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WEAKENED FIELD EQUATIONS IN GENERAL RELATIVITY ADMITTING THE "GENERALIZED PLANE SYMMETRIC" METRIC

[1]S R Bhoyar* & [2]A. G. Deshmukh

[1]*Department of Mathematics, College of Agriculture, Darwha -445202 (India)*
E-mail: sbhoyar68@yahoo.com

[2]*Ex. Reader and Head, Department of Mathematics, Govt. Vidarbha Institute of Science and Humanities, Amravati, Joint Director, Higher Education, Nagpur Division, Nagpur - 440 022(India)*
E-mail: dragd2003@yahoo.co.in

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ABSTRACT

*In this paper, we propose to obtain the generalized plane gravitational waves in a set of five vacuum weakened equations in the generalized space-time introduced by us having plane symmetry in the sense of Taub [**Ann.**Math.**53**,472 (1951)]. These vacuum field equations has been suggested as alternatives to the Einstein vacuum field equations of general relativity. Furthermore we have discussed some particular cases.*

Keywords: *Plane symmetry, Plane gravitational waves, Curvature tensor, Ricci tensor, weakened field equations etc.*

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1. INTRODUCTION

The theory of plane gravitational waves in general relativity has been introduced by many investigators like Einstein and Rosen (1937); Bondi, Pirani and Robinson (1959); Takeno (1957, 1961). Takeno (1961) has discussed the mathematical theory of plane gravitational waves and classified them into two categories, namely, $Z = (z - t)$ or (t/z)- type wave according as the phase function $Z = (z - t)$ or (t/z)- type wave respectively. According to him, a plane wave g_{ij} is a non-flat solution of Ricci tensor $R_{ij} = 0$ in general relativity and in some suitable coordinate system; all the components of the metric tensor are functions of a single variable $Z = Z(z, t)$ (i.e. a phase function).

A generalization of plane gravitational waves has been sought in this paper by assuming a general phase function $Z = Z(z, t)$ and taking the line element represented by us (2011) in the form

$$ds^2 = -A\left(dx^2 + dy^2\right) - \phi^2 B dz^2 + B dt^2, \tag{1.1}$$

where A and B are functions of a single variable *Z* and $\phi = \left(Z_{,3}/Z_{,4}\right)$. The quadratic form $A\left(dx^2 + dy^2\right)$ is positive definite $B > 0$ and ϕ is real.

In general, ϕ is not necessarily taken as functions of *Z*. The metric (1.1) does not confirm the Takeno's concept of plane gravitational waves completely; otherwise it admits plane wave solutions. The aim of generalized plane gravitational waves has been described by assuming space-time metric (1.1) wherein a certain components g_{ij} (i.e. gravitational potentials) are not necessarily a function of $Z = (z - t)$ or (t/z)- type wave. This approach has advantage that the general solutions so obtained reduce to plane wave solutions under certain conditions and the results corresponding to plane gravitational waves can be deduced by suitable choice of the phase function *Z*. It is not difficult to verify that the metric (1.1) admits a parallel null vector field and is also a gravitational null vector field, that conditions which characterize the generalized plane gravitational waves. Accordingly the space-time represented by the

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***Corresponding author: S R Bhoyar*, *E-mail: sbhoyar68@yahoo.com**

metric (1.1) posses the plane symmetry in the sense of Taub [**Ann.**Math.**53**,472 (1951)], hence called as "generalized plane symmetric" metric and it admits generalized plane gravitational waves which is different from the Takeno's (1957,1961) concept of plane wave.

Lovelock (1967a, b) has considered a set of five weakened field equations (WFE) in vacuum, namely,

$$R^h_{ijk;h} = 0, \tag{I}$$

$$(-g)^{\frac{1}{4}}\left[g^{ih}R_{kj;ih} - g^{ih}R_{ij;kh} + \frac{1}{6}R_{;kj} - \frac{1}{6}g_{jk}g^{ih}R_{;ih} - R^{ih}C_{jhik} + \frac{R}{6}g^{ih}C_{jhik}\right] = 0, \tag{II}$$

$$(-g)^{\frac{1}{2}}\left[g^{hj}g^{ki}(2R_{jlim}R^{ml} + g^{ml}R_{ij;lm} - R_{;ij}) - \frac{1}{2}g^{hk}(R^l_m R^m_l - g^{lm}R_{;lm})\right] = 0, \tag{III}$$

$$(-g)^{\frac{1}{2}}\left[(g^{hk}g^{rs} - \frac{1}{2}g^{hr}g^{ks} - \frac{1}{2}g^{hs}g^{kr})R_{;rs} + R(R^{kh} - \frac{1}{4}g^{kh}R)\right] = 0, \tag{IV}$$

and $R^{ij}_{;k} = 0,$ (V)

where a semicolon (;) followed by an index denotes covariant differentiation and C_{jhik} is the Weyl curvature tensor defined by

$$C_{jhik} = R_{jhik} - \frac{1}{2}(R_{ji}g_{hk} - R_{hi}g_{jk} - R_{jk}g_{hi} + R_{hk}g_{ij}) + \frac{R}{6}(g_{ij}g_{hk} - g_{hi}g_{jk}). \tag{1.2}$$

Kilmister and Newman (1961) has originally proposed the vacuum weakened field equations (I)-(V). These field equations are suggested as various alternatives to the Einstein field equation of general relativity in vacuum. The Einstein vacuum field equation of general relativity is given by

$$R_{ij} = 0. \tag{1.3}$$

The solution of (1.3) together with the trajectories of test particles (geodesics hypothesis) give agreement with experiment. Lovelock (1967a, b) obtained the solutions of WFE field equations (I)-(V) in a spherically symmetric space-time and he proved to be gravitationally unphysical metric by geodesics hypothesis in the sense that these solutions correspond to the static situation of an isolated mass at origin which repels the test particles. Consequently the physical aspects of weakened field equations are not well established through many researchers (for examples: Thompson 1963; Kilmister 1966; Rund 1967; Lovelock 1967a, b; Swami 1970; Lal and Singh 1973; Lal and Pandey 1975; Pandey 1975; Rane and Katore 2009; Bhoyar and Deshmukh 2011 etc.) have tried to investigate the solutions to interpret the useful results. Thompson (1963) made detailed study of these field equations and concluded that they are too weak. The various alternative vacuum weakened field equations are weaker than the Einstein vacuum field equations in the sense that they each admit (1.3) as a solutions and hence they have been called WFE. Swami (1970) has solved three solutions of the weakened field equations $R_{ij;k} - R_{ik;j} = 0$ with $R_{ij} \neq 0,\ R_{ij} \neq \lambda g_{ij}$ and has discussed the geometrical and dynamical properties of these solutions.

Furthermore R^h_{ijk} satisfies the Bianchi identities

$$R^h_{ijk;m} + R^h_{ikm;j} + R^h_{imj;k} = 0. \tag{1.4}$$

From which

$$R^i_{j;i} = \frac{1}{2}R_{;j}\ , \tag{1.5}$$

where $R^i_j = g^{ih}R_{hj}$.

Recently we have investigated that the metric (1.1) be the plane wave solutions of field equations of general relativity. In this paper, we have studied the generalized plane gravitational waves of vacuum weakened field equations (I)-(V) in the metric (1.1). We have obtained some useful results in the form of theorems under curvature properties with some particular cases and conclusions.

2. GENERALIZED PLANE WAVE METRIC AND CURVATURE PROPERTIES

The components of contravariant tensor g^{ij} from the metric (1.1) are

$$g^{11}=g^{22}=-\frac{A}{m},\,,\; g^{33}=-\frac{1}{\phi^2 B},\; g^{44}=\frac{1}{B} \tag{2.1}$$

where $m=A^2$ and $\varphi=\frac{Z_{,3}}{Z_{,4}}=\frac{u}{v}(u=Z_{,3}\neq 0$ and $v=Z_{,4}\neq 0)$.

The non vanishing components of Christoffel symbol of second kind are

$$\Gamma^3_{11}=-\frac{\overline{A}u}{2B\phi^2},\; \Gamma^4_{11}=\frac{\overline{A}v}{2B},\; \Gamma^3_{22}=-\frac{\overline{A}u}{2B\phi^2},\Gamma^4_{22}=\frac{\overline{A}v}{2B},\; \Gamma^1_{13}=\frac{(\overline{A}A)u}{2m},$$
$$\Gamma^1_{14}=\frac{(\overline{A}A)v}{2m},\; \Gamma^2_{23}=\frac{(A\overline{A})u}{2m},\; \Gamma^2_{24}=\frac{(A\overline{A})v}{2m},\; \Gamma^3_{34}=\frac{\overline{(\phi^2 B)}v}{2B\varphi^2},$$
$$\Gamma^4_{33}=\frac{\overline{(\phi^2 B)}v}{2B},\; \Gamma^3_{33}=\frac{\overline{(\phi^2 B)}u}{2B\phi^2},\; \Gamma^4_{34}=\frac{\overline{B}u}{2B},\; \Gamma^3_{44}=\frac{\overline{B}u}{2B\phi^2} \text{ and } \Gamma^4_{44}=\frac{\overline{B}v}{2B}. \tag{2.2}$$

Using (1.1), (2.1) and (2.2), the components of curvature tensor R_{ijkl} are as follows:

$$\frac{R_{1313}}{u^2}-\frac{\overline{A}}{2}\left(\frac{\overline{u}}{u}\right)+\overline{A}\left(\frac{\overline{\phi}}{\phi}\right)=\frac{R_{1314}}{uv}-\frac{\overline{A}}{2}\left(\frac{\overline{v}}{v}\right)+\frac{\overline{A}}{2}\left(\frac{\overline{\phi}}{\phi}\right)=\frac{R_{1414}}{v^2}-\frac{\overline{A}}{2}\left(\frac{\overline{v}}{v}\right)$$
$$=\frac{R_{2323}}{u^2}-\frac{\overline{A}}{2}\left(\frac{\overline{u}}{u}\right)+\overline{A}\left(\frac{\overline{\phi}}{\phi}\right)=\frac{R_{2324}}{uv}-\frac{\overline{A}}{2}\left(\frac{\overline{v}}{v}\right)+\frac{\overline{A}}{2}\left(\frac{\overline{\phi}}{\phi}\right)=\frac{R_{2424}}{v^2}-\frac{\overline{A}}{2}\left(\frac{\overline{v}}{v}\right)$$
$$=\frac{\overline{\overline{A}}}{2}-\frac{\overline{A}^2}{2A}-\frac{(A\overline{A}^2)}{4m}. \tag{2.3}$$

The components of covariant and contravariant Ricci tensor, from (2.1) and (2.2) are as follows:

$$\frac{R_{33}}{u^2}=\frac{R_{34}}{uv}=\frac{R_{43}}{uv}=\frac{R_{44}}{v^2}=\kappa \text{ (Say)},$$
$$\frac{R^{33}}{v^2}=\frac{\kappa}{\phi^2 A^2},\; \frac{R^{34}}{uv}=\frac{R^{43}}{uv}=-\frac{\kappa}{\phi^2 A^2},\; \frac{R^{44}}{v^2}=\frac{\kappa}{A^2} \tag{2.4}$$

and all other $R_{ij}=R^{ij}=0$, where

$$\kappa=\frac{\overline{\overline{m}}}{2m}-\frac{\overline{m}^2}{4m^2}-\frac{\overline{m}\,\overline{B}}{2mB}-\frac{(\overline{A}^2)}{2m}+\frac{\overline{m}}{2m}\left(\frac{v_{,4}}{v^2}\right). \tag{2.5}$$

From (2.4), we obtain

$$\frac{R_{33;33}}{u^2}=\frac{R_{33;34}}{uv}=\frac{R_{34;33}}{uv}=\frac{R_{33;44}}{v^2}=u^2\psi$$

and $$\frac{R_{34;34}}{u^2}=\frac{R_{34;44}}{uv}=\frac{R_{44;34}}{uv}=\frac{R_{44;44}}{v^2}=v^2\psi\,, \tag{2.6}$$

where $$\psi=\overline{\overline{\kappa}}+5\overline{\kappa}\left(\frac{v_{,4}}{v^2}\right)+6\kappa\left(\frac{v_{,4}}{v^2}\right)^2+\frac{2\kappa}{v}\left(\frac{v_{,4}}{v^2}\right)_{,4}-\frac{12\kappa\overline{B}}{B}\left(\frac{v_{,4}}{v^2}\right)$$
$$-\frac{5\overline{\kappa}\overline{B}}{B}-\frac{2\kappa\overline{\overline{B}}}{B}+\frac{8\kappa\overline{B}^2}{B^2}. \tag{2.7}$$

Here a bar (−) overhead letter denotes the differentiation with respect to Z

(i.e. $\overline{\kappa}=\frac{\partial\kappa}{\partial Z}$ and $\overline{\overline{\kappa}}=\frac{\partial^2\kappa}{\partial Z^2}$).

Also for metric (1.1), we deduce, from (2.4), that

(a) The scalar curvature R defined by $R=g^{ij}R_{ij}$ is zero i.e $R = 0$,

(b) $g = \det(g_{ij})=-mB^2\phi^2$,

(c) $R^l_m R^m_l = R_{im}R^{im}=0$ and

(d) $R_{jlim}R^{ml}=0$. (2.8)

3. SOLUTIONS OF WEAKENED FIELD EQUATIONS

The metric (1.1) is non-conformally flat which implies that Weyl curvature tensor (1.2) in view of (2.8a) (i.e. $R = 0$) reduces to

$$C_{jhik}=R_{jhik}-\frac{1}{2}(R_{ji}g_{hk}-R_{hi}g_{jk}-R_{jk}g_{hi}+R_{hk}g_{ij})\,. \tag{3.1}$$

From Bianchi identities (1.4), we find

$$R^h_{ijk;h}=R_{ij;k}-R_{ik;j} \tag{3.2}$$

where $R_{ij;k}=R_{ij,k}-\Gamma^m_{jk}R_{im}-\Gamma^m_{ik}R_{mj}$.

On substituting the components of Γ^k_{ij} and R_{ij} from (2.2) and (2.4) respectively in R.H.S. of equation (3.2), it is seen that

$$R_{ij;k}-R_{ik;j}=0 \tag{3.3}$$

or equivalently, from (3.2),

$$R^h_{ijk;h}=0, \tag{3.4}$$

which is weakened field equation (I).

Also it follows from (3.3) that

$$R_{ij;kh}=R_{ik;jh}. \tag{3.5}$$

In view of equations (2.8a) and (3.5), the weakened field equation (II) reduces to

$$(-g)^{\frac{1}{4}}R^{ih}C_{jhik}=0. \tag{3.6}$$

Using (3.1) in (3.6), we obtain

$$(-g)^{\frac{1}{4}}\left[R^{ih}R_{jhik}-\frac{1}{2}(R_{ji}R^{i}_{k}-R^{ih}R_{hi}g_{jk}-R^{i}_{i}R_{jk}+R^{h}_{j}R_{hk})\right]=0 \qquad (3.7a)$$

which reduces to

$$(-g)^{\frac{1}{4}}\left[R^{ih}R_{jhik}-\frac{1}{2}(g_{jk}R^{k}_{i}R^{i}_{k}-g_{jk}R^{ih}R_{hi}-RR_{jk}+g_{jk}R^{j}_{h}R^{h}_{j})\right]=0\,. \qquad (3.7b)$$

Using (2.8a, c, d) in (3.7b), it is seen that no term remains in LHS and hence (3.7b) i.e. (3.6) is identically satisfied.

It is observed that the weakened field equation (IV) is satisfied by $R = 0$ (i.e. 2.8a) alone. Hence the following theorem:

Theorem 1: The g_{ij} given by (1.1) is a solution of weakened field equation (I), (II) and (IV).

Theorem 2: A necessary and sufficient condition that g_{ij} given by (1.1) be a solution of weakened field equation (III) is $\psi = 0$, where ψ is given by (2.7).

Proof: The weakened field equation (III) for $R = 0$ becomes

$$(-g)^{\frac{1}{2}}\left[g^{hj}g^{ki}(2R_{jlim}R^{ml}+g^{ml}R_{ij;lm})-\frac{1}{2}g^{hk}R^{l}_{m}R^{m}_{l}\right]=0\,. \qquad (3.8)$$

In view of (2.8c, d), the equation (3.8) reduces to

$$(-g)^{\frac{1}{2}}g^{hj}g^{ki}g^{ml}R_{ij;lm}=0\,. \qquad (3.9)$$

Equation (3.9) is identically satisfied for all values of h, k expect h, k = 3 and 4. For h, k = 3, 4 equation (3.9) after simplification yields

$$\psi = 0\,.$$

Conversely if $\psi = 0$, then in view of equations (2.4) and (2.6), the weakened field equation (III) is identically satisfied. This proves the theorem.

Theorem 3: A necessary and sufficient condition that g_{ij} given by (1.1) be a solution of weakened field equation (V) is $\overline{\left(\frac{\kappa}{B^2}\right)}+\frac{2\kappa}{B^2}\left(\frac{\nu_{,4}}{\nu^2}\right)=0$.

Proof: Let g_{ij} given by (1.1) be a solution of weakened field equation (V) i.e.

$$R^{ij}_{;k}=0, \qquad (3.10)$$

where $R^{ij}_{;k}=\frac{\partial R^{ij}}{\partial x^{k}}+\Gamma^{i}_{sk}R^{sj}+\Gamma^{j}_{sk}R^{is}$.

In view of equations (2.2) and (2.4), the equation (3.10) is identically satisfied for all values of i, j, k except i, j, k = 3, 4.

For i, j, k = 3, 4 equation (3.10) gives on simplification

$$\overline{\left(\frac{\kappa}{B^2}\right)}+\frac{2\kappa}{B^2}\left(\frac{\nu_{,4}}{\nu^2}\right)=0\,.$$

Conversely, if $\overline{\left(\frac{\kappa}{B^2}\right)} + \frac{2\kappa}{B^2}\left(\frac{\nu_{,4}}{\nu^2}\right) = 0$, then the weakened field equation (V) is identically satisfied and hence the theorem is proved.

4. SOME PARTICULAR CASES

Case-1: If we take *A* and *B* are functions of $Z = (z - t)$ and $\varphi = -1$, then the metric (1.1) reduces to Takeno's [7] plane symmetric metric

$$ds^2 = -A(dx^2 + dy^2) - Bdz^2 + Bdt^2 \tag{4.1}$$

and non-vanishing components of R_{ij} turn out to be

$$R_{33} = -R_{34} = -R_{43} = R_{44} = \frac{\overline{\overline{A}}}{A} - \frac{\overline{A}^2}{2A^2} - \frac{\overline{A}\,\overline{B}}{AB}. = \alpha \text{ (say).}$$

In such case, plane wave metric satisfied the curvature properties given in (2.8). Hence we have the following results:

(1) The g_{ij} given by metric (4.1) is a solution of weakened field equations (I), (II) and (IV).

(2) The g_{ij} given by metric (4.1) is a solution of weakened field equation (III) if and only if $\overline{\overline{\alpha}} = 0$, where $\alpha = R_{33}$.

(3) The g_{ij} given by metric (4.1) is a solution of weakened field equation (V) if and only if $\overline{\alpha} = \frac{2\alpha\overline{B}}{B}$.

Case-2: If we take *A* and *B* are functions of $Z = (t/z)$ and $\phi = -Z$, then the metric (1.1) reduces to our deduced plane symmetric metric [8],

$$ds^2 = -A(dx^2 + dy^2) - Z^2Bdz^2 + Bdt^2 \tag{4.2}$$

and non-vanishing components of R_{ij} turn out to be

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = -\frac{R_{43}}{Z} = R_{44} = \frac{1}{z^2}\left[\frac{\overline{\overline{A}}}{A} - \frac{\overline{A}^2}{2A^2} - \frac{\overline{A}\,\overline{B}}{AB}.\right] = \frac{\alpha}{z^2} \text{ (say).}$$

In such case, plane wave metric satisfied the curvature properties given in (2.8). Hence we have the following results:

(1) The g_{ij} given by metric (4.2) is a solution of weakened field equations (I), (II) and (IV).

(2) The g_{ij} given by metric (4.2) is a solution of weakened field equation (III) if and only if $\overline{\overline{\alpha}} = 0$, where $\alpha = \frac{z^2 R_{33}}{Z^2}$.

(3) The g_{ij} given by metric (4.1) is a solution of weakened field equation (V) if and only if $\overline{\alpha} = \frac{2\alpha\overline{B}}{B}$.

5. CONCLUSIONS

In this paper,

(1) The generalized plane wave g_{ij} given by metric (1.1) representing a non-conformally flat space-time with the scalar curvature zero (*R* = 0) is a solution of the weakened field equation (IV).
(2) It is also a solution of WFE (I), (II) and (III) under the curvature properties (2.8).
(3) In non-conformally flat space-time (1.1), the solution of (1.4) follows from Bianchi identity gives (1.5) and (3.2).

(4) When $R = 0$, then $R_{;j} = 0$ and (1.5) implies that $R^{i}_{j;i} = 0$. It is observed that the metric (1.1) is a non-flat and it is an exact solution of $R_{ij} = 0$ if and only if $\kappa = 0$.

REFERENCES

[1] H. Takeno **(1961):** The mathematical theory of plane gravitational waves in general relativity, *Sci. Rep. Inst. Theo. Phy .Hiroshima University, Japan.*

[2] **D. Lovelock (1967):** Weakened Field Equations in general relativity admitting an "unphysical" metric, *Commun. Math. Phys., 5, 264-214.*

[3] S.N.Pandey **(1975):** Plane wave solutions of Weakened Field Equations in Peres space-time, *Tensor N.S., 29, 264.*

[4] **Lal K.B and Pandey S.N (1975):** On Plane wave-like solutions of Weakened Field Equations in general relativity, *Tensor N.S., 29, 297-298.*

[5] **Chirde.V, Metkar. A and Deshmukh A.** (2011): $(z-t)$-Type Plane wave solutions of Weakened Field Equations.

[6] **Rane. R.S and Katore S. D (2009):** Generalized Plane gravitational wave of Weakened Field Prespace-time Journal, Nov.2011, Vol. 2. Issue 11 1845-1851 general relativity, *J.Math.Phys.50.053504* (2009).

[7] **H. Takeno (1957):** *Tensor, N.S; Vol.7 No.2pp. 97-102 and 34-58.*

[8] **Bhoyar S.R and Deshmukh A.G (2011):** The Z=Z (t/z)-type Plane wave solutions of the field equations of general relativity in a Plane symmetric space-time. *Int. Jour .of Theo. Phy. DOI 10.1007/s10773-011-0853-4.*

[9] **H. Rund (1967):** Invariant theory of variational problems for geometric objects, *Tensor (N.S) 12.*

[10] **Thompson A.H (1963):** The investigations of set of weakened field equations for general relativity, Aerospace*Research Laboratories, U.S.A.F.*

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