International Journal of Mathematical Archive-3(4), 2012, Page: 1372-1378
Available online through www.ijma.info ISSN 2229 – 5046

STABILITY OF JENSEN TYPE QUADRATIC FUNCTIONAL EQUATIONS IN MULTI-BANACH SPACES

Manoj Kumar*& Ashish Kumar**
Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, India
*E-mail: *manoj.ahlawat393@gmail.com, **akrmsc@gmail.com*

(Received on: 20-03-12; Accepted on: 05-04-12)

ABSTRACT

In this paper, we investigate the Hyers-Ulam-Rassias stability of a Jensen-type quadratic functional equations in Multi-Banach Spaces using Direct approach.

***Mathematics Subject Classification**: 39B82, 39B52, 39B72.*

***Keywords and Phrases:** Jensen-Type Quadratic functional equations, Multi-Banach spaces.*

---

1. INTRODUCTION

One of the interesting questions in the theory of non- linear functional analysis involved is the stability problem of functional equations as follows: Under what conditions is there a homomorphism near an approximately homomorphism between a group and a metric group, which was first given by S. M. Ulam [11]. In 1941, D. H. Hyers [1] gave the first affirmative answer to this question for approximately additive functions under the assumption of Banach spaces. Th. M. Rassias [13] gave the generalized version of Hyer's result for approximately linear mappings.

In1994, P. Gavruta [9] provided a further generalization of Th. M. Rassias [13] result in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function $\phi(x, y)$ for the existence of unique linear mapping. During last decades, Hyers-Ulam-Rassias stability of various functional equations have been extensively introduced by a number of mathematicians ([12], [14-16]) on various spaces such as normed spaces, Banach space, Fuzzy normed space, RN-space, IRN-space, Non-Archimedean space etc.

In 1983, F. Skof [2] first proved the stability of the quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ for the mapping $f : X \to Y$, where X is a normed space and Y is a Banach space. P. Cholewa [8] again generalized the Skof's result for abelian groups. Lator on, Skof's [2] result was generalized by many mathematicians on various spaces. The functional equations

$$D(fx, fy) = 2f((x \pm y)/2) - f(x) - f(y) \tag{1.1}$$

and

$$D^{'}(fx, fy) = f(ax \pm ay) - 2a^2[f(x) + f(y)] \tag{1.2}$$

for all x, y∈X are called Jensen- Type Quadratic functional equations. In 2009, S.Y.Jang, Rye Lee, Choonkil Park, and Dong Yun Shin [10] proved the Fuzzy stability of equation (1.1) and (1.2).

In the section 2, we adopt some usual terminology, notion and conventions of the theory of Multi-Banach spaces. In the last section, we prove the stability problem in the sense of Hyers-Ulam-Rassias for the functional equations (1.1) and (1.2) on Multi-Banach spaces. We also present some corollaries in reference to our results.

2. PRELIMINARIES

The multi-Banach space was first investigated by Dales and Polyakov [3]. Theory of multi-Banach spaces is similar to the operator sequence space and has some connections with operator spaces and Banach spaces. In 2007, H. G. Dales

---

Corresponding author: Manoj Kumar*, ***E-mail:** manoj.ahlawat393@gmail.com*

and M. S. Moslehian [4] first proved the stability of mappings on multi-normed spaces and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces was investigated by M. S. Moslehian, K. Nikodem, and D. Popa [6] in 2009. In last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians ([5], [7], [17]).

Now, we adopt some usual terminology, notion and convention of the theory of multi-Banach spaces from [3] and [4].

Let $(E, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbf{N}$. We denote by E^k the linear space $E\oplus\ldots\oplus E$ consisting of k-tuples $(x_1,\ldots, x_k)$, where $x_1,\ldots, x_k \in E$. The linear operations on E^k are defined coordinate-wise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbf{N}_k$ the set $\{1, 2,\ldots, k\}$ and by S_k the group of permutations on k symbols.

Definition 2.1:(Multi - norm) A multi-norm on $\{E^k : k\in N\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k\in N)$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbf{N}$, $\|x\|_1 = \|x\|$ for each $x\in E$, and the following axioms are satisfied for each $k\in\mathbf{N}$ with $k \geq 2$:

(N1) $\|(x_{\sigma(1)},\ldots, x_{\sigma(k)})\|_k = \|(x_1,\ldots, x_k)\|_k$, for $\sigma \in S_k$, $x_1,\ldots, x_k \in E$;

(N2) $\|(\alpha_1 x_1,\ldots, \alpha_k x_k)\|_k \leq (\max_{i\in N_k} |\alpha_i|)\,\|(x_1,\ldots, x_k)\|_k$, for $\alpha_1,\ldots, \alpha_k\in C$, $x_1,\ldots, x_k\in E$;

(N3) $\|(x_1,\ldots, x_{k-1}, 0)\|_k = \|(x_1,\ldots, x_{k-1})\|_{k-1}$, for $x_1,\ldots, x_{k-1} \in E$;

(N4) $\|(x_1,\ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1,\ldots, x_{k-1})\|_{k-1}$, for $x_1,\ldots, x_{k-1}\in E$

In this case, we say that $((E^k, \|\cdot\|_k) : k\in\mathbf{N})$ is a multi-normed space (see [3], [4]).

Suppose that $((E^k, \|\cdot\|_k) : k\in\mathbf{N})$ is a multi-normed space, and take $k\in N$. We need the following two properties of multi-norms. They can be found in [3].

(a) $\|(x,\ldots, x)\|_k = \|x\|$, for $x\in E$,

(b) $\max_{i\in N_k} \|x_i\| \leq \|(x_1,\ldots, x_k)\|_k \leq \sum_{i=1}^{k}\| x_i \| \leq k \max_{i\in N_k} \|x_i\|$, for $x_1,\ldots,x_k \in E$.

It follows from (b) that if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k\in N$; in this case, $((E^k, \|\cdot\|_k) : k\in\mathbf{N})$ is a multi-Banach space.

Lemma 2.2: Suppose that $k\in\mathbf{N}$ and $(x_1,\ldots, x_k) \in E^k$. For each $j \in \{1,\ldots, k\}$, let $(x_n^j)_{n=1,2,\ldots}$ be a sequence in E such that $\lim_{n\to\infty} x_n^j = x_j$. Then

$$\lim_{n\to\infty} (x_n^1 - y_1,\ldots,x_n^k - y_k) = (x_1 - y_1,\ldots, x_k - y_k)$$

holds for all $(y_1,\ldots, y_k) \in E^k$ (see [3], [4]).

Definition 2.3: Let $((E^k, \|\cdot\|_k) : k\in\mathbf{N})$ be a multi-normed space. A sequence $\{x_n\}$ in E is a multi-null sequence if for each $\varepsilon > 0$, there exists $n_0\in N$ such that

$$\sup_{k\in N} \|(x_n,\ldots, x_{n+k-1})\|_k \leq \varepsilon \quad (n \geq n_0).$$

Let $x \in E$, we say that the sequence $\{x_n\}$ is multi-convergent to x in E and write $\lim_{n\to\infty} x_n = x$ if $(x_n - x)$ is a multi-null sequence (see [3], [4]).

Lemma 2.4: If a mapping $f : X\to Y$ satisfies the functional equation (1.1) and (1.2) then f is a Quadratic mapping.

3. MAIN RESULTS

In this section, we prove the Hyers – Ulam – Rassias stability of functional equations (1.1) and (1.2). Throughout this section, let E be a linear space and $((F^n, \|\cdot\|_n): n\in N)$ be a multi-Banach space.

3.1 Stability of the functional equation (1.1) by Direct Approach

Theorem 3.1: Let E be a linear space and $((F^n, \|\cdot\|_n): n\in N)$ be a multi-Banach space. Let $f: E\to F$ is a mapping satisfying $f(0) = 0$ such that

$$\sup_{k\in N} \| D(f(x_1, y_1),\ldots, D f(x_k, y_k)\|_k \le \varepsilon \tag{3.1}$$

for all $x_1, \ldots, x_k, y_1,\ldots, y_k \in E$ and $\varepsilon \ge 0$. Then there exist a unique mapping $C : E\to F$ such that

$$\sup_{k\in N} \|(f(x_1) - C(x_1),\ f(x_2) - C(x_2),\ldots, f(x_k) - C(x_k))\|_k \le \frac{\varepsilon}{3} \tag{3.2}$$

for all $x_1, \ldots, x_k \in E$.

Proof: Let $y_1,\ldots, y_k = 0$ and replacing $x_1, \ldots, x_k$ with $2x_1, \ldots, 2x_k$ in (3.1), we obtain

$$\sup_{k\in N} \left\| \left(\frac{f(2x_1)}{4} - f(x_1)\ldots, \frac{f(2x_k)}{4} - f(x_k) \right) \right\|_k \le \frac{\varepsilon}{4} \tag{3.3}$$

again replacing $x_1,\ldots, x_k$ by $2x_1 ,\ldots, 2x_k$ and dividing by 4 to the inequality (3.3), we get

$$\sup_{k\in N} \left\| \left(\frac{f(2^2 x_1)}{4^2} - f(x_1),\ldots, \frac{f(2^2 x_k)}{4^2} - f(x_k) \right) \right\|_k \le \frac{\varepsilon}{4^2} + \frac{\varepsilon}{4}$$

By using induction for a positive integer 'n', we obtain

$$\sup_{k\in N} \left\| \left(\frac{f(2^n x_1)}{4^n} - f(x_1),\ldots, \frac{f(2^n x_k)}{4^n} - f(x_k) \right) \right\|_k \le \sum_{i=0}^{n-1} \frac{\varepsilon}{4^{i+1}} \le \sum_{i=0}^{\infty} \frac{\varepsilon}{4^{i+1}} \tag{3.4}$$

Now, to prove that the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is a Cauchy sequence, we fix $x\in E$ and replacing $x_1, x_2,\ldots, x_k$ with $x, 2x,\ldots, 2^{k-1}x$ such that

$$\sup_{k\in N} \left\| \left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m},\ldots, \frac{f(2^{n+k-1} x)}{4^{n+k-1}} - \frac{f(2^{m+k-1} x)}{4^{m+k-1}} \right) \right\|_k$$
$$\le \sup_{k\in N} \left\| \left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m},\ldots, \frac{1}{4^{k-1}} \left(\frac{f(2^n (2^{k-1} x)}{4^n} - \frac{f(2^m (2^{k-1} x))}{4^m} \right) \right\|_k$$

Now, applying the condition N3 of definition (2.1) we get

$$\le \sup_{k\in N} \left\| \left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m},\ldots, \frac{f(2^n (2^{k-1} x)}{4^m} - \frac{f(2^m (2^{k-1} x)}{4^n} \right) \right\|_k \le \sum_{i=m}^{n-1} \frac{\varepsilon}{4^{i+1}} \tag{3.5}$$

Hence inequality (3.5) shows that $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is a Cauchy sequence as $n\to\infty$, since Y is complete space, thus, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is convergent to a fixed point $C(x)\in Y$, such that

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{3.6}$$

Therefore, as $n\to\infty$ inequality (3.4) implies the inequality (3.2),

$$\sup_{k\in N} \|(C(x_1) - f(x_1) ,\ldots, C(x_k) - f(x_k))\|_k \le \sum_{i=0}^{\infty} \frac{\varepsilon}{4^{i+1}} \le \frac{\varepsilon}{3}$$

Now, to prove that the mapping C: $X \to Y$ is additive, putting $x_1 = x_2 = ,\dots, = x_k = 2^n x$ and $y_1, = , \dots , = y_k = 2^n y$ in (3.1) and dividing both sides by 4^n, we get

$$\left\| \frac{1}{4^n} f\left(\frac{2^n(x+y)}{2}\right) + \frac{1}{4^n} f\left(\frac{2^n(x-y)}{2}\right) - \frac{f(2^n x) + f(2^n y)}{4^n} \right\| \le \frac{\varepsilon}{4^n}$$

which upon taking the limit as $n \to \infty$, yields

$$C\left(\frac{x+y}{2}\right) + C\left(\frac{x-y}{2}\right) - \frac{C(x)+C(y)}{2} = 0$$

Hence C is quadratic mapping which satisfies the inequality (3.1).

Now, To prove the uniqueness of mapping C, let as consider another mapping C' which satisfies (3.1), then we have $C'(2^n x) = 4^n C(x)$, such that

$$\begin{aligned} \| C'(x) - C(x) \| &\le \frac{1}{4^n} \|C'(2^n x) - C(2^n x)\| \\ &\le \frac{1}{4^n} \|C'(2^n x) - f(2^n x)\| + \frac{1}{4^n} \|f(2^n x) - C(2^n x)\| \\ &\le \frac{2\varepsilon}{3.4^n} \end{aligned}$$

Using the property (a) of multi-norms, we have $C = C'$

This proves the uniqueness. This evidently completes the proof of Theorem 3.1.

Corollary 3.1: Let E be a linear-space and $((F^n, \|\cdot\|_n) : n \in N)$ be a multi-Banach space. Let $f : E \to F$ be a mapping satisfying $f(0) = 0$ such that

$$\sup_{k\in N} \|D f(x_1, y_1) ,\dots, Df(x_k, y_k)\|_k \le \phi(x_1, y_1,\dots, x_k, y_k) \tag{3.7}$$

for all $x_1,\dots, x_k, y_1,\dots, y_k \in E$ and $\psi : E^{2k} \to [0, \infty)$, $k \in N$. Then, there exists a unique Quadratic mapping $C : E \to F$ such that

$$\sup_{k\in N} \|f(x_1) - C(x_1),\dots, f(x_k) - C(x_n)\|_k \le \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \phi(2^i x_1, 0 ,\dots, 2^i x_k, 0) \tag{3.8}$$

for all $x_1 ,\dots, x_k \in E$.

Proof: Proof is similar to that of Theorem 3.1 by replacing the condition $\phi(x_1, y_1,\dots, x_k, y_k)$ in place of ε.

Corollary 3.2: Let $(E, \|\cdot\|)$ be a normed space and let $((F^n, \|\cdot\|_n) : n \in N)$ be a multi-Banach space.

Let $0<p < 2$, $\theta \ge 0$ and let $f : E \to F$ be a mapping satisfying $f(0) = 0$ and

$$\sup_{k\in N} \|D f(x_1, y_1) ,\dots, D f(x_k, y_k)\|_k \le \theta (\|x_1\|^p + \|y_1\|^p,\dots, \|x_k\|^p + \|y_k\|^p) \tag{3.9}$$

for all $x_1 ,\dots, x_k, y_1 ,\dots, y_k \in E$. Then, there exists unique Quadratic mapping $C : E \to F$ such that

$$\sup_{k\in N} \|(f(x_1) - C(x_1) ,\dots, f(x_k) - C(x_k))\|_k \le \frac{\theta}{4-2^p} (\|x_1\|^p,\dots, \|y_k\|^p) \tag{3.10}$$

for all $x_1 ,\dots, x_k \in E$.

Proof: Proof is similar to that of Theorem 3.1 by replacing the condition $\theta (\|x_1\|^p + \|y_1\|^p,\dots, \|x_k\|^p + \|y_k\|^p)$ in place of ε.

3.1 Stability of the functional equation (1.2) by Direct Approach

Theorem 3.2: Let E be a linear space and $((F^n, \|\cdot\|_n): n\in N)$ be a multi-Banach space. Let f: E→F is a mapping satisfying f(0) = 0, such that

$$\sup_{k\in N} \|D' f(x_1, y_1), \ldots, D' f(x_k, y_k)\|_k \le \varepsilon \tag{3.11}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exists a unique mapping C : E→F such that

$$\sup_{k\in N} \|(f(x_1) - C(x_1), \ldots, f(x_k) - C(x_k))\|_k \le \frac{\varepsilon}{2(a^2-1)} \tag{3.12}$$

for all $x_1, \ldots, x_k \in E$.

Proof: Let $y_1, \ldots, y_k = 0$ in (3.11), we get

$$\sup_{k\in N} \left\| \left(\frac{f(ax_1)}{a^2} - f(x_1), \ldots, \frac{f(ax_k)}{a^2} - f(x_k) \right) \right\|_k \le \frac{\varepsilon}{2a^2} \tag{3.13}$$

Replacing $x_1, \ldots, x_k$ with $ax_1, ax_2, \ldots, ax_k$ and dividing by a^2 in (3.13), we get

$$\sup_{k\in N} \left\| \left(\frac{f(a^2x_1)}{a^4} - f(x_1), \ldots, \frac{f(a^2x_k)}{a^4} - f(x_k) \right) \right\|_k \le \frac{\varepsilon}{2a^4} + \frac{\varepsilon}{2a^2}$$

for all $x_1, \ldots, x_k \in E$. Using induction on a positive integer 'n', we obtain that

$$\sup_{k\in N} \left\| \left(\frac{f(a^nx_1)}{a^{2n}} - f(x_1), \ldots, \frac{f(a^nx_k)}{a^{2n}} - f(x_k) \right) \right\|_k \le \frac{1}{2}\sum_{i=1}^{n} \frac{\varepsilon}{a^{2i}} \tag{3.14}$$

To prove that the sequence $\left\{\frac{f(a^nx)}{a^{2n}}\right\}$ is Cauchy sequence, we fix $x\in E$ and replacing $x_1\ x_2, \ldots, x_k$ with $x, ax, \ldots, a^{2(k-1)}x$.

$$\sup_{k\in N} \left\| \left(\frac{f(a^nx)}{a^{2n}} - \frac{f(a^mx)}{a^{2m}}, \ldots, \frac{f(a^{n+k-1}x)}{a^{2(n+k-1)}} - \frac{f(a^{m+k-1}x)}{a^{2(m+k-1)}} \right) \right\|_k$$

$$\le \sup_{k\in N} \left\| \left(\frac{f(a^nx)}{a^{2n}} - \frac{f(a^mx)}{a^{2m}}, \ldots, \frac{1}{a^{2(k-1)}} \left(\frac{f(a^{n+k-1}x)}{a^{2n}} - \frac{+(a^m(a^{k-1}x)}{a^{2m}} \right) \right) \right\|_k$$

Using the condition N 3 of definition 2.1, we get

$$\le \frac{1}{2} \sum_{i=m+1}^{n} \frac{\varepsilon}{a^{2i}} \tag{3.15}$$

Hence inequality (3.15) shows that $\left\{\frac{f(a^nx)}{a^{2n}}\right\}$ is a Cauchy sequence as $n\to\infty$, since F is complete space. Thus, the sequence $\left\{\frac{f(a^nx)}{a^{2n}}\right\}$ is convergent to a fixed point $C(x)\in F$ such that

$$C(x) = \lim_{n\to\infty} \frac{f(a^nx)}{a^{2n}} \tag{3.16}$$

Therefore, as $n\to\infty$ inequality (3.14) implies inequality (3.12),

$$\sup_{k\in N} \|(C(x_1) - f(x_1), \dots, C(x_k) - f(x_k))\|_k \le \frac{\varepsilon}{2(a^2 - 1)}$$

rest of the proof is similar to the proof of theorem 3.1.

Corollary 3.3: Let E be a linear space and $(F^n, \|\cdot\|_n)$; $n\in N$) be a multi-Banach space. Let $\phi : E^{2n} \to [0, \infty)$ be a function such that for some $0 < \alpha < a^2$; $n\in N$.

$$\phi(ax_1, 0, \dots, ax_k, 0) \le \alpha\, \phi(x_1, 0, \dots, x_k, 0)$$

for all $x_1, \dots, x_k \in E$. If $f : E\to F$ is a mapping satisfying $f(0) = 0$ such that

$$\sup_{k\in N} \|D'(f(x_1, y_1), \dots, D'(f(x_k, y_k))\|_k \le \phi(x_1, y_1, \dots, x_k, y_k)$$

Then there exists a unique mapping $C : E\to F$ such that

$$\sup_{k\in N} \|(f(x_1) - C(x_1), \dots, f(x_k) - C(x_k))\| \le \frac{\phi(x_1, 0, \dots, x_k, 0)}{2(a^2 - \alpha)}$$

for all $x_1, \dots, x_k \in E$.

Corollary 3.4: Let E be a linear space and $((F^n, \|\cdot\|_n) : n\in N)$ be a multi-Banach space. Let $0 < p < 2$ and $\theta \ge 0$ let $f : E\to F$ be a mapping satisfying $f(0) = 0$, such that

$$\sup_{k\in N} \|(D'f(x_1, y_1), \dots, D'f(x_k, y_k))\|_k \le \theta(\|x_1\|^p + \|y_1\|^p, \dots, \|x_k\|^p + \|y_k\|^p)$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E$. Then, there exists a unique mapping $C : E\to F$ such that

$$\sup_{k\in N} \|(f(x_1) - C(x_1), \dots, f(x_k) - C(x_k))\|_k \le \frac{\theta}{2(a^2 - a^p)} (\|x_1\|^p, \dots, \|x_k\|^p)$$

for all $x_1, \dots, x_k \in E$.

REFERENCES

[1] D. H. Hyers, On the Stability of the Linear Functional Equation, Proc. Nat.Acad. Sci. U.S.A. (27) (1941), pp. 222–224.

[2] F. Skof, Proprietà locali e approssimazione di operatori, Rend. Sem. Mat. Fis. Milano, (53)(1983), pp. 113–129.

[3] H. G. Dales and M. E. Polyakov, Multi-normed spaces and multi-Banach algebras, preprint.

[4] H. G. Dales and M. S. Moslehian, Stability of mappings on multi-normed paces, Glasgow Mathematical Journal, (49)(2)(2007), pp. 321–332.

[5] L. Wang, B. Liu and R. Bai, Stability of a Mixed Type Functional Equation on Multi-Banach Spaces: A Fixed Point Approach, Fixed Point Theory and Applications, vol. 2010, Art. ID 283827, 9 pages doi:10.1155/2010/283827

[6] M. S. Moslehian, K. Nikodem, and D. Popa, Asymptotic aspect of the quadratic functional equation in multi-normed spaces, Journal of Mathematical Analysis and Applications, (355) (2) (2009), pp. 717–724.

[7] M. S. Moslehian and H. M. Srivastava, Jensens functional equations in multi-normed spaces, Taiwnese J. of Math., (14) (2) (2010), pp. 453-462.

[8] P. W. Cholewa, Remarks on the stability of functional equations, Aequationes Math., (27) (1984), pp. 76–86.

[9] P. Gavruta, A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings, J. Math. Anal. Appl. (184) (1994), pp. 431– 436.

[10] S.Y. Jang, Rye Lee, C. Park and Dong Yun Shin, Fuzzy stability of Jensen – Type Quadratic functional equations, Abstract and AppliedAnalysis, vol.2009, Article ID 535678

[11]S. M. Ulam, A Collection of the Mathematical Problems, Interscience Publ. New York, 1960.

[12] S. Czerwik, On the stability of the quadratic mappings in normed spaces, Abh. Math. Sem. Univ. Hamburg, (62) (1992), pp. 59–64.

[13] Th. M. Rassias, On the stability of the Linear mapping in Banach spaces, Procc. Of the American Mathematical Society, (72) (2) (1978), pp. 297-300.

[14] Th. M. Rassias, On the stability of the functional equations in Banach spaces, J. Math. Anal. Appl., (251) (2000), pp. 264–284.

[15] T. Aoki, On the stability of the linear transformation in Banach spaces, Journal of the Mathematical Society, (2) (1-2) (1950), pp. 64-66.

[16] V. Radu, The fixed point alternative and the stability of functional equations, Fixed Point Theory, (4) (1) (2003), pp. 91–96.

[17] Z. Wang, X. Li, and Th. M. Rassias, Stability of an Additive-Cubic-Quartic Functional Equation in Multi-Banach Spaces, Abstract and Applied Analysis, vol. 2011, Article ID 536520, 11 pages doi:10.1155/2011/536520

****