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CONTRACTION OF IDEALS IN BOOLEAN LIKE SEMI RING

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ABSTRACT

In this paper, we introduce the concept of contraction of an ideal in a Boolean like semi ring and obtain certain properties of these special classes of ideals .We obtain this in theorem 2.5 below

Mathematics Subject Classification:*16Y30, 16 Y 60.*

Keywords: *Boolean like semi rings and Contraction of an ideal.*

INTRODUCTION

The concept of Boolean like semi rings is due to Venkateswarlu, Murthy and Amaranth in [6].

The present paper is divided into 2 sections. In section 1, we give preliminary concepts and results regarding Boolean like semi rings. In section 2, we establish that Let R and S be two Boolean like semi rings. Let I be an ideal of S and let $f : R \rightarrow S$ be a homomorphism then $f^{-1}(I)$ is an ideal of R (see theorem 2.3). Further we introduce the concept of contraction of an ideal and study of of its properties in Boolean like semi ring (see theorems 2.5)

1. PRELIMINARIES: BOOLEAN LIKE SEMI RINGS AND ITS PROPERTIES

We recall certain definitions and results concerning Boolean like semi rings from [6]

Definition 1.1: A non empty set R together with two binary operations + and . satisfying the following conditions is called a Boolean like semi ring

1. (R, +) is an abelian group
2. (R, .) is a semi group
3. $a.(b+c) = a.b + a.c$ for all $a, b, c \in R$
4. $a + a = 0$ for all $a \in R$
5. $ab(a+b+ab) = ab$ for all $a, b \in R$.

Example 1.2: Let R = {0, a, b, c}. The binary operations + and . are defined as follows

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	a	b	c

Then (R, +, .) is a Boolean like semi ring. We observe that $cab \neq cba$.

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Example 1.3: Let R = {0, x, y, z}. The binary operations + and. are defined as follows

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

.	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Then (R, +,.) is a Boolean like semi ring. We note that abc = acb for all a, b, c $\in$ R.

Example 1.4: Let R={0,p,q,1 }. The binary operations + and. are defined as follows

+	0	p	q	1
0	0	p	q	1
p	p	0	1	q
q	q	1	0	p
1	1	q	p	0

.	0	p	q	1
0	0	0	0	0
p	0	0	p	p
q	0	0	q	q
1	0	p	q	1

Then R is a Boolean like semi ring. It is clear that a.1 = 1.a = a for all a $\in$ R.

The following are properties of Boolean like semi rings

Let R be a Boolean like semi ring. Then

Lemma 1.5: For a $\in$ R, a.0 =0

Lemma 1.6: For a $\in$ R, $a^4 = a^2$ (weak idempotent law)

Remark 1.7: From the above lemma 1.6, we have $a^{2n} = a^2$, for any integer n > 0

Lemma 1.8: If R is a Boolean like semi ring then, $a^n = a$ or a^2 or a^3 for any integer n > 0 and $0a^2 = 0$ and $0(a + a^2) = 0a$ for all a $\in$ R.

Definition 1.9: A Boolean like semi ring R is said to be **weak commutative** if abc = acb, for all a, b, c $\in$ R.

Lemma 1.10: If R is a Boolean like semi ring with weak commutative then 0.a = 0.for all a $\in$ R

Lemma 1.11: Let R be Boolean like semi ring then for any a, b $\in$ R and for any integers m, n > 0, 1. $a^m a^n = a^{m+n}$ 2. $(a^m)^n = a^{mn}$ 3. $(ab)^n = a^n b^n$ if R is weak commutative.

Definition 1.12: An element 1$\in$ R is said to be unity if a1=1a = a, for all a$\in$R. If a1 = a, then 1 is called right unity and if 1a = a, then 1 is called left unity.

Definition 1.13: A non empty subset I of R is said to be an ideal if

1. (I,+) is a sub group of (R,+) , i.e , for a,b $\in$ I $\Rightarrow$ a + b $\in$ I
2. ra $\in$I, for all a $\in$ I , r $\in$ R, i.e RI $\subseteq$ I
3. (r + a)s + rs $\in$ I , for all r, s $\in$R , a $\in$ I

Theorem 1.14: The set of all nilpotent elements of a weak commutative Boolean like semi ring form an ideal.

Theorem 1.15: If I is an ideal of a weak commutative Boolean like semi ring R then the radical of I, denoted by r(I) and is defined as{ x$\in$R / x^n $\in$I, for some positive integer n } is an ideal of R.

Theorem 1.16: If I and J are ideals of a Boolean like semi ring R then I + J is an ideal of R.

Theorem 1.17: If I and J are ideals of a Boolean like semi ring R then I ∩ J is an ideal of R.

Theorem 1.18: If I and J are left ideals of a Boolean like semi ring R then the product $IJ = \{a_1b_1+a_2b_2+\text{----}+a_nb_n \,/ a_i\in I ,\, b_i\in J\}$ is a left ideal of R.

Theorem 1.19:. If I and J are ideals of a weak commutative Boolean like semi ring R then

i. $I \subseteq r(I)$
ii. $r(I\cap J) = r(I) \cap r(J)$
iii. If $I \subseteq J$ then $r(I) \subseteq r(J)$
iv. $r(r(I)) = r(I)$
v. $r(I + J) = r(r(I) + r(J))$
vi. If R has right unity 1 then $r(I) = R$ if and only if $I = R$
vii. $r(IJ) = r(I \cap J)$

Definition 1.20: Let R be a Boolean like semi ring. Let I and J be ideals of R. Then their ideal quotient is denoted by (I: J) and is defined by $(I: J) = \{x \in R / Jx \subseteq I\}$.

Now we have the following

Theorem 1.21: If R is a weak commutative Boolean like Semi ring and I, J and K are ideals of R then the following hold.

1. $(I:J) = \{x \in R/ Jx \subseteq I\}$ is an ideal of R.
2. $I \subseteq (I:J)$
3. $((I:J):K) = (I:JK)$
4. $(\cap I_i : J) = \cap_i (I_i :J)$

2. CONTRACTION OF IDEALS IN A BOOLEAN LIKE SEMI RING

We now introduce the concept of contraction of an ideal in a Boolean like semi ring

Definition 2.1: If R and R′ are Boolean like semi rings. A mapping $f: R\to R'$ is said to be a Boolean like semi ring homomorphism (or simply homomorphism) of R into R′if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Theorem 2.2: A mapping $f: R \to R'$ is a homomorphism of a Boolean like semi ring R into R′ then $f(0) = 0'$ and f(R) is weak commutative if R is weak commutative.

Theorem 2.3: Let R and S be two Boolean like semi rings . Let I be an ideal of S and let $f : R \to S$ be a homomorphism then $f^{-1}(I)$ is an ideal of R.

Proof: $f^{-1}(I) = \{x \in R / f(x) \in I\}$

From theorem 2.2, we have $0' = f(0) \in I \Rightarrow 0 \in f^{-1}(I)$

Hence $f^{-1}(I)$ is non empty and $f^{-1}(I) \subseteq R$.

Let $x , y \in f^{-1}(I) \Rightarrow f(x) , f(y) \in I \Rightarrow f(x) + f(y) \in I$

$$\Rightarrow f(x + y) \in I \Rightarrow x + y \in f^{-1}(I)$$

Hence $(f^{-1}(I), +)$ is a sub group of $(R , +)$.

Let $x \in f^{-1}(I)$ and $r , s \in R \Rightarrow f(x) \in I$

Consider $f(rx) = f(r) f(x) \in I$ (I is an ideal of S)
$\Rightarrow rx \in f^{-1}(I)$

Consider $f((r + x) s + rs) = f((r + x) s) + f(rs)$
$= f(r +x) f(s) + f(r) f(s)$
$= [f(r) + f(x)] f(s) + f(r) f(s) \in I$ (I is an ideal of S)

Thus $f^{-1}(I)$ is an ideal of R.

Definition 2.4: Let R and S be two Boolean like semi rings . If I is an ideal of S and $f : R \rightarrow S$ is a homomorphism then $f^{-1}(I)$ is an ideal of R, called the contraction of I and is denoted by I^c .

Theorem 2.5: Let I and J be two ideals of S then

(i) $(I^c + J^c) \subseteq (I + J)^c$
(ii) $(I \cap J)^c = (I^c \cap J^c)$
(iii) $I^c J^c \subseteq (IJ)^c$
(iv) $[r(I)]^c = r (I^c)$
(v) $(I : J)^c \subseteq (I^c : J^c)$

Proof:

(i) We have $I \subseteq I + J \Rightarrow f^{-1} (I) \subseteq f^{-1}(I + J) \Rightarrow I^c \subseteq (I + J)^c$

Also $J \subseteq I + J \Rightarrow f^{-1} (J) \subseteq f^{-1}(I + J) \Rightarrow J^c \subseteq (I + J)^c$

Hence $(I^c + J^c) \subseteq (I + J)^c$

(ii) $I \cap J \subseteq I$ and $I \cap J \subseteq J \Rightarrow f^{-1}(I \cap J) \subseteq f^{-1}(I)$ and $f^{-1}(I \cap J) \subseteq f^{-1}(J)$
$\Rightarrow (I \cap J)^c \subseteq (I^c \cap J^c)$

Suppose $x \in (I^c \cap J^c) \Rightarrow x \in I^c$ and $x \in J^c$
$\Rightarrow x \in f^{-1}(I)$ and $x \in f^{-1}(J)$
$\Rightarrow f(x) \in I$ and $f(x) \in J$
$\Rightarrow f(x) \in I \cap J$
$\Rightarrow x \in f^{-1}(I \cap J)$
$\Rightarrow x \in (I \cap J)^c$

Hence $(I^c \cap J^c) \subseteq (I \cap J)^c$, Thus $(I \cap J)^c = (I^c \cap J^c)$.

(iii) Let $x \in (I^c J^c)$ then $x = a_1b_1 + a_2b_2 + ---- + a_nb_n$, $a_i \in I^c$, $b_i \in J^c$

Since $a_i \in I^c , b_i \in J^c \Rightarrow a_i \in f^{-1}(I) , b_i \in f^{-1}(J)$
$\Rightarrow f(a_i) \in I$ and $f(b_i) \in J$

$\Rightarrow f(a_i) f(b_i) \in IJ$, for all $i = 1,2,3, \dots , n$

$\Rightarrow f(a_i b_i) \in IJ$, for all $i = 1,2,3, \dots , n$

$\Rightarrow \Sigma_i f(a_i b_i) \in IJ$ (by definition of IJ)

$\Rightarrow f(\Sigma_i a_i b_i) \in IJ$

$\Rightarrow f(x) \in IJ$

$\Rightarrow x \in f^{-1}(IJ)$

$\Rightarrow x \in (IJ)^c$

Thus $I^c J^c \subseteq (IJ)^c$

(iv) Let $x \in [r(I)]^c \Leftrightarrow f (x) \in r (I)$
$\Leftrightarrow [f (x)]^n \in I$ by definition of radical of I)
$\Leftrightarrow f (x^n) \in I$
$\Leftrightarrow x^n \in f^{-1}(I) \Leftrightarrow x^n \in I^c \Leftrightarrow x \in r(I^c)$

Thus $[r(I)]^c = r (I^c)$.

$(I : J)^c \subseteq (I^c : J^c)$

(v) Let $x \in (I : J)^c \Rightarrow f(x) \in (I:J) \Rightarrow Jf(x) \subseteq I$ (A)

We now prove that $J^c x \subseteq I^c$

Let $z \in J^c x \Rightarrow z = yx$, for some $y \in J^c$.

If $y \in J^c \Rightarrow y \in f^{-1}(J) \Rightarrow f(y) \in J$

Now $f(z) = f(yx) = f(y) f(x) \in Jf(x) \subseteq I$ (from A)

Hence $f(z) \in I \Rightarrow z \in f^{-1}(I) \Rightarrow z \in I^c$. Hence $J^c x \subseteq I^c$

$\Rightarrow x \in (I^c : J^c)$. Thus $(I : J)^c \subseteq (I^c : J^c)$

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