G with n vertices and $\delta(G) \geq 2$, then, we have $d_t(G, n-1) = n$. Hence $d_t(C_n, n-1) = n$.

(iii) Proof by induction on n. First, suppose that n = 5, then $d_t(C_5, 3) = 5$. Now suppose that the result is true for all natural numbers less than n. From theorem 2.8,

(iv) $d_t(C_n, n-2) = d_t(C_{n-1}, n-3) + d_t(C_{n-2}, n-3)$, $n \geq 6$

$$= \frac{1}{2}(n-1)(n-4) + n-2$$

$$= \frac{1}{2} n(n-3).$$

(v) Proof by induction on n. First, suppose that n = 7, then $d_t(C_7, 7-3) = 7$. Now suppose that the result is true for all the natural numbers less than n. Therefore,

$d_t(C_m, m-3) = m(m^2 - 9m+20)$, $7 \leq m \leq n-1$.

From theorem 2.8, $d_t(C_n, n-3) = d_t(C_{n-1}, n-4) + d_t(C_{n-2}, n-4)$

$$= \frac{1}{6}(n-1)((n-1)^2 - 9(n-1) + 20) + ½ (n-2)(n-5)$$

$$= \frac{1}{6}(n-1)(n^2 - 11n+30) + \frac{1}{2}(n^2 - 7n+10)$$

$$= \frac{1}{6}[n^3 - 9n^2 + 20n]$$

$$= \frac{1}{6} n(n^2 - 9n+20).$$

Hence the result is true for all n.

(vi) $\mathfrak{D}_t$ (C_{4k}, 2k) has the only four total dominating sets, such as, {1, 2, 5, 6, ……,4k-3, 4k-2}, {2,3,6,7, …..,4k-2,4k-1}, {3,4,7,8,…… ,4k-1,4k}, {1,4,5,8, ……….., 4k-3,4k}.

Hence d_t (C_{4k}, 2k) = 4.

(vii) Consider a cycle C_{2k+1}. Then it has 2k+1 vertices, The total dominating sets of C_{2k+1} of cardinality k+1 are {1,2,5,6,9,10,……… 2k-3, 2k-2, 2k+1}, {2,3,6,7,10,11,…….. 2k-2, 2k-1,1} ,{3,4,7,8,11,12,…….2k-1, 2k, 2},….. ………{2k+1, 1,4,5,…….2k-4, 2k-3, 2k}. Therefore, we have 2k+1 total dominating sets C_{2k+1} cardinality k+1. Hence d_t (C_{2k+1}. k+1) = 2k+1.

By observation, easily we can see (vii) and (viii).

3. TOTAL DOMINATION POLYNOMIAL OF CYCLE

Definition 3.1: Let $\mathfrak{D}_t$ (C_n, i) be the family of total dominating sets of C_n with cardinality i, and let d_t (C_n, i) = |$\mathfrak{D}_t$ (C_n, i)|. Then the total dominating polynomial D_t (C_n, x) of C_n is defined as

$D_t (C_n, x) = \sum_{i=\gamma t(Cn)}^{n} d_t (C_n, i)\, x^i$.

Particularly, the total domination polynomial of C_n is defined by
$D_t (C_n, x) = \sum_{i=1+n/2}^{n} d_t (C_n, i)\, x^i$, if n ≡ 2(mod4).
$D_t (C_n, x) = \sum_{i=\lceil n/2\rceil}^{n} d_t (C_n, i)\, x^i$, if n ≢ 2(mod4).

Theorem 3.2: Let C_n, n ≥ 3 be a cycle with |V (C_n)| = n. Then, for any k ≥ 1
(i) $D_t (C_{4k}, x) = 4x^{2k} + x [D_t (C_{4k-1} , x) + D_t (C_{4k-2} , x)]$
(ii) $D_t (C_{4k+1} , x) = - 2x^{2k+1} + x [D_t (C_{4k}, x) + D_t (C_{4k-1} , x)]$
(iii) $D_t (C_{4k+2}, x) = - 4x^{2k+1} + x [D_t (C_{4k+1} , x) + D_t (C_{4k} , x)]$
(iv) $D_t (C_{4k+3}, x) = 2x^{2k+2} + x [D_t (C_{4k+2} , x) + D_t (C_{4k+1} , x)]$.

Proof: Proof of (i).
From theorem 2.7 and 2.8, d_t (C_{4k}, 2k) = 4, k ≥ 1

$d_t (C_{4k}, 2k+m) = d_t (C_{4k-1} , 2k+m-1) + d_t (C_{4k-2}, 2k+m-1)$, 2 ≤ m ≤ 2k.

Summing all the equalities, we get,

$D_t (C_{4k}, x) = d_t(C_{4k} , 2k)\, x^{2k} + \sum_{m=1}^{2k} d_t (C_{4k}, 2k+m)\, x^{2k+m}$.
$= 4 x^{2k} + \sum_{m=1}^{2k} [d_t (C_{4k-1} , 2k+m-1) + d_t (C_{4k-2} , 2k+m-1)]\, x^{2k+m}$
$= 4 x^{2k} + \sum_{m=1}^{2k} d_t (C_{4k-1}, 2k+m-1)\, x^{2k+m} + \sum_{m=2}^{2n} d_t (C_{4k-2}, 2k+m-1)\, x^{2k+m}$.
$= 4 x^{2k} + x \sum_{m=1}^{2k} d_t (C_{4k-1}, 2k+m-1)\, x^{2k+m-1} + x \sum_{m=1}^{2k} d_t (C_{4k-2}, 2k+m-1)\, x^{2k+m-1}$
$= 4 x^{2k} + x [D_t(C_{4k-1}, x) + D_t (C_{4k-2}, x)]$.

Hence, $D_t (C_{4k}, x) = 4 x^{2k} + x [D_t (C_{4k-1}, x) + D_t (C_{4k-2}, x)]$

Proof of (ii).

From theorem 2.7 and 2.8, we have $D_t(C_{4k+1}, 2k+1) = 4k +1$ and

$d_t(C_{4k+1}, 2k+m) = d_t (C_{4k}, 2k+m-1) + d_t(C_{4k-1}, 2k+m-1)$, 2 ≤ m ≤ 2k+1.

Now, $D_t (C_{4k+1}, x) = d_t(C_{4k+1}, 2k+1)\, x^{2k+1} + \sum_{m=2}^{2k+1} d_t (C_{4k+1}, 2k+m)\, x^{2k+m}$
$= d_t (C_{4k+1}, 2k+1)\, x^{2k+1} + \sum_{m=2}^{2k+1} [d_t (C_{4k}, 2k+m-1) + d_t (C_{4k-1}, 2k+m-1)]\, x^{2k+m}$

$= d_t (C_{4k+1}, 2k+1)\, x^{2k+1} + \sum_{m=2}^{2k+1} d_t (C_{4k}, 2k+m-1)\, x^{2k+m} + \sum_{m=2}^{2k+1} d_t (C_{4k-1}, 2k+m-1)\, x^{2k+m}$
$= (4k+1)\, x^{2k+1} + x [\sum_{m=1}^{2k+1} d_t (C_{4k}, 2k+m-1)\, x^{2k+m-1} - d_t(C_{4k}, 2k)\, x^{2k}] + x [\sum_{m=1}^{2k+1} d_t(C_{4k-1}, 2k+m-1)\, x^{2k+m-1} - d_t(C_{4k-1}, 2k)x^{2k}]$
$= (4k+1)\, x^{2k+1} + x [D_t(C_{4k}, x) - 4 x^{2k}] + x [D_t(C_{4k-1}, x) - (4k-1)^{2k}]$
$= [(4k+1) - 4 - (4k-1)]\, x^{2k+1} + x D_t (C_{4k},x) + x D_t (C_{4k-1}, x)$

$D_t (C_{4k+1}, x) = - 2x^{2k+1} +x [D_t (C_{4k},x) + D_t (C_{4k-1},x)]$.

Proof of (iii):

From theorem 2.7 and 2.8, $d_t(C_{4k+2}, 2k+2) = (2k+1)^2$, $1 \leq k$ and

$d_t(C_{4k+2}, 2k+m) = d_t(C_{4k+1}, 2k+m-1) + d_t(C_{4k}, 2k+m-1)$, $3 \leq m \leq 2k+2$

Now $D_t(C_{4k+2}, x) = d_t(C_{4k+2}, 2k+2)\, x^{2k+2} + \sum_{m=3}^{2k+2} d_t(C_{4k+2}, 2k+m)\, x^{2k+m}$

$= d_t(C_{4k+2}, 2k+2)\, x^{2k+2} + \sum_{m=3}^{2k+2} [d_t(C_{4k+1}, 2k+m-1) + d_t(C_{4k}, 2k+m-1)\, x^{2k+m}$

$= (2k+1)^2 x^{2k+2} + \sum_{m=3}^{2k+2} d_t(C_{4k+1}, 2k+m-1)\, x^{2k+m} + \sum_{m=3}^{2k+2} d_t(C_{4k}, 2k+m-1)\, x^{2k+m}$

$= (2k+1)^2 x^{2k+2} + x[\sum_{m=2}^{2k+2} d_t(C_{4k+1}, 2k+m-1)\, x^{2k+m-1} - d_t(C_{4k+1}, 2k+1)\, x^{2k+1}]$

$+ x[\sum_{m=1}^{2k+2} d_t(C_{4k}, 2k+m-1)\, x^{2k+m-1} - d_t(C_{4k}, 2k)\, x^{2k} - d_t(C_{4k}, 2k+1)\, x^{2k+1}]$

$= (2k+1)^2 x^{2k+2} - (4k+1)\, x^{2k+2} - 4\, x^{2k+1} - 4k^2 x^{2k+2} + x\, D_t(C_{4k+1}, x) + x\, D_t(C_{4k}, x)$

$= - 4x^{2k+1} + (4k^2+4k+1-4k-1- 4k^2)\, x^{2k+2} + x\, [D_t(C_{4k+1}, x) + D_t(C_{4k}, x)]$

$D_t(C_{4k+2}, x) = - 4x^{2k+1} + x\, [D_t(C_{4k+1}, x) + D_t(C_{4k}, x)]$.

Proof of (iv)

From theorem 2.7 and 2.8, $d_t(C_{4k+3}, 2k+2) = 4k+3$ and

$d_t(C_{4k+3}, 2k+m) = d_t(C_{4k+2}, 2k+m-1) + d_t(C_{4k+1}, 2k+m-1)$, $3 \leq m \leq 2k+3$

Now $D_t(C_{4k+3}, x) = d_t(C_{4k+3}, 2k+2)\, x^{2k+2} + \sum_{m=3}^{2k+3} d_t(C_{4k+3}, 2k+m)\, x^{2k+m}$

$= d_t(C_{4k+3}, 2k+2)\, x^{2k+2} + \sum_{m=3}^{2k+3} [d_t(C_{4k+2}, 2k+m-1) + d_t(C_{4k+1}, 2k+m-1)]x^{2k+m}$

$= d_t(C_{4k+3}, 2k+2)\, x^{2k+2} + \sum_{m=3}^{2k+3} d_t(C_{4k+2}, 2k+m-1)\, x^{2k+m} + \sum_{m=3}^{2k+3} d_t(C_{4k+1}, 2k+m-1)\, x^{2k+m}$

$= (4k+3)\, x^{2k+2} + x\, [\sum_{m=3}^{2k+3} d_t(C_{4k+2}, 2k+m-1) x^{2k+m-1}]$

$+ x\, [\sum_{m=2}^{2k+3} d_t(C_{4k+1}, 2k+m-1)\, x^{2k+m-1} - d_t(C_{4k+1}, 2k+1)\, x^{2k+1}]$

$= (4k+3)\, x^{2k+2} + x\, [\sum_{m=3}^{2k+3} d_t(C_{4k+2}, 2k+m-1) x^{2k+m-1}]$

$+ x\, [\sum_{m=2}^{2k+3} d_t(C_{4k+1}, 2k+m-1)\, x^{2k+m-1} - (4k+1)\, x^{2k+1}]$

$= [(4k+3) - (4k+1)]\, x^{2k+2} + x\, D_t(C_{4k+2}, x) + x\, D_t(C_{4k+1}, x)$

$D_t(C_{4k+3}, x) = 2x^{2k+2} + x[D_t(C_{4k+2}, x) + D_t(C_{4k+1}, x)]$.

Using all the above theorems and lemmas, we obtain the coefficients of $D_t(C_k, i)$ for $2 \leq n \leq 18$ in Table 1.

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
C_3	3	1															
C_4	4	4	1														
C_5	0	5	5	1													
C_6	0	0	9	6	1												
C_7	0	0	7	14	7	1											
C_8	0	0	4	16	20	8	1										
C_9	0	0	0	9	30	27	9	1									
C_{10}	0	0	0	0	25	50	35	10	1								
C_{11}	0	0	0	0	11	55	77	44	11	1							
C_{12}	0	0	0	0	4	36	105	112	54	12	1						
C_{13}	0	0	0	0	0	13	91	182	156	65	13	1					
C_{14}	0	0	0	0	0	0	49	196	294	210	77	14	1				
C_{15}	0	0	0	0	0	0	15	140	378	450	275	90	15	1			
C_{16}	0	0	0	0	0	0	4	64	336	672	660	352	104	16	1		
C_{17}	0	0	0	0	0	0	0	17	204	714	1122	935	442	119	17	1	
C_{18}	0	0	0	0	0	0	0	0	81	540	1386	1782	1287	546	135	18	1

Table 1: $d_t(C_n, i)$, the number of total dominating sets of C_n with cardinality i.

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