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ON LORENTZİAN BCV SPACES

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ABSTRACT

Begining with the works of Sasakian, Riemannian or Lorentzian cases, many studies were devoted to the metric differential geometry of almost contact manifolds and related structures(see e.g. [1], [2], [3], [5], [6]). Two of them are M. Belkhelfa, I.E. Hirica, R. Rosca, L. Verstlraelen, (see e.g. [3]) made a study of On Legendre curves in Riemanniann and Lorentzian Sasaki Spaces and A. Yildirim, H. H. Hacisalihoğlu (see e.g. [6]) On BCV(Bianchi-Cartan-Vranceanu)-Sasakian Manifolds. Inspired by these studies, we examined the structures of Lorentzi-an BCV Sasakian spaces.

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1. INTRODUCTION

From(see [2]), we remind the following properties and definitions. Let M be a (2n+1) manifold and ϕ,ξ,η are (1,1), (1,0), (0,1) tensorson M, respectively.

Definition 1: *If η,ξ,ϕ tensors satisfy the following conditions*

$$\phi^2 = -I+\eta\xi,$$
$$\phi\xi = 0,$$
$$\eta o\phi = 0,$$
$$\eta(\xi) = 1,$$

then the structure $(\eta,\xi,\phi)$and (M,η,ξ,ϕ) are called almost contact structure and called almost contact manifold on M, respectively.

Proposition 1: *The linear endomorphism*

$$\phi : \chi(M) \to \chi(M)$$
$$X \to \phi(X)$$

has rank 2n.

Definition 2: *Let g a Riemannian or a Lorentzian metric such that $g(\xi,\xi)=\varepsilon,\varepsilon=1,\varepsilon=-1$, according as ε is spacelike or timelike, if it satisfies the following equations*

$$g(\phi X,\phi Y) = g(X,Y)-\varepsilon\,\eta(X)\eta(Y),$$

$$\eta(X) = \varepsilon\, g(\xi,X),$$

the structure $(\eta,\xi,\phi,g,\varepsilon)$ is called almost contact metric structure on M.

Definition 3: *If the metric g satisfies the equation*

$$g(\phi X,Y)=-\,\varepsilon d\eta(X,Y),$$

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then the structure (η,ξ,ϕ,g) is called a contact metric structure on M.

Proposition 2: *An almost contact metrik manifold (M,η,ξ,ϕ,g) is Sasakian manifold if and ony if*

$$(\nabla_X \phi)Y=\varepsilon g(X,Y)\xi-\eta(Y)X,$$

for $\forall X,Y\in\chi(M)$.

Proposition 3: *If a global differential 1-form η is a contact structure on M, it satisfies the equation $\eta\wedge(d\eta)^n\neq 0$.*

Definition 4: *Let (M, η, ξ, ϕ) be a almost contact structure. Lie brackets operator* [,] *define the following*

$$[,]:\chi(M\times R)\times\chi(M\times R)\rightarrow\chi(M\times R),$$

so that

$$\left[\left(X,f\frac{d}{dt}\right),\left(Y,g\frac{d}{dt}\right)\right]=\left([X,Y],(Xg-Yf)\frac{d}{dt}\right).$$

for $\forall X,Y\in\chi(M\times R)$, $\forall f,g\in C^\infty(M,R)$.

Definition 5: *The Nijenhuis torsion N_J of a tensor field J of type (1,1) is the the tensor field of type (1,2) given by*

$$N_J:\chi(M\times R)\times\chi(M\times R)\rightarrow\chi(M\times R),$$

so that

$$N_J(U,V)=J^2[U,V]+[JU,JV]-J[JU,V]-J[U,JV], \tag{1.1}$$

for $\forall U,V\in\chi(M\times R)$.

Proposition 4. *If* $N_J\big((X,0),(Y,0)\big)=\left(N^{(1)}(X,Y),N^{(2)}(X,Y)\frac{d}{dt}\right)$ *and* $N_J\left((X,0),\left(0,\frac{d}{dt}\right)\right)=\left(N^{(3)}(X),N^{(4)}(X)\frac{d}{dt}\right)$ *at those cases*

$$N^{(1)}(X,Y)=N_\phi(X,Y)+2d\eta(X,Y)\xi, \tag{1.2}$$

$$N^{(2)}(X,Y)=\big(L_{\phi X}\eta\big)(Y)-\big(L_{\phi Y}\eta\big)(X)\quad, \tag{1.3}$$

$$N^{(3)}(X)=\big(L_\xi\phi\big)(X), \tag{1.4}$$

$$N^{(4)}(X)=\big(L_\xi\eta\big)(X), \tag{1.5}$$

for $\forall X,Y\in\chi(M)$, where L denoting Lie derivative.

Proposition 5: *For an almost contact metric structure $(\eta,\xi,\phi,g,)$ the covariant derivative of ϕ is given by*

$$2g((\nabla_X\phi)Y,Z) = 3d\Phi(X,\phi Y,\phi Z)-3d\Phi(X,Y,Z)+g(N^{(1)}(Y,Z),\phi X) +N^{(2)}(Y,Z)\eta(X)+2d\eta(\phi Y,X)\eta(Z) - 2d\eta(\phi Z,X)\eta(Y),$$

where $\Phi(X,Y)=\varepsilon g(X,\phi Y)$. In the particular case of a contact structure $(\Phi=d\eta)$ we have,[3],

$$2g((\nabla_X\phi)Y,Z)=g(N^{(1)}(Y,Z),\phi X)+2d\eta(\phi Y,X)\eta(Z)-2d\eta(\phi Z,X)\eta(Y).$$

Definition 6: *If the tensor $N^{(1)}(X,Y)=[\phi,\phi](X,Y)+2d\eta(X,Y)\xi$ on the Sasakian manifold (M,η,ξ,ϕ,g) vanishes then the tensor $N^{(1)}$ is called Sasakian tensor.*M^3

2. LORENTZİAN BCV SPACES

$\{R^3,g_{\lambda,\mu}\}$ is called a Riemannian or a Lorentzian BCV space which denoted by $\mathfrak{M}^3$. $g_{\lambda,\mu}$ is a Riemannian or a Lorentzian Bianchi -Cartan-Vranceanu (BCV) metric in R^3 and denoted by

$$g_{\lambda,\mu}=\frac{dx_1^2+dx_2^2}{\{1+\mu(x_1^2+x_2^2)\}^2}+\varepsilon\left(dx_3+\frac{\lambda}{2}\frac{x_2dx_1-x_1dx_2}{1+\mu(x_1^2+x_2^2)}\right)^2 \tag{2.1}$$

where (x_1,x_2,x_3) denoting standart coordinates of R^3, $\varepsilon=\pm1$ and $\lambda,\mu\in R$ such that $1+\mu(x_1^2+x_2^2)>0$. It is defined that a BCV space $\mathfrak{M}^3$ is isomorphic to the following homogeneous a Riemannian or a Lorentzian 3-manifolds:

- If $\mu=0$, $\lambda=0$, $\varepsilon=1$, then $\mathfrak{M}^3\cong E^3$ (Euclidean 3-space),
- If $\mu=0$, $\lambda=0$, $\varepsilon=-1$, then $\mathfrak{M}^3\cong E_1^3$ (Minkowski 3-space),
- If $\mu=0$, $\lambda\neq0$, $\varepsilon=1$, then $\mathfrak{M}^3\cong N^3$ (Heisenberg 3-space),
- If $\mu=0$, $\lambda\neq0$, $\varepsilon=-1$, then $\mathfrak{M}^3\cong N_1^3$ (Lorentzian Heisenberg 3-space).

We can see (2.1) that the matrix of components of $g_{\lambda,\mu}$ is

$$g_{\lambda,\mu}=\begin{bmatrix} \frac{4+\lambda^2\varepsilon x_2^2}{4\left(1+\mu(x_1^2+x_2^2)\right)^2} & -\frac{\lambda^2\varepsilon x_1x_2}{4\left(1+\mu(x_1^2+x_2^2)\right)^2} & \frac{\lambda\varepsilon x_2}{2\left(1+\mu(x_1^2+x_2^2)\right)} \\ -\frac{\lambda^2\varepsilon x_1x_2}{4\left(1+\mu(x_1^2+x_2^2)\right)^2} & \frac{4+\lambda^2\varepsilon x_1^2}{4\left(1+\mu(x_1^2+x_2^2)\right)^2} & -\frac{\lambda\varepsilon x_1}{2\left(1+\mu(x_1^2+x_2^2)\right)} \\ \frac{\lambda\varepsilon x_2}{2\left(1+\mu(x_1^2+x_2^2)\right)} & -\frac{\lambda\varepsilon x_1}{2\left(1+\mu(x_1^2+x_2^2)\right)} & \varepsilon \end{bmatrix}$$

with respect to standard basis of R^3. Standard base in this space

$\psi=\left\{\frac{\partial}{\partial x_i}=(\delta_{1i},\delta_{2i},\delta_{3i});1\le i\le 3\right\}$ is not orthonormal. If we denote a new base

$\varphi=\{e_1,e_2,e_3\}$ of $\{R^3,g_{\lambda,\mu}\}$, we can write as

$$\left.\begin{aligned} e_1&=\{1+\mu(x_1^2+x_2^2)\}\frac{\partial}{\partial x_1}-\frac{1}{2}\lambda x_2\frac{\partial}{\partial x_3}\\ e_2&=\{1+\mu(x_1^2+x_2^2)\}\frac{\partial}{\partial x_2}+\frac{1}{2}\lambda x_1\frac{\partial}{\partial x_3}\\ e_3&=\frac{\partial}{\partial x_3}\end{aligned}\right\} \tag{2.2}$$

Theorem 1: *Let* (x_1,x_2,x_3) *be standart coordinates of* R^3. *Using orthonormal base* φ (see 2.2), *we get*

$$\begin{aligned} &\nabla_{e_1}e_1=-2\mu x_2e_2, && \nabla_{e_1}e_2=-2\mu x_2e_1+\frac{\lambda}{2}e_3, && \nabla_{e_1}e_3=-\frac{\lambda\varepsilon}{2}e_2,\\ &\nabla_{e_2}e_1=-2\mu x_1e_2-\frac{\lambda}{2}e_3, && \nabla_{e_2}e_2=2\mu x_1e_1, && \nabla_{e_2}e_3=\frac{\lambda\varepsilon}{2}e_1,\\ &\nabla_{e_3}e_1=-\frac{\lambda\varepsilon}{2}e_2, && \nabla_{e_3}e_2=\frac{\lambda\varepsilon}{2}e_1, && \nabla_{e_3}e_3=0,\end{aligned}$$

and $[e_1,e_2]=-2\mu x_2e_1+2\mu x_1e_2+\lambda e_3$, $[e_3,e_2]=[e_1,e_3]=0$,

where ∇ *and* $[,]$ *are Levi-Civita connection and brackets operator on* $\mathfrak{M}^3$, *respectively.*

Theorem 2: *Vector field* $e_3\in\varphi$ *is Killing vector field on* $\mathfrak{M}^3$.

Proof:. If we use Lie derivative for $\forall X,Y\in\chi(\mathfrak{M}^3)$

$$\begin{aligned}\left(L_{e_3}g_{\lambda,\mu}\right)(X,Y)&=e_3g_{\lambda,\mu}(X,Y)-g_{\lambda,\mu}([e_3,X],Y)-g_{\lambda,\mu}(X,[e_3,Y])\\ &=g_{\lambda,\mu}\left(\nabla_{e_3}X,Y\right)+g_{\lambda,\mu}\left(X,\nabla_{e_3}Y\right)-g_{\lambda,\mu}\left(\nabla_{e_3}X,Y\right)+g_{\lambda,\mu}\left(X,\nabla_{e_3}Y\right)-g_{\lambda,\mu}\left(X,\nabla_{e_3}Y\right)+g_{\lambda,\mu}(X,\nabla_Xe_3)\\ &=g_{\lambda,\mu}\left(\nabla_{e_3}X,Y\right)+g_{\lambda,\mu}\left(X,\nabla_{e_3}Y\right)\\ &=0\end{aligned}$$

which completes the proof and hereafter we show $e_3:=\xi$

The dual basis θ of φ is given by

$$\theta=\left\{\theta^1=\frac{dx_1}{1+\mu(x_1^2+x_2^2)},\ \theta^2=\frac{dx_2}{1+\mu(x_1^2+x_2^2)},\theta^3=dx_3+\frac{\lambda}{2}\frac{x_2dx_1-x_1dx_2}{1+\mu(x_1^2+x_2^2)}\right\}$$

and we can easily see $X=\sum_{j=1}^{3}\theta^j\,e_j$ for $\forall X\in\chi(\mathfrak{M}^3)$.

Theorem 3: *The connection forms*

$$\omega_{g_{\lambda,\mu}}=\begin{bmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{bmatrix}$$

of the metric $g_{\lambda,\mu}$ *relative to* θ *on the space* $\mathfrak{M}^3$ *are*

$$\left.\begin{aligned} \omega_{12} &= 2\mu x_2\theta^1 - 2\mu x_2\theta^2 - \tfrac{\lambda\varepsilon}{2}\theta^3 \\ \omega_{13} &= -\tfrac{\lambda\varepsilon}{2}\theta^2 \\ \omega_{23} &= -\tfrac{\lambda\varepsilon}{2}\theta^1 \end{aligned}\right\} \tag{2.3}$$

Now let we take 1-form η

$$\eta = \theta^3 = dx_3 + \frac{\lambda}{2}\frac{x_2dx_1 - x_1dx_2}{1+\mu(x_1^2+x_2^2)} \tag{2.4}$$

on $\mathfrak{M}^3$ for $\lambda \neq 0$. We can provide

$\eta(e_1)=\eta(e_2)=0,\ \eta(e_3)=1$.

On the other hand, we can give the following matrix of endomorphism ϕ with respect to standard basis of R^3

$$\phi = \begin{bmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ -\frac{\lambda\epsilon x_1}{2\left(1+\mu(x_1^2+x_2^2)\right)} & -\frac{\lambda\epsilon x_2}{2\left(1+\mu(x_1^2+x_2^2)\right)} & 0 \end{bmatrix} \tag{2.5}$$

So we can easily show that

$$\phi^2(X) = -X + \eta(X)\xi \tag{2.6}$$

$$\phi(e_1) = \varepsilon e_2, \phi(e_2) = -\varepsilon e_1, \phi(e_3) = 0. \tag{2.7}$$

Theorem 4: *Let us* $\forall X,Y \in \chi(\mathfrak{M}^3)$ *and* $X = \sum_{i=1}^{3} u_i\, e_i,\ Y = \sum_{i=1}^{3} v_i\, e_i$

$$\nabla_X\xi = -\frac{\lambda\phi(X)}{2} \tag{2.8}$$

$$d\eta(X,Y) = \frac{\lambda\varepsilon}{2}g_{\lambda,\mu}\big(X,\phi(Y)\big). \tag{2.9}$$

Proof: Using 2.5 we obtain the following equality

$$\phi(X) = \varepsilon u_2 e_1 - \varepsilon u_1 e_2.$$

Now then

$$\begin{aligned} \nabla_X\xi &= \nabla_{\sum_{i=1}^{3} u_i e_i}\ \xi, \\ &= \textstyle\sum_{i=1}^{3} u_i\, \nabla_{e_i}\ \xi, \\ &= u_1\nabla_{e_1}\ \xi + u_2\nabla_{e_2}\ \xi + u_3\nabla_{e_3}\ \xi, \\ &= u_1\left(-\tfrac{\lambda\varepsilon}{2}e_2\right) + u_2\left(\tfrac{\lambda\varepsilon}{2}e_1\right) + u_3(0), \\ &= -\tfrac{\lambda}{2}(\varepsilon u_2 e_1 - \varepsilon u_1 e_2), \\ &= -\tfrac{\lambda}{2}\phi(X), \end{aligned}$$

and then using Ricci equation we can be provide in the following equation

$$d\eta(X,Y) = \frac{\lambda\varepsilon}{2}g_{\lambda,\mu}\big(X,\phi(Y)\big),$$

which completes the proof

Using 2.4, we can obtain $\eta\wedge d\eta \neq 0$. Moreover, using 1.1, 1.2, 1.3, 1.4 and 1.5 the following equations can be proved by direct calculation on $\mathfrak{M}^3$

$$N^{(1)} = N^{(3)} = 0,\ N^{(2)} = N^{(4)} = 0. \tag{2.10}$$

Theorem 5: For an almost contact metric structure $(\eta,\xi,\phi, g_{\lambda,\mu},\varepsilon)$ there exists the following equation

$$g_{\lambda,\mu}(\phi X,\phi Y) = g_{\lambda,\mu}(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{2.11}$$

for $\forall X,Y\in\chi(\mathfrak{M}^3)$.

Proof. $\forall X,Y\in\chi(\mathfrak{M}^3)$

$$\begin{aligned} g_{\lambda,\mu}(\phi X,\phi Y) &= -g_{\lambda,\mu}(X,\phi^2 Y) \\ &= -g_{\lambda,\mu}(X,-Y+\eta(Y)\xi) \\ &= g_{\lambda,\mu}(X,Y) - \varepsilon\eta(X)\eta(Y) \end{aligned}$$

Theorem 6: For the orthonormal basis $\varphi=\{e_1,e_2,\xi=e_3\}$ *of* $\chi(\mathfrak{M}^3)$, then we have

$$(\nabla_X\phi)Y = \frac{\lambda\varepsilon}{2}\{g_{\lambda,\mu}(X,Y)\xi - \eta(Y)X\}. \tag{2.12}$$

Proof: Let us $\forall X,Y\in\chi(\mathfrak{M}^3)$ and $X=\sum_{i=1}^3 u_i e_i$, $Y=\sum_{i=1}^3 v_i e_i$. Using 1.1, 1.6 and 2.10 we obtain

$$2g_{\lambda,\mu}\big((\nabla_X\phi)Y,Z\big) = 2d\eta(\phi Y,X)\eta(Z) - 2d\eta(\phi Z,X)\eta(Y)$$

and then using 2.9

$$\begin{aligned} 2g_{\lambda,\mu}\big((\nabla_X\phi)Y,Z\big) &= \lambda\varepsilon g_{\lambda,\mu}(\phi Y,\phi X)\eta(Z) - \lambda\varepsilon g_{\lambda,\mu}(\phi Z,\phi X)\eta(Y) \\ &= \lambda\varepsilon\big[g_{\lambda,\mu}(X,Y) - \varepsilon\eta(X)\eta(Y)\big]\eta(Z) - \lambda\varepsilon\big[g_{\lambda,\mu}(X,Z) - \varepsilon\eta(X)\eta(Z)\big]\eta(Y) \\ &= \lambda\varepsilon g_{\lambda,\mu}(X,Y)\eta(Z) - \lambda\varepsilon g_{\lambda,\mu}(X,Z)\eta(Y) \\ &= \lambda\varepsilon\big[g_{\lambda,\mu}\big(g_{\lambda,\mu}(X,Y)\xi,Z\big)\big] \quad - \lambda\varepsilon\big[g_{\lambda,\mu}(\eta(Y)X,Z)\big] \\ &= \lambda\varepsilon g_{\lambda,\mu}\big[g_{\lambda,\mu}\big((X,Y)\xi - \eta(Y)X,Z\big)\big] \end{aligned}$$

Hence we obtain

$$(\nabla_X\phi)Y = \frac{\lambda\varepsilon}{2}\{g_{\lambda,\mu}(X,Y)\xi - \eta(Y)X\}$$

CONCLUSİON

In veiw of 2.10, 2.11, 2.12 *we can say that* $(\mathfrak{M}^3,\eta,\xi,\phi, g_{\lambda,\mu},\varepsilon)$ *is a* 3*-dimensional Sasaki space.*

1. Curvatures And Torsions Of Lorentzian BCV Spaces

Let Riemannian curvature be

$$R(e_j,e_k)e_i = \sum_{l=1}^{3} R_{ijk}^l\, e_l, (1\le i,j,k,l\le 3)$$

on $\mathfrak{M}^3$. *All of* R_{ijk}^l *described as follows:*

$R_{121}^1 = 0,\qquad R_{121}^2 = 4\mu - \frac{3\varepsilon}{4}\lambda^2,\qquad R_{121}^3 = 0,$

$R_{313}^1 = \frac{\lambda^2}{4},\qquad R_{313}^2 = 0,\qquad R_{313}^3 = 0,$

$R_{323}^1 = 0,\qquad R_{323}^2 = \frac{\lambda^2}{4},\qquad R_{323}^3 = 0$

$R_{221}^1 = -4\mu + \frac{3\varepsilon}{4}\lambda^2,\qquad R_{221}^2 = 0,\qquad R_{221}^3 = 0$

$R_{331}^1 = -\frac{\lambda^2}{4},\qquad R_{331}^2 = 0,\qquad R_{331}^3 = 0,$

$R_{112}^1 = 0,\qquad R_{112}^2 = -4\mu + \frac{3\varepsilon}{4}\lambda^2,\qquad R_{112}^3 = 0,$

$R_{223}^1 = 0,\qquad R_{223}^2 = 0,\qquad R_{223}^3 = -\frac{\lambda^2}{4}$

$R_{212}^{1} = 4\mu - \frac{3\varepsilon}{4}\lambda^2, \qquad R_{212}^{2} = 0, \qquad R_{212}^{3} = 0$

$R_{332}^{1} = 0, \qquad R_{332}^{2} = -\frac{\lambda^2}{4}, \qquad R_{332}^{3} = 0$

$R_{113}^{1} = 0, \qquad R_{113}^{2} = 0, \qquad R_{113}^{3} = -\frac{\lambda^2}{4}$

Theorem 8: *Sectional curvature of all plane sections ortogonal to* ξ *is equal to*

$$4\mu - \frac{3\varepsilon}{4}\lambda^2$$

on $\mathfrak{M}^3$.

Proof: Sectional curvature of a plane independent of the selected bases [3]. Let us take plane $\Pi = Sp\{e_1, e_2\}$ orthogonal to ξ. Now then

$$\begin{aligned} K(e_1,e_2) &= \frac{g_{\lambda,\mu}(R(e_1,e_2)e_2,e_1)}{g_{\lambda,\mu}(e_1,e_1)g_{\lambda,\mu}(e_2,e_2) - g_{\lambda,\mu}(e_2,e_1)^2} \\ &= g_{\lambda,\mu}(R_{212}^1 e_1 + R_{212}^2 e_2 + R_{212}^3 e_3, e_1) \\ &= g_{\lambda,\mu}\left(\left(4\mu - \frac{3\varepsilon}{4}\lambda^2\right)e_1 + 0e_2 + 0e_3, e_1\right) \\ &= 4\mu - \frac{3\varepsilon}{4}\lambda^2 \end{aligned}$$

which completes the proof.

We can easily show that $K(e_1,e_3)=K(e_2,e_3)=\frac{\lambda^2}{4}$.

Remark: *So we note* ($\mathfrak{M}^{3\,3}, \eta, \xi, \phi, g_{\lambda,\mu}, \varepsilon$) *by* $R^3(4\mu - \frac{3\varepsilon}{4}\lambda^2)$.

Theorem 9: *Let us give structure* ($M^3, \eta, \xi, \phi, g_{\lambda,\mu}, \varepsilon$)*. The Ricci operator is* $\tilde{S} = \left(-4\mu + \frac{3\varepsilon\lambda^2}{4} - \frac{\lambda^2}{4}\right)\phi^2$*.and the Ricci curvature is*

$$Ricc(X,Y) = \left(-4\mu + \frac{3\varepsilon\lambda^2}{4} - \frac{\lambda^2}{4}\right) g_{\lambda,\mu}(\phi Y, \phi X)$$

for $\forall X, Y \in \chi(\mathfrak{M}^3)$.

Example: *Let us give structure* ($\mathfrak{M}^3, \eta, \xi, \phi, g_{\lambda,\mu}, \varepsilon$) *and a ortogonal base of* M^3 *is*

$$\begin{aligned} e_1 &= \{1 + \mu(x_1^2 + x_2^2)\}\frac{\partial}{\partial x_1} - \frac{1}{2}\lambda x_2 \frac{\partial}{\partial x_3} \\ e_2 &= \{1 + \mu(x_1^2 + x_2^2)\}\frac{\partial}{\partial x_2} + \frac{1}{2}\lambda x_1 \frac{\partial}{\partial x_3} \\ e_3 &= \frac{\partial}{\partial x_3} \end{aligned}$$

now then

$\tilde{S}(e_1) = \left(-4\mu + \frac{3\varepsilon\lambda^2}{4} - \frac{\lambda^2}{4}\right)e_1,\ Ricc(e_1,e_1) = 4\mu - \frac{3\varepsilon\lambda^2}{4} + \frac{\lambda^2}{4},$

$\tilde{S}(e_2) = \left(-4\mu + \frac{3\varepsilon\lambda^2}{4} - \frac{\lambda^2}{4}\right)e_2,\ Ricc(e_2,e_2) = 4\mu - \frac{3\varepsilon\lambda^2}{4} + \frac{\lambda^2}{4},$

$\tilde{S}(e_3) = 0, \qquad Ricc(e_3,e_3) = 0,$

and then scalar curvature

$$\delta(P) = \sum_{i=1}^{3} Ricc\left(e_{i_p}, e_{i_p}\right) = 8\mu - \frac{3\varepsilon\lambda^2}{4} + \frac{\lambda^2}{4}$$

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