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FIXED POINT THEOREMS IN G- METRIC SPACES VIA RATIONAL TYPE CONTRACTIVE CONDITION

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ABSTRACT
I*n this paper, we prove some fixed point result for a self mapping on a G- metric space satisfying rational type contractive conditions. Our results are generalized and extended many known previous results in G- metric space.*

Mathematics subject classification (2010): *47H10, 54H25.*

Key words: *G- metric space, Fixed point, rational contraction mapping.*

1. INTRODUCTION and PRELIMINIRES

It is well known that the contractive type conditions play an important role in the study of fixed point theory. The first intrusting result on fixed point for contractive type mappings was the well known Banach Caccioppoli theorem, published for the first time in 1922. After the invention of Banach contraction principle, it has been generalized by the many mathematicians. During the sixties, 2- metric spaces were introduced by Ghaler [7], and claimed that a 2- metric space is a generalization of the usual notion of metric space, but different another proved that there is no relation between these two functions. For instance Ha et al in [9] show that a 2- metric need not be a continuous function of its variables, whereas an ordinary metric is, further there is no easy relationship between results obtained in the two settings, in the two setting, in particular the contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated.

These considerations led Bapure Dhage in his Ph. D thesis [1992] to introduce a new class of generalized metrics called D-metrics. After some time, in 2006, Mustafa is collaboration with Sims introduced a new notation of generalized metric space called G- metric space. In fact, Mustafa et al. studied many fixed point results for a self mapping in G- metric space under certain conditions.

In the present work we study some fixed point results for a self mapping in a complete G- metric space X under weakly contractive conditions related to altering distance functions.

We present now the necessary definitions and results in G- metric spaces, which will be useful for the rest.

Definition 1.2: Let X be a non empty set, and let $G: X \times X \times X \rightarrow R^+$ be a function satisfying the following properties:
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $G(x, x, y) > 0$ *for all* $x, y \in X$, with $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,
(G_4) $G(x, y, z) = G(y, z, x) = G(z, x, y) = \cdots \ldots$ (symmetry in all three variables),
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)

Then the function G is called a generalized metric space, or more specially a G- metric on X, and the pair (X, G) is called a $G -$ metric space.

Definition 1.3: Let (X, G) be a G- metric space and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\rightarrow\infty} G(x, x_n, x_m) = 0$ and we say that the sequence $\{x_n\}$ is G- convergent to x or $\{x_n\}$ G- converges to x.

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Thus $x_n \to x$ in a G- metric space (X, G) if for any $\epsilon > 0$ there exists $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq k$.

Proposition 1.4: Let (X, G) be a G- metric space. Then, the following are equivalent
i. $\{x_n\}$ is G- convergent to x
ii. $G(x_n, x_n, x) \to 0$ as $n \to \infty$
iii. $G(x_n, x, x) \to 0$ as $n \to \infty$
iv. $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$

Definition 1.5: Let (X, G) be a G- metric space. A sequence $\{x_n\}$ is called a G- Cauchy sequence if for any $\epsilon > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq k$, that is $G(x_n, x_m, x_l) \to 0$ as $l, m, n \to \infty$.

Proposition 1.6: Let (X, G) be a G- metric space. Then, the following are equivalent
i. The sequence $\{x_n\}$ is G- Cauchy sequence
ii. For any $\epsilon > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $m, n \geq k$.

Proposition 1.7: Let (X, G) be a G- metric space. Then $f: X \to X$ is G- continuous at $x \in X$,if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G- convergent to x, $\left(f(x_n)\right)$ is G-convergent to f(x).

Proposition 1.8: Let (X, G) be a G- metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 1.9: Let (X, G) be a G- metric space. Then (X, G) is said to be G- complete if every G- Cauchy sequence is G- convergent in(X, G).

2. MAIN RESULT

Theorem 2.1: Let X be a complete G –metric space. Suppose the map $T: X \to X$ satisfies for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq \alpha \frac{\max\{G^2(x,Tx,Ty), G^2(y,Ty,Tz), G^2(z,Tz,Tx)\}}{G(x,y,z)} \tag{2.1.1}$$

For all $x, y, z \in X$ and $0 \leq \alpha < 1$. Then T has a unique fixed point and T is G- continuous at u.

Proof: Let x_0 be an arbitrary point in X , and let $x_{n+1} = Tx_n$, for any $n \in N$. Assume $x_n \neq x_{n-1} \neq x_{n+1}$, for $n \in N$, then from 2.1.1,

$$G(x_n, x_{n+1}, x_{n+2}) \leq G(Tx_{n-1}, Tx_n, Tx_{n+1})$$

$$G(Tx_{n-1}, Tx_n, Tx_{n+1}) \leq \alpha \frac{\max\{G^2(x_{n-1},Tx_{n-1},Tx_n), G^2(x_n,Tx_n,Tx_{n+1}), G^2(x_{n+1},Tx_{n-1},Tx_{n+1})\}}{G(x_{n-1},x_n,x_{n+1})} \tag{2.1.2}$$

$$G(x_n, x_{n+1}, x_{n+2}) \leq \alpha \frac{\max\{G^2(x_{n-1},x_n,x_{n+1}), G^2(x_n,x_{n+1},x_{n+1}), G^2(x_{n+1},x_n,x_{n+2})\}}{G(x_{n-1},x_n,x_{n+1})}$$

$$G(x_n, x_{n+1}, x_{n+2}) \leq \alpha\, G(x_{n-1}, x_n, x_{n+1}) \tag{2.1.3}$$

Similarly we show that,

$$G(x_{n-1}, x_n, x_{n+1}) \leq \alpha\, G(x_{n-2}, x_{n-1}, x_n)$$

By induction we can write,

$$G(x_n, x_{n+1}, x_{n+2}) \leq \alpha^n\, G(x_0, x_1, x_2) \text{ for } n \geq 1$$

Rewrite this as $n \to \infty$,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0 \tag{2.1.4}$$

Next we prove that $\{x_n\}$ is a G- Cauchy sequence. We argue by contradiction. Assume that $\{x_n\}$ is not a G- Cauchy sequence. Then the following proposition 1.6, there exists $\epsilon > 0$ for which we can find subsequence $\left(x_{m(k)}\right)$ and $\left(x_{n(k)}\right)$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) \geq \epsilon \tag{2.1.5}$$

Further corresponding to m(k), we can choose n(k) in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying 2.1.5. then

$$G\left(x_{n(k)-1}, x_{m(k)}, x_{m(k)}\right) < \epsilon \qquad (2.1.6)$$

We have using 2.1.6 and the condition (G_5), that

$$\epsilon \leq G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) \leq G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) + G\left(x_{n(k)-1}, x_{m(k)}, x_{m(k)}\right)$$

$$< G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) + \epsilon \qquad (2.1.7)$$

In other words, from the condition $(G_3) - (G_4)$

$$0 \leq G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) = G\left(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}\right)$$

Letting $k \to \infty$ and using 2.1.4, we find $G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) \to 0$. We take this in 2.1.7

$$\lim_{n\to\infty} G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) = \epsilon \qquad (2.1.8)$$

Moreover, we have thanks to condition (G_4)

$$G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) \leq G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) + G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right) + G\left(x_{m(k)-1}, x_{m(k)}, x_{m(k)}\right)$$

$$G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right) \leq G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right) + G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) + G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right)$$

Let $k \to \infty$ in the two above inequalities and using 2.1.4 and 2.1.8

$$\lim_{n\to\infty} G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right) = \epsilon \qquad (2.1.9)$$

Setting $x = x_{n(k)-1}$ and $y = y_{m(k)-1}$ in 2.1.1, which contradiction the fact since $\epsilon > 0$. This show that $\{x_n\}$ is a G-Cauchy sequence and since X is a G- complete space, hence $\{x_n\}$ is G- convergent to some $u \in X$, that is from proposition 1.6

$$\lim_{n\to\infty} G(x_n, x_n, u) = \lim_{n\to\infty} G(x_n, u, u) = 0 \qquad (2.1.10)$$

We show now that u is a fixed point of the map T. from 2.1.1 and on taking

$$G(x_{n+1}, x_{n+1}, Tu) \leq \alpha\, G(Tx_n, Tx_n, Tu)$$

From 2.1.10, we have

$$\lim_{n\to\infty} G(x_{n+1}, x_{n+1}, Tu) = 0 \qquad (2.1.11)$$

Again, using the condition (G_4) and (G_5) given by definition 1.2, we can write

$$G(u, u, Tu) \leq G(u, u, x_{n+1}) + G(x_{n+1}, x_{n+1}, Tu)$$

Let $k \to \infty$ in the above inequalities and having in the mind 2.1.10 and 2.1.11, we have $G(u, u, Tu) = 0$ and then$= u$.

Hence u is a fixed point of T. let us show the uniqueness of u.

For this let us assume that v is another fixed point of T, then

$$G(u, u, v) = G(Tu, Tu, Tv)$$

By using 2.1.1 we have $G(u, u, v) = 0$, yielding that $u = v$.

Following proposition 1.7, to show that T is G- continuous at u, let $\{y_n\}$ be any sequence in X such that $\{y_n\}$ is G-convergent to u. for $n \in N$, we have

$$\psi\left(G(u, u, Ty_n)\right) = \psi\left(G(Tu, Tu, Ty_n)\right)$$

By using 2.1.1 and as $n \to \infty$ then write hand side of the above inequality tends to 0, then we obtain

$$\lim_{n\to\infty} G(u,u,Ty_n) = 0$$

Hence $\{Ty_n\}_n$ is G- convergent to $u = Tu$, so T is G- continuous at u, and u is unique fixed point of T in G.

Corollary 2.2: Let X be a complete G –metric space. Suppose the map $T: X \to X$ satisfies for all $x, y, z \in X$

$$\int_0^{G(Tx,Ty,Tz)} \xi(t)\ dt \leq \alpha \int_0^{\frac{\max\{G^2(x,Tx,Ty),G^2(y,Ty,Tz),G^2(z,Tz,Tx)\}}{G(x,y,z)}} \xi(t)\ dt \quad (2.2.1)$$

For each $x, y, z \in X$, where $\xi : \mathcal{R}^+ \to \mathcal{R}^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $\mathcal{R}^+$, non negative, and such that

$$\text{for each } \epsilon > 0, \int_0^{\epsilon} \xi(t)\ dt \quad (2.2.2)$$

Then T has a unique fixed point $z \in X$ and for each $x \in X, \lim_{n\to\infty} T^n x = z$.

Proof: To prove of the above result, it is immediate to see that, if we take $\xi(t) = 1$ then result is follows Theorem 2.1, and nothing to prove.

Theorem 2.3: Let X be a complete G –metric space. Suppose the map $S, T: X \to X$ satisfies for all $x, y, z \in X$ and $T(X) \subseteq S(X)$

$$G(Tx,Ty,Tz) \leq \alpha \frac{\max\{G^2(Sx,Tx,Ty),G^2(Sy,Ty,Tz),G^2(Sz,Tz,Tx)\}}{G(Sx,Sy,Sz)} \quad (2.3.1)$$

For all $x, y, z \in X$ and $0 \leq \alpha < 1$. Then S and T have a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X, and let $y_{n+1} = Tx_n = Sx_{n+1}$, for any $n \in N$. Assume $x_n \neq x_{n-1} \neq x_{n+1}$, and $y_n \neq y_{n-1} \neq y_{n+1}$ for $n \in N$, then from 2.1.1,

$$G(y_n,y_{n+1},y_{n+2}) \leq G(Tx_{n-1},Tx_n,Tx_{n+1})$$

From 2.3.1, we have

$$G(Tx_{n-1},Tx_n,Tx_{n+1}) \leq \alpha \frac{\max\{G^2(Sx_{n-1},Tx_{n-1},Tx_n),G^2(Sx_n,Tx_n,Tx_{n+1}),G^2(Sx_{n+1},Tx_{n-1},Tx_{n+1})\}}{G(Sx_{n-1},Sx_n,Sx_{n+1})} \quad (2.3.2)$$

$$G(y_n,y_{n+1},y_{n+2}) \leq \alpha \frac{\max\{G^2(y_{n-1},y_n,y_{n+1}),G^2(y_n,y_{n+1},y_{n+1}),G^2(y_{n+1},y_n,y_{n+2})\}}{G(y_{n-1},y_n,y_{n+1})}$$

$$G(y_n,y_{n+1},y_{n+2}) \leq \alpha\, G(y_{n-1},y_n,y_{n+1}) \quad (2.3.3)$$

Similarly we show that,

$$G(y_{n-1},y_n,y_{n+1}) \leq \alpha\, G(y_{n-2},y_{n-1},y_n)$$

By induction we can write,

$$G(y_n,y_{n+1},y_{n+2}) \leq \alpha^n\, G(y_0,y_1,y_2) \text{ for } n \geq 1$$

Rewrite this as $n \to \infty$,

$$\lim_{n\to\infty} G(y_n,y_{n+1},y_{n+1}) = 0 \quad (2.3.4)$$

Remaining prove of the theorem, similarly the prove of Theorem 2.1.

Corollary 2.4: Let X be a complete G –metric space. Suppose the map $S, T: X \to X$ satisfies for all $x, y, z \in X$

$$\int_0^{G(Tx,Ty,Tz)} \xi(t)\ dt \leq \alpha \int_0^{\alpha\frac{\max\{G^2(Sx,Tx,Ty),G^2(Sy,Ty,Tz),G^2(Sz,Tz,Tx)\}}{G(Sx,Sy,Sz)}} \xi(t)\ dt \quad (2.4.1)$$

For each $x, y, z \in X$, where $\xi : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $\mathcal{R}^+$, non negative, and such that

for each $\epsilon > 0, \int_0^\epsilon \xi(t)\ dt$ (2.4.2)

Then S and T have a unique fixed point $z \in X$ and for each $x \in X$, that is

$$\lim_{n\to\infty} S^n x = \lim_{n\to\infty} T^n x = z.$$

Proof: To prove of the above result, it is immediate to see that, if we take $\xi(t) = 1$ then result is follows Theorem 2.3, and nothing to prove.

REMARK: In Theorem 2.3, if we take, S = I (identity mapping) then we get Theorem 2.1.

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