International Journal of Mathematical Archive-3(4), 2012, Page: 1354-1364
Available online through www.ijma.info ISSN 2229 – 5046

LIE GROUP TREATMENT FOR THE FLOW OF NON-NEWTOWNION FLUID THROUGH POROUS MEDIA

Dr. Heenaben A. Raj*
Department of Mathematics, Veer Narmad South Gujarat University, Surat (Gujarat), India
E-mail: hinaalpesh@yahoo.co.in
&
Dr. M. G. Timol
Professor, Department of Mathematics, Veer Narmad South Gujarat University, Surat (Gujarat), India
E-mail: mgtimol@gmail.com

(Received on: 09-03-12; Accepted on: 28-03-12)

ABSTRACT

This paper shows the solution of non-linear equation describing the transient flow of power law fluids through porous medium. The solution is obtained by infinitesimal group transformation group method. At first, the one parameter lie group transformation in terms of characteristics function W is taken. Then the solution of the characteristics function W for the infinitesimals was obtained. The next step is to find the absolute invariants. Using these similarity variables the partial differential equation can be transformed into an ordinary differential equation. Analytical solution of Ordinary Differential Equation is presented.

Key words: *Power law fluids, non- Newtonian fluids, Infinitesimal transformation.*

---

1 INTRODUCTION:

The solution of the nonlinear equation describing the transient flow of power law fluids through a porous medium is presented in this paper. These are obtained by infinitesimal group transformations for several cases of practical interest in interpretation of well flow test of short duration, used currently in oil reservoir engineering for obtaining the reservoir properties. Obviously, from an oil reservoir engineering point of view, it is essential to have an adequate understanding of the flow behavior of non-Newtonian fluids, in particular of the pseudo-plastic type, through a porous medium. In Pascal, H.[4], on this subject has shown the rheological effects of non-Newtonian displacing fluids of power law with yield stress on the dynamics of a moving interface separating oil from water. Several relevant conclusions, obtained there, indicated the conditions in which the viscous fingering effect in oil displacement could be eliminated and a piston-like displacement may be possible.

To adequately describe the flow of non-Newtonian fluids in porous media, a modified Darcy's law is required. In Pascal, H. [3], this basic relationship between the flow rate and the pressure drop for power law fluids with yields stress has been shown. However, a basic requirement for understanding is knowledge of the coefficients occurring in the modified Darcy's law for this class of non-Newtonian fluids. While for an unrealistic geometrical model, for examples the capillary tube model, these coefficients may be determined and expressed in terms of rheological parameters and geometrical properties of the porous medium, in the case of real porous media the appropriate approach, as it was shown in Pascal, H. [5], remains the determination of the coefficients from in situ measurements on the pressure or flow rate behavior in time, recorded in wells during the transient flow of short duration. Knowledge of this behavior may permit us the obtaining of useful information regarding the permeability of the porous medium, provided the oil is Newtonian, in which case Darcy's law holds (Refer. Matthews at el [2], Earloughes, R.C. [1]).

The main object of this paper is to find the solution for the problem in terms of infinitesimal transformation.

The basic equations describing the steady and unsteady flow of power law fluids with yield stress were derived in our (Refer. Pascal, H. [3] & [4]). According to the results obtained there, the basic equations for power law fluids in the absence of yield stress may be written as

$$\left(\frac{\partial p}{\partial x}\right)^{(1-n)/n} \frac{\partial^2 p}{\partial x^2} = na^2 \frac{\partial p}{\partial t} \tag{1}$$

---

for one dimensional flow and for a plane redial flow.

The coefficient a^2 in (1) is given by the relation

$$a^2 = \left(\frac{\mu_{ef}}{k}\right)^{1/n} (C_0\phi + C_p) \tag{2}$$

Once the pressure distributions are determined from previous equations, the velocity distributions may be obtained from the modified Darcy's law and expressed as

$$\upsilon(x, t) = \left[\frac{k}{\mu_{ef}} \frac{\partial p}{\partial x}\right]^{1/n} \tag{3}$$

2. MATHEMATICAL FORMULATION:

The problems considered in this paper will be designed by means of a pair of symbols describing the oil reservoir boundary conditions. The first symbol will describe the condition at the outface flow, i.e. in wells, while the second will describe that at the inlet end. For example, the case where the pressure at both ends of the reservoir is specified will be termed the P_α - P_e problem. The case in which the flow rates are specified at both ends will be termed the Q_w - Q_e problem, while the case where the pressure is specified at one end and the flow rate at the other end will be termed the Q_w - P_e problem. The P_α - P_e problem corresponds to the first boundary value problem, the Q_w - Q_e problem represents the second boundary value problem, and the Q_w - P_e problem is a mixed problem.

3 THE P_α – P_e PROBLEM (ONE DIMENSIONAL FLOW):

The flow system to be analyzed is a linear oil reservoir having a closed outer boundary. It is assumed that at the initial moment t = 0 the pressure in the reservoir is constant, say P_e, and for t > 0 the reservoir is depleted by a constant pressure of production, say P_w, at the outface flow. As result, the flow is initiated and maintained in the reservoir provided that P_w < P_e. This case corresponds to a reservoir producing by natural depletion. I.e. by elastic decompression, in which transient flow response appears in reservoir for t > 0.

It should be pointed out that during the depletion period we have two phases concerning the reservoir behavior from a fluid mechanics point of view. For example, by reducing the pressure at the outface flow from P_e to P_w, a decompression front will be generated in the reservoir, which requires a certain time to reach the closed boundary of the reservoir. Obviously, this time depends, as we will see further on, on the reservoir permeability, fluid compressibility, and rheological parameters of the reservoir fluids. The time interval required for the front to reach the closed boundary will be denoted by t_m and referred to as the duration of the first phase.

At any moment $0 < t < t_m$ the decompression front will be located at a certain position denoted in the article by the equation x = l(t). Taking into account these notations, we can state that the pressure distribution p(x, t) in the reservoir will be, for $0 < t < t_m$ and $0 \le x \le l(t)$, the solution of the basic equation (1) satisfying the double inequality $P_w \le p(x,t) \le P_e$, while for $x \ge l(t)$ it has a constant value, namely $p(x,t) = P_e$. For $t > t_m$, the pressure at the closed boundary starts to decline in time and after a certain time of production, termed as the depletion time, the pressure tends to equalize its value at the outface flow. This is the second phase and the corresponding pressure distribution for $t > t_m$ will be $P_w < p(x,t)$ in the range 0 < x < L, where L is the length of the reservoir. The reservoir is considered to be completely depleted when its energy is insufficient to maintain the production at an imposed economical flow rate.

From an oil reservoir engineering point of view, the first phase is of particular interest in well-test analysis, since information obtained in this phase, i.e. prior to when the decompression front reaches the closed boundary, may provide the reservoir properties, which are used in predicting the reservoir performance for a long period of time. Consequently, in well-test analysis the knowledge of the flow parameters in wells for a very short period of time, i.e. $t < t_m$, is required. In these circumstances, the solution of equation (1) may be determined by assuming that the reservoir is of infinite extent, in which case the following boundary and initial conditions may be used for $t < t_m$

$$\begin{aligned} t = 0 \qquad & p(x, 0) = P_e \\ t > 0 \qquad & p(x, t) = P_e; \quad 0 < t < t_m \\ t > 0 \qquad & p(0, t) = P_w \end{aligned} \tag{4}$$

However, it may be shown that, for $t < t_m$ and n < 1, the pressure distribution p(x,t) corresponding to the conditions will be identical with that obtained when it is determined from consideration of a moving boundary problem, since $p(x, t) = P_e$ for $x \geq l(t)$.

On the other hand, the pressure p(x,t) and its derivative $\frac{\partial p}{\partial x}$ should be continuous functions of x and t for any x > 0 and t > 0, but because p(x, t) is constant for $x \geq l(t)$, then $\left.\frac{\partial p}{\partial x}\right|_{x = l(t)} = 0$. As a result, instead of (4), we have, the following conditions:

$$\begin{aligned} & p[l(t), t] = P_e \qquad 0 < t < t_m \\ & \left.\frac{\partial p}{\partial x}\right|_{x = l(t)} = 0, \ 0 < t < t_m \\ & p(0, t) = P_w \\ & l(0) = 0 \qquad ; 0 < t < t_m \end{aligned} \tag{5}$$

Specially, the pressure distribution p(x,t) for $0 < x < l(t)$ is the solution of equation (1), satisfying the conditions specified in (5). The nature of these conditions is motivated by the physical reality of the flow mechanism appearing in oil reservoirs producing by natural depletion, i.e. by elastic decompression, where for $l(t) < x < \infty$ the pressure remains constant to its value corresponding to the initial moment t = 0. As a result, no flow occurs in the reservoir behind the decompression front.

4 SOLUTION OF THE EQUATION USING INFINITESIMAL GROUP TRANSFORMATIONS:

The basic equation for Power law fluids

$$\left(\frac{\partial p}{\partial x}\right)^{\frac{1-n}{n}} \cdot \frac{\partial^2 p}{\partial x^2} = na^2 \frac{\partial p}{\partial t} \tag{6}$$

Using the notation

$$p_1 = \frac{\partial p}{\partial t}, \quad p_{22} = \frac{\partial^2 p}{\partial x^2}, \quad p_2 = \frac{\partial p}{\partial x}$$

$$p_{12} = \frac{\partial^2 p}{\partial x \ \partial t}, \quad p_{11} = \frac{\partial^2 p}{\partial t^2} \tag{7}$$

The equation can be written as

$$(p_2)^{\frac{1-n}{n}} . \ p_{22} = na^2 . \ p_1 \tag{8}$$

We define the infinite group of transformation as follows.

$$\begin{aligned} \bar{t} &= t + \sum \xi(t, x, u) + o(\varepsilon^2) \\ \bar{x} &= x + \sum \eta(t, x, u) + o(\varepsilon^2) \\ \bar{p} &= p + \sum \varsigma(t, x, u) + o(\varepsilon^2) \\ \bar{p}_1 &= p_1 + \sum \pi_1(t, x, u, p_1, p_2) + o(\varepsilon^2) \\ \bar{p}_2 &= p_2 + \sum \pi_2(t, x, u, p_1, p_2) + o(\varepsilon^2) \end{aligned}$$

$$\overline{p_{22}} = p_{22} + \sum \pi_{22}(t, x, u, p, p_1, p_2, p_{11}, p_{12}, p_{22},) + o(\varepsilon^2) \tag{9}$$

The transformation function $\xi, \eta, \pi_1, \pi_2, \pi_{22}$ can be expressed in terms of characteristic function W as,

$$\xi = \frac{\partial W}{\partial p_1} \tag{10}$$

$$\eta = \frac{\partial W}{\partial p_2} \tag{11}$$

$$\varsigma = p_1 \frac{\partial W}{\partial p_1} + p_2 \frac{\partial W}{\partial p_2} - W \tag{12}$$

$$\pi_1 = -\frac{\partial W}{\partial t} - p_1 \frac{\partial W}{\partial u} \tag{13}$$

$$\pi_2 = -\frac{\partial W}{\partial x} - p_2 \frac{\partial W}{\partial u} \tag{14}$$

$$\pi_{22} = \frac{\partial^2 W}{\partial x^2} + \left(\frac{n+1}{n}\right) p_2 \frac{\partial^2 W}{\partial x \partial u} + p_2^2 \frac{\partial^2 W}{\partial u^2} + \left(\frac{n+1}{n}\right) p_{12} \left(\frac{\partial^2 W}{\partial x \partial p_1} + p_1 \frac{\partial^2 W}{\partial u \partial p_1}\right)$$

$$+ \left(\frac{n+1}{n}\right) p_{22} \left(\frac{\partial^2 W}{\partial x \partial p_2} + p_2 \frac{\partial^2 W}{\partial u \partial p_2}\right) + p_{12}^2 \frac{\partial^2 W}{\partial p_1^2} + \left(\frac{n+1}{n}\right) p_{12} \cdot p_{22} \cdot \frac{\partial^2 W}{\partial p_1 \partial p_2} + p_{22}^2 \frac{\partial^2 W}{\partial p_2^2} + p_{22} \frac{\partial W}{\partial u} \tag{15}$$

The expanded form

$$\xi \frac{\partial U}{\partial t} + \eta \frac{\partial U}{\partial x} + \varsigma \frac{\partial U}{\partial u} + \pi_1 \frac{\partial U}{\partial p_1} + \pi_2 \frac{\partial U}{\partial p_2} \pi_{11} \frac{\partial U}{\partial p_{11}} + \pi_{12} \frac{\partial U}{\partial p_{12}} + \pi_{22} \frac{\partial U}{\partial p_{22}} = 0 \tag{16}$$

From (8), we have

$$p_2^{\frac{1-n}{n}} \pi_{22} + p_{22} \left(\frac{1-n}{n}\right) (\pi_2)^{\frac{1-n}{n} - 1 = \frac{1-2n}{n}} = na^2 \pi_1 \tag{17}$$

Using equations (16) & (17),

$$\therefore \quad p_2^{\frac{1-n}{n}} \frac{\partial^2 W}{\partial x^2} + \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{\partial^2 W}{\partial x \partial u} + p_2^{\frac{1+n}{n}} \frac{\partial^2 W}{\partial u^2}$$

$$+ \left(\frac{n+1}{n}\right) p_2^{\frac{1-n}{n}} \cdot p_{12} \cdot \frac{\partial^2 W}{\partial x \partial p_1} + \left(\frac{n+1}{n}\right) p_2^{1/n} \cdot p_{12} \cdot \frac{\partial^2 W}{\partial u \partial p_1}$$

$$+ \left(\frac{n+1}{n}\right) p_2^{\frac{1-n}{n}} \cdot p_{22} \cdot \frac{\partial^2 W}{\partial x \partial p_2} + \left(\frac{n+1}{n}\right) p_2^{1/n} \cdot p_{22} \cdot \frac{\partial^2 W}{\partial u \partial p_2}$$

$$+ p_2^{\frac{1-n}{n}} p_{12}^2 \frac{\partial^2 W}{\partial p_1^2} + \left(\frac{n+1}{n}\right) p_2^{\frac{1-n}{n}} p_{12} \cdot p_{22} \cdot \frac{\partial^2 W}{\partial p_1 \partial p_2}$$

$$+ p_2^{\frac{1-n}{n}} p_{22}^2 \frac{\partial^2 W}{\partial p_2^2} + p_2^{\frac{1-n}{n}} p_{22} \frac{\partial W}{\partial u} - \left(\frac{1-n}{n}\right) p_2 \left(\frac{\partial W}{\partial x}\right)^{\frac{1-2n}{n}}$$

$$+ \ p_2 \left(\frac{1-2n}{n}\right) \left(\frac{\partial W}{\partial x}\right)^{\frac{1-3n}{n}} \left(\frac{\partial W}{\partial u}\right) + p_2 \frac{\left(\frac{1-2n}{n}\right)\left(\frac{1-3n}{n}\right)}{2n^2} \tag{18}$$

$$. \left(\frac{\partial W}{\partial x}\right)^{\frac{1-4n}{n}} \left(\frac{\partial W}{\partial u}\right)^2 - na^2 \frac{\partial W}{\partial t} - p_1 na^2 \frac{\partial W}{\partial u} = 0$$

This equation is solved for W.

Since W is not a function of p_{12}, the co-efficient involving $p_{12} \ . \ p_{12}^2$ should be zero.

Take coefficient of p_{12}^2 is zero.

$$W = W_1(t, x, u, p_2) + p_1 W_2(t, x, u, p_2) \tag{19}$$

Coefficient of terms involving p_{12} -

$$\left(\frac{n+1}{n}\right)\frac{\partial W_2}{\partial x} + \left(\frac{n+1}{n}\right) p_2 \frac{\partial W_2}{\partial u} + \left(\frac{n+1}{n}\right) p_2 \frac{\partial W_2}{\partial p_2} = 0 \tag{20}$$

W_2 is not a function of p_1, The coefficient of p_1 is zero.

$$\left(\frac{n+1}{n}\right)\frac{\partial W_2}{\partial p_2} = 0 \tag{21}$$

$\frac{n+1}{n} \neq 0, \ \frac{\partial W_2}{\partial p_2} = 0$, W_2 is independent of p_2.

$$W_2 = W_2(t, u, x) \tag{22}$$

The remaining two terms lead to condition that W_2 is independent of y and u.

$$W = W_1(t, y, u, p_2) + p_1 W_2(t) \tag{23}$$

W_1 and W_2 are independent of p_1.

$$p_1^0 : \ - na^2 \frac{\partial W_1}{\partial t} - p_2^{\frac{1-n}{n}} \frac{\partial^2 W}{\partial x^2} - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{\partial^2 W}{\partial x \partial u} + p_2^{\frac{1+n}{n}} \frac{\partial^2 W}{\partial x^2} \tag{24}$$

$$p_1^1 : \ na^2 \frac{\partial W_2}{\partial t} - \left(\frac{n+1}{n}\right) p_1 \ p_2^{\frac{1-n}{n}} \frac{\partial^2 W_1}{\partial x \partial p_2} - p_2^{\frac{1+n}{n}} \frac{\partial^2 W}{\partial u \partial p_2} \tag{25}$$

$$p_1^2 : \ - p_2^{\frac{1-n}{n}} \frac{\partial^2 W_1}{\partial p_2^2} = 0 \tag{26}$$

The coefficient of p_1^2 gives

$$W_1 = W_{11}(t, x, u) + p_2 W_{12}(t, x, u) \tag{27}$$

The coefficient of p_1^1 gives

$$na^2 \frac{\partial W_2}{\partial t} - \left(\frac{n+1}{n}\right) p_2^{\frac{1-n}{n}} \frac{\partial W_{12}}{\partial x} - p_2^{1/n} \frac{\partial W_2}{\partial u}$$

$\Rightarrow$ W_{12} and W_2 are independent of p_2.

$$\therefore \quad na^2 \frac{\partial W_2}{\partial t} - \left(\frac{n+1}{n}\right) p_2^{\frac{1-n}{n}} \frac{\partial W_{12}}{\partial x} = 0 \qquad (28)$$

W_2 is a function of t only. W_2 depends linearly on x.

$$w = w_{121}(t) + w_{122}(t)$$

$$x - \frac{\partial W_2}{\partial t} + \left(\frac{n+1}{n}\right) p_2^{\frac{1-n}{n}} W_{122} = 0$$

$$W = W_{11} + [W_{121} + W_{122}\, x]\, p_2 + W_2\, p_1 \qquad (29)$$

Now, setting to zero terms with different Power of p_2.

$$P_2^0 : \frac{\partial W_{11}}{\partial t} + \frac{\partial^2 W_{11}}{\partial x^2} = 0 \qquad (30)$$

$$p_2^1 : \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{\partial^2 W_1}{\partial x \partial u} = 0$$

$$\Rightarrow \left(\frac{n+1}{n}\right) p_2^{1/n} \left(- \frac{dw_{121}}{dt} - \frac{dw_{122}}{dt} \right) + \left(\frac{n+1}{n}\right) \frac{\partial^2 W_{11}}{\partial x \partial u} = 0$$

$$\Rightarrow - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{121}}{dt} - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{122}}{dt} + \left(\frac{n+1}{n}\right) \frac{\partial^2 u}{\partial x \partial u} = 0$$

$$\Rightarrow - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{121}}{dt} - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{122}}{dt} x + \left(\frac{n+1}{n}\right) \frac{\partial W_{112}}{\partial y} = 0$$

$$W_{11} = W_{111} + W_{112}.u \qquad (31)$$

$$W_{112} = W_{1121} + W_{1122} x + W_{1123}.x^2 \qquad (32)$$

$$\therefore \ - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{121}}{dt} - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{122}}{dt} x + \left(\frac{n+1}{n}\right) [W_{1122} + 2\, W_{1123} x] \qquad (33)$$

$$\therefore \ - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{121}}{dt} - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{122}}{dt} x + \left(\frac{n+1}{n}\right) W_{1122} + 2 \left(\frac{n+1}{n}\right) W_{1123} x$$

$$\Rightarrow - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{121}}{dt} + \left(\frac{n+1}{n}\right) W_{1122} = 0 \ \& \ - \left(\frac{n+1}{n}\right) p_2^{1/n} \frac{dW_{122}}{dt} x + 2 \left(\frac{n+1}{n}\right) W_{1123} x = 0$$

$$- p_2^{1/n} \frac{dW_{121}}{dt} + W_{1122} = 0 \qquad (34)$$

$$\& \; - p_2^{1/n} \frac{dW_{122}}{dt} x + 2W_{1123} = 0 \tag{35}$$

Now for, $p_2^0 = \frac{-\partial W_{11}}{\partial t} + \frac{\partial^2 W_{11}}{\partial x^2} = 0$ (36)

$$\Rightarrow \left(\frac{-\partial W_{111}}{\partial t} + \frac{\partial^2 W_{111}}{\partial x^2} \right) - \left(\frac{dW_{1121}}{\partial t} - 2W1123 \right) u - \frac{dW_{1122}}{\partial t} x\, u - \frac{dW_{1123}}{\partial t} x^2 u = 0$$

The coefficient of u^0, u, yu, x^2u should be zero, because W_{111} is a function of to x only.

$$W_{1122} = C_1 \tag{37}$$

$$W_{1123} = C_2 \tag{38}$$

$$W_{1121} = 2C_2 t + C_3 \tag{39}$$

$$\& \; \frac{\partial W_{111}}{\partial t} - \frac{\partial^2 W_{111}}{\partial x^2} = 0$$

Now, $W_{121} = 2\, W_{1122}\, t + C_4$

$$W_{121} = 2\, C_1\, t + C_4 \tag{40}$$

$$W_{122} = 2 \left(\frac{n+1}{n} \right) W_{1123}\, t + C_5$$

$$W_{122} = 2 \left(\frac{n+1}{n} \right) C_2\, t + C_5 \tag{41}$$

also $W_2 = \left(\frac{n+1}{n} \right) W_{122} t + C_6 \;\therefore\; W_2 = 2 \left(\frac{n+1}{n} \right)^2 C_2 t + \left(\frac{n+1}{n} \right) C_5 t + C_6$ (42)

$$W = W_1 \; [2C_2 t + C_3 + C_1 x + C_2 x^2]\, u + [2C_1 t + C_4 + 2 \left(\frac{n+1}{n} \right) C_2 t + C_5] p_2 + [\left(\frac{n+1}{n} \right) C_2 t^2 + \left(\frac{n+1}{n} \right) C_5 t + C_6]\; p_1$$

$$\frac{dt}{\xi} = \frac{dx}{\eta} = \frac{du}{\zeta} \tag{44}$$

$$\therefore \; \eta = x\, t^{\frac{-n}{n+1}} \tag{45}$$

$$\therefore \; f(\eta) = \frac{p}{t^{\alpha}} \qquad \text{Where} \quad \alpha = \left(\frac{n}{n+1} \right) \left(\frac{c_3}{c_5} \right) \tag{46}$$

Using this transformation the basic equation (1) is reduced, for $0 < x < l(t)$, to the nonlinear ordinary differential equation

$$\frac{d^2 p}{d\eta^2} + \frac{n^2 a^2}{n+1} \eta \left(\frac{dp}{d\eta} \right)^{(2n-1)/n} = 0 \tag{47}$$

for $0 < \eta < \eta_1$ where $\eta_1 = \frac{l(t)}{t^{n/(n+1)}}$ (48)

The boundary conditions (5) now become

$$p(\eta_1) = P_e; \; \left. \frac{dp}{d\eta} \right|_{\eta = \eta_1} = 0 \text{ and } p(0) = P_w \tag{49}$$

These three conditions determine, along with the solution of equation (47), the front location l(t) and pressure distribution p(x, t) in the range $0 < x < l(t)$, as it will be shown further on.
And its solution may be written as follows:

$$\frac{P_e - p(x,t)}{P_e - P_w} = 1\text{-Erf}\left(\frac{xa}{2t^{-n/n+1}}\right)$$

where $$\text{Erf}\left(\frac{xa}{2t^{-n/n+1}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{xa/2t^{-n/n+1}} e^{-\xi^2}d\xi$$

For a Newtonian fluid, we have n = 1 in (47), and in this particular case equation (47) becomes

$$\frac{d^2p}{d\eta^2} + \frac{1}{2}a^2\eta\frac{dp}{d\eta} = 0 \tag{50}$$

and its solution may be written as follows:

$$\frac{P_e - p(x,t)}{P_e - P_w} = 1 - \text{Erf}\left(\frac{xa}{2\sqrt{t}}\right) \tag{51}$$

where $$\text{Erf}\left(\frac{xa}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{xa/2\sqrt{t}} e^{-\xi^2}d\xi \tag{52}$$

For n < 1, from (47) we obtains

$$\frac{dp}{d\eta} = \left[C_1 - \frac{na^2}{2}\frac{1-n}{1+n}\eta^2\right]^{n/(1-n)} \tag{53}$$

And from (9) we have $\left.\frac{dp}{d\eta}\right|_{\eta=\eta_1} = 0$, so that C_1 in (53) is expressed as $C_1 = \frac{na^2}{2}\frac{1-n}{1+n}\eta^2$. Integration of equation (53) with condition $p(0) = P_w$ yields a relation for determining the pressure distribution in the range $0 < x < l(t)$. This may be written as

$$p(\eta) - P_w = \left[\frac{na^2}{2}\frac{1-n}{1+n}\right]^{n/(1-n)} \eta_1^{(1+n)/(1-n)} J_n(\eta) \tag{54}$$

where $$J_n(\eta) = \int_0^{\eta/\eta_1}(1-\xi^2)^{n/(1-n)}d\xi;\ \ 0 < \frac{\eta}{\eta_1} < 1 \tag{55}$$

To determine η_1, one can use the condition $p(\eta_1) = P_e$. As a result, from (54) we have

$$P_e - P_w = \left[\frac{na^2}{2}\frac{1-n}{1+n}\right]^{n/(1-n)} \eta_1^{(1+n)/(1-n)} J_n(\eta_1) \tag{56}$$

in which $$J_n(\eta_1) = \int_0^1 (1-\xi^2)^{n/(1-n)}d\xi \tag{57}$$

Does not depend on η_1. Using the following notations

$$\Delta_p = P_e - P_w \quad \text{and} \quad L_n = \int_0^1 (1-\xi^2)^{n/(1-n)}d\xi \tag{58}$$

From (56) one obtains

$$\eta_1 = \left[\frac{\Delta_p}{L_n}\right]^{(1+n)/(1-n)} \left[\frac{na^2}{2}\frac{1-n}{1+n}\right]^{-n/(1-n)} = \text{constant} \tag{59}$$

Where as from (45), l (t) is

$$l(t) = \left[\frac{\Delta_p}{L_n}\right]^{(1+n)/(1-n)} \left[\frac{na^2}{2}\frac{1-n}{1+n}\right]^{-n/(1-n)} t^{n(1+n)} \tag{60}$$

Relation (60) determines the front location at any time t, L_n may be obtained from (58) exactly for certain values of n, as it will be shown further , or approximately for any n, using some numerical integration formulas.

As equation (60) indicates, the front location depends strongly on the rheological parameters n and a^2 i.e. on k/η_{ef}

For the capillary tube model k/η_{ef} is related to n and H by relation (2).

$$\frac{k}{\mu_{ef}} = \frac{1}{2H}\left(\frac{n\phi}{1+3n}\right)^n \left(\frac{8k}{\phi}\right)^{(n+1)/2} \tag{61}$$

in which the rheological parameter H denotes the consistency index, k denotes the permeability , and ϕ denotes the porosity.

As is well known, for power law fluids the apparent viscosity is shear rate dependent. Consequently, the rheological effects on the front location will depend on the pressure drop Δp. We again find this dependence by the relation (60).

The following relations illustrate how these effects appear on l(t), for some particular cases.

$$n = \frac{1}{3};\ l(t) = \frac{\left(\frac{2\Delta p}{\pi}\right)^{1/2} t^{1/4}}{\left(\frac{a^2}{24}\right)^{1/4}}$$

$$n = \frac{1}{2};\ l(t) = \frac{\left(\frac{3\Delta p}{2}\right)^{1/3} t^{1/3}}{\left(\frac{a^2}{12}\right)^{1/3}}$$

$$n = \frac{3}{4};\ l(t) = \frac{\left(\frac{35\Delta p}{16}\right)^{1/7} t^{3/7}}{\left(\frac{3a^2}{56}\right)^{3/7}}$$

For a Newtonian fluid, i.e. n = 1, from (51) we have

$$l(t) = \sqrt{\frac{4t}{a^2}}\ ;\ a^2 = \frac{\mu}{k}\ (C_0\varphi + C_p)$$

This relation shows that the front location is independent of Δp. Once η_1 is determined by relation (59), the pressure distribution may be obtained and expressed as

$$\frac{p(\eta)\ -\ P_w}{p_e\ -\ P_w} = \frac{J_n(\eta)}{J_n(\eta_1)};\ 0 < \frac{\eta}{\eta_1} < 1 \tag{62}$$

in which $J_n(\eta)$, given by relation (55), may be calculated for any n and η, using an adequate numerical integral formula, but for certain values of n an exact solution in closed form can be obtained.

For example, using the transformation $\xi = \sin\alpha$ in (55), we have

$$J_n(\eta) = \int_0^{\arcsin \eta/\eta_1} (\cos\alpha)^{(1+n)/(1-n)} d\alpha \tag{63}$$

Where as one can see that $J_n(\eta)$ may be determined by means of the formulas

$$\int (\cos\alpha)^{2k} d\alpha = \frac{1}{2^{2k}}\binom{2k}{k}\alpha + \frac{1}{2^{2k-1}}\sum_{m=0}^{k-1}\binom{2k}{m}\frac{\sin(2k-2m)\alpha}{2k-2m}$$

for $\frac{1+n}{1-n} = 2k$ or $n = \frac{2k-1}{2k+1}$, k =1, 2, ... (64)

and $\int (\cos\alpha)^{2k+1} d\alpha = \frac{1}{2^{2k}}\sum_{m=0}^{k}\binom{2k+1}{m}\frac{\sin(2k-2m+1)\alpha}{2k-2m+1}$

for $\frac{1+n}{1-n} = 2k+1$ or $n = \frac{k}{k+1}$, k = 1, 2, 3, ... (65)

In order to illustrate the flow deviations from Newtonian behavior, where n = 1, several illustrative examples of practical interest will be considered, namely $n = \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{3}{5}$, and $\frac{5}{7}$. For these cases, the function $J_n(\eta)$ defined by (63) may be expressed as

$$J_{1/3}(\eta) = \frac{1}{2}\left[\arcsin\frac{\eta}{\eta_1} + \frac{\eta}{\eta_1}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^{1/2}\right] \quad (66)$$

$$J_{1/2}(\eta) = \frac{\eta}{\eta_1}\left(1 - \frac{1}{3}\frac{\eta^2}{\eta_1^2}\right)^{1/2} \quad (67)$$

$$J_{3/4}(\eta) = \frac{24}{35}\frac{\eta}{\eta_1} - \frac{8}{35}\frac{\eta^3}{\eta_1^3} + \frac{6}{35}\frac{\eta}{\eta_1}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^2 + 5\frac{\eta}{\eta_1}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^3 \quad (68)$$

$$J_{3/5}(\eta) = \arcsin\frac{\eta}{\eta_1} + \frac{3}{8}\frac{\eta}{\eta_1}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^{1/2} + \frac{1}{4}\frac{\eta}{\eta_1}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^{3/2} \quad (69)$$

$$J_{5/7}(\eta) = \frac{5}{16}\arcsin\frac{\eta}{\eta_1} + \frac{\eta}{\eta_1}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^{1/2}\left[\frac{5}{16} + \frac{5}{24}\left(1 - \frac{\eta^2}{\eta_1^2}\right) + \frac{1}{6}\left(1 - \frac{\eta^2}{\eta_1^2}\right)^2\right] \quad (70)$$

From these relation for $\eta = \eta_1$, we have the values of $L_n = J_n(\eta_1)$ as :

$$L_{1/3} = \frac{\pi}{4};\ L_{1/2} = \frac{2}{3};\ L_{3/4} = \frac{16}{35};\ L_{3/5} = \frac{3\pi}{16};\ \text{and } L_{5/7} = \frac{5\pi}{32} \quad (71)$$

A problem of special interest in oil reservoir engineering is the prediction of the flow-rate decline in time at the outface flow, i.e. at x = 0, during the natural depletion mechanism under constant pressure. Since the pressure distribution is known from (54), then the modified Darcy's law allows the knowledge of the flow-rate variation expressed in terms of variables x and t. This variation may be written as

$$Q(x, t) = F\left(\frac{k}{\mu_{ef}}\right)^{1/n} t^{-1/(1+n)}\left[\frac{n(1-n)a^2}{2(1+n)}\left(\eta_1^2 - \frac{x^2}{t^{2n/(1+n)}}\right)\right]^{1/(1-n)} \quad (72)$$

in which η_1 is known from (59). At the outface flow, i.e. at x = 0, the relation (72) determines the flow-rate decline in time.

5. CONCLUSION:

The transformation $\eta = x^{-n/n+1}$ is used to find the non-linear ordinary differential equation for the power law fluids associated with the interpretation of the well-known flow test of short duration. This transformation is obtained by infinitesimal group method. The analytical solution of the equation is given in the form of complimentary error function. Also the flow deviations from Newtonian behavior are illustrated with several cases in terms of Bessel's function and the flow variation is expressed in equation (72).

REFERENCES:

[1] Earloughes, R.C., Jr. Advances in well test analysis. Monograph Series. Soc. Petroleum Eng. (1977).

[2] Matthews, S.C. and Russel, G.D., Pressure build up and flow tests in well. Monograph Series Vol. 1, Soc. Petroleum Eng., (1967).

[3] Pascal, H. Int. J., Numerical Methods in Geomechanics 7(3), (1983), 289.

[4] Pascal, H., Dynamics of Moving Interface in Porous Media for Power Law Fluids With Yield Stress. Int. J. Engg Sci. 22(5), (1984), 577.

[5] Pascal, H., Some Problems Related to The Quantitative Evaluation Of Physical Properties Of The Porous Medium From Flow Tests With Non-Newtonian Fluids. Int. J. Engg Sci. 23(3), (1985), 307.

[6] Seshadri, R. & Na T.Y., Group Invariance In Engineering Boundary Value Problems, Springer-verlay, New York, Berlin Heidelberg, Tokyo,(1985). 157-168.

****

