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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACE

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ABSTRACT

In this paper, we prove some common fixed point theorems for a pair of self mappings using M.S. property and satisfy certain sufficient conditions in setting of Menger space.

***Keywords:** Menger space, Probabilistic distance, Probabilistic metric space, Distribution function, t-norm, weakly compatible, common fixed point, M.S. property.*

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INTRODUCTION:

There have been a number of generalizations of Metric space. One such generalization is Menger space introduced in 1942 by Menger [2] who used distribution functions instead of non negative real numbers as values of the metric. Schweizer and Sklar [7] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [4].

Sessa [5] introduced weakly commuting maps in metric spaces. Jungck [1] enlarged this concept to compatible maps. The notion of compatible maps in Menger spaces has been introduced by Mishra [3]. Recently Singh and Jain [6] generalized the results of Mishra [3] using the concept of weak compatibility and compatibility of pair of self maps. In this paper, we extend the result of Sisodia, K.S. [8] to obtain a common fixed point for a pair of self mappings in Menger space.

PRELIMINARIES:

Definition1: A binary operation $*$: $[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a*1=a$ for all $a\epsilon[0,1]$;
(d) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\ \epsilon[0,1]$.

Examples of t-norms are $a*b=\max\{a+b-1, 0\}$ and $a*b=\min\{a, b\}$.

Definition2: A distribution function is a function $F:[-\infty,\infty]\to[0,1]$ which is left continuous on R, non-decreasing and $F(-\infty)=0$, $F(\infty)=1$.

We will denote Δ by the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t)=\begin{cases}0, & if\ t\le 0,\\ 1, & if\ t>0\end{cases}.$$

If X is a non empty set, $F: X\times X\to\Delta$ is called a probabilistic distance on X and F(x, y) is usually denoted by F_{xy}.

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Definition3: The ordered pair (X, F) is called a probabilistic metric space (shortly PM-space) if X is a non empty set and F is a probabilistic distance satisfying the following conditions: for all x, y, z∊X and t, s>0;

(i) $F_{xy}(t)=1 \Leftrightarrow x=y$;
(ii) $F_{xy}(0)=0$;
(iii) $F_{xy}= F_{yx}$;
(iv) $F_{xz}(t)=1$; $F_{zy}(s)=1 \Rightarrow F_{xy}(t+s)=1$.

The ordered triple (X, F,∗) is called Menger space if (X, F) is a PM-space, ∗ is a t-norm and the following condition is also satisfied: for all x, y, z∊X and t, s>0;

(v) $F_{xy}(t+s) \geq F_{xz}(t) * F_{zy}(s)$.

Proposition1 [4]: Let (X, d) be a metric space, then the metric d induces a distribution function F defined by $F_{xy}(t)=H(t-d(x, y))$ for all x, y∊X and t>0. If t-norm ∗ is a∗b=min {a, b} for all a,b ∊[0,1] then (X, F,∗) is a Menger space. Further, (X, F,∗) is a complete Menger space if (X, d) is complete.

Definition4: Let (X, F,∗) be a Menger space and ∗ be a continuous t-norm.

(a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \to x$) iff for every ϵ>0 and $\lambda \in(0, 1)$, there exists an integer $n_0 = n_0(\epsilon, \lambda)$ such that $F_{x_n x}(\epsilon)>1-\lambda$ for all $n \geq n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every ϵ>0 and $\lambda \in(0, 1)$, there exists an integer $n_0 = n_0(\epsilon, \lambda)$ such that $F_{x_n x_{n+p}}(\epsilon)>1-\lambda$ for all $n \geq n_0$ and p>0.
(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition5: Self maps A and B of a Menger space (X, F,∗) are said to be weakly compatible (or coincidently commuting) if they commute at their coincidence points, i.e. if Ax=Bx for some x∊X then ABx=BAx.

Definition6: Self maps A and B of a Menger space (X, F,∗) are said to be compatible if $F_{ABx_n BAx_n}(t) \to 1$ for all t>0, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X as n→ ∞.

Definition6: Self maps A and B of a Menger space (X, F,∗) are called semi compatible if $F_{ABx_n Bx}(t) \to 1$ for all t>0, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X.

Lemma1 [6]: Let $\{x_n\}$ be a sequence in a Menger space (X, F,∗) with continuous t-norm ∗ and $t * t \geq t$. If there exists a constant k ∊(0, 1) such that $F_{x_n x_{n+1}}(kt) \geq F_{x_{n-1} x_n}(t)$ for all t>0 and n=1, 2… Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma2 [6]: Let (X, F,∗) be a Menger space. If there exists k ∊(0, 1) such that $F_{xy}(kt) \geq F_{xy}(t)$ for all $x, y \in X$ and t>0, then $x = y$.

Definition7 [8]: Let A and B be two self maps of a Menger space (X, F,∗), we say that A and B satisfy M.S. property, if there exists a sequence $\{x_n\}$ in X such that $Ax_n, Bx_n \to x_0$ for some $x_0 \in X$ as n→ ∞.
i.e. $\lim_{n\to\infty} F_{Bx_n x_0}(t) = \lim_{n\to\infty} F_{Ax_n x_0}(t) = 1$ for all $t \in (0, \infty)$.

Example1: Let X= [0, ∞). Let $F_{xy}(t)=\frac{t}{t+|x-y|}$ for all t>0.
Define A, B: X→ $[0,\infty)$ by $Ax = \frac{x}{5}$ and $Bx = \frac{2x}{5}$ for all $x \in X$. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 0$, where $x_n = \frac{1}{n}$.

MAIN RESULTS:

Theorem1: Let A and B be two weakly compatible self mappings of a Menger space (X, F,∗) with $t * t \geq t$ such that

(i) A and B satisfy the M.S. property,
(ii) For each $x \neq y$ in X and t>0 $F_{AxAy}(qt) \geq \min\{F_{BxBy}(t), F_{AxBx}(t) * F_{AyBy}(t), F_{AyBx}(t) * F_{AxBy}(t)\}$, where for 0<q<1
(iii) $A(X) \subset B(X)$.
(iv) $B(X)$ or A(X) is a complete subspace of X

Then A and B have a unique common fixed point.

Proof: Since A and B satisfy the M.S. property, there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = x_0$ for some $x_0 \in X$. Suppose that B(X) is complete, then $lim_{n\to\infty} Bx_n = Ba$ for some $a \in X$.

∴ $\lim_{n\to\infty} Ax_n = Ba$ by →(i).

Now, we show that $Aa = Ba$.

Condition (ii) implies that

$F_{Ax_n Aa}(qt) \geq \min\{F_{Bx_n Ba}(t), [F_{Ax_n Bx_n}(t) * F_{AaBa}(t)], [F_{AaBx_n}(t) * F_{Ax_n Ba}(t)]\}$

Letting limit $n \to \infty$

$F_{BaAa}(qt) \geq \min\{1, F_{BaBa}(t) * F_{AaBa}(t), F_{AaBa}(t) * 1\}$

$= \{1, F_{AaBa}(t), \ F_{AaBa}(t)\}$

$F_{BaAa}(qt) \geq F_{AaBa}(t)$

$\therefore Aa = Ba.$

Now we show that Aa is the common fixed point of A and B.

Since A and B are weakly compatible,

$BAa = ABa = BBa = AAa$

$F_{AaAAa}(qt) \geq \min\{F_{BaBAa}(t), F_{AaBa}(t) * F_{AAaBAa}(t), F_{AAaBa}(t) * F_{AaBAa}(t)\}$

$= \min\{F_{AaAAa}(t), F_{AaAa}(t) * F_{AAaAAa}(t), F_{AAaAa}(t) * F_{AaAAa}(t)\}$

$= \min\{(t), 1, F_{AaAAa}(t) * F_{AaAAa}(t)\}$

$F_{AaAAa}(qt) \geq F_{AaAAa}(t)$

$\therefore AAa = Aa.$

Hence Aa is the common fixed point of A and B.

Even, if we assume that A(X) is complete and proceed as above the result will be same.

Now it is left to prove that the fixed point is unique.

Let $x_0$and y_0 be two common fixed points of A and B. then

$F_{x_0 y_0}(qt) = F_{Ax_0 Ay_0}(qt)$

$\geq \min\{ F_{Bx_0 By_0}(t), [F_{Ax_0 Bx_0}(t) * F_{Ay_0 By_0}(t)], [F_{Ay_0 Bx_0}(t) * F_{Ax_0 By_0}(t)]\}$

$= \min\{ (t), [F_{x_0 x_0}(t) * F_{y_0 y_0}(t)], [F_{y_0 x_0}(t) * F_{x_0 y_0}(t)]\}$

$F_{x_0 y_0}(qt) \geq F_{x_0 y_0}(t)$.

$\therefore x_0 = y_0$.

Corollary1: Let A and B be two non compatible weakly compatible self mappings of a Menger space (X, F,$*$) with $t * t \geq t$ such that

(i) $F_{AxAy}(qt) \geq \min\{ F_{BxBy}(t), F_{AxBx}(t) * F_{AyBy}(t), F_{AyBx}(t) * F_{AxBy}(t)\}$
(ii) $A(X) \subset B(X)$.

If B(X) or A(X) is complete subspace of X, then A and B have a unique common fixed point.

Theorem2: Let A and B be two weakly compatible self mappings of a Menger space (X, F,$*$) with $t *$ t $\geq t$ such that

(i) A and B satisfy the M.S. property,
(ii) For each $x \neq y$ in X and t>0 $F_{AxAy}(qt) \geq \min\{F_{BxBy}(t), \frac{[F_{AxBx}(t) + F_{AyBy}(t)]}{2}, \frac{[F_{AxBy}(t) + F_{AyBx}(t)]}{2}\}$, where for 0<q<1.

(iii) $A(X) \subset B(X)$.
(iv) $B(X)$ or A(X) is a complete subspace of X

Then A and B have a unique common fixed point.

Proof: Since A and B satisfy the M.S. property, there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = x_0$ for some $x_0 \in X$. Suppose that B(X) is complete, then $\lim_{n\to\infty} Bx_n = Ba$ for some $a \in X$.

$\therefore \lim_{n\to\infty} Ax_n = Ba$ by (i).

Now, we claim that $Aa = Ba$.

If $Aa \neq Ba$, then $F_{AaBa}(t) < 1$ for all t.

Condition (ii) implies that

$$F_{Ax_n Aa}(qt) \geq \min\{ F_{Bx_n Ba}(t), \frac{[F_{Ax_n Bx_n}(t) + F_{AaBa}(t)]}{2}, \frac{[F_{Ax_n Ba}(t) + F_{AaB\,x_n}(t)]}{2}\}$$

Letting limit $n \to \infty$

$$F_{BaAa}(qt) \geq \min\{ 1, \frac{[1+F_{AaBa}(t)]}{2}, \frac{[1+F_{AaBa}(t)]}{2}\}$$

$$F_{BaAa}(qt) \geq \frac{[1+F_{AaBa}(t)]}{2}$$

$$\geq F_{AaBa}(t) \quad \text{for all t.}$$

Because, if $\frac{[1+F_{AaBa}(t)]}{2} < F_{AaBa}(t)$ then $F_{AaBa}(t) > 1$, which is a contradiction.

Hence $Aa = Ba$.

Let us now prove that Aa is the common fixed point of A and B.

Suppose that $Aa \neq AAa$. Since A and B are weakly compatible, $BAa = ABa$ and therefore

$BBa = ABa$ and $BAa = AAa$.

Then by (ii), we have

$$F_{AaAAa}(qt) \geq \min\{F_{AaBAa}(t), \frac{[F_{AaBa}(t) + F_{AAaBAa}(t)]}{2}, \frac{[F_{AaBAa}(t) + F_{AAaBa}(t)]}{2}\}$$

$$= \min\{F_{AaAAa}(t), \frac{[1+F_{AAaAAa}(t)]}{2}, \frac{[F_{AaAAa}(t) + F_{AAaAa}(t)]}{2}\}$$

$$= \min\{F_{AaAAa}(t), \frac{[1+F_{AAaAAa}(t)]}{2}, F_{AaAAa}(t)\}$$

$$= F_{AaAAa}(t)$$

Because $F_{AaAAa}(t) \leq \frac{[1+F_{AAaAAa}(t)]}{2}$

Thus $F_{AaAAa}(t) \geq F_{AaAAa}(t)$ for all t.

This implies that $AAa = Aa$.

Hence Aa is the common fixed point of A and B.

Even, if we assume that A(X) is complete and proceed as above the result will be same.

Finally we show that the fixed point is unique.

If possible, let x_0 and y_0 be two common fixed points of A and B. then

$F_{x_0y_0}(qt) = F_{Ax_0Ay_0}(qt)$

$\geq \min\{F_{Bx_0By_0}(t), \frac{[F_{Ax_0Bx_0}(t)+F_{Ay_0By_0}(t)]}{2}, \frac{[F_{Ax_0By_0}(t)+F_{Ay_0Bx_0}(t)]}{2}\}$

$= \min\{F_{x_0y_0}(t), \frac{[F_{x_0x_0}(t)+F_{y_0y_0}(t)]}{2}, \frac{[F_{x_0y_0}(t)+F_{y_0x_0}(t)]}{2}\}$

$= \min\{F_{x_0y_0}(t), 1, F_{x_0y_0}(t)\}$

$\Rightarrow F_{x_0y_0}(qt) \geq F_{x_0y_0}(t)$, for all t.

$\therefore x_0 = y_0$.

Hence the theorem.

Corollary2: Let A and B be two non-compatible weakly compatible self mappings of a Menger space (X, F,$*$) with $t * t \geq t$ such that

(i) $F_{AxAy}(qt) \geq \min\{F_{BxBy}(t), \frac{[F_{AxBx}(t)+F_{AyBy}(t)]}{2}, \frac{[F_{AxBy}(t)+F_{AyBx}(t)]}{2}\}$, where for 0<q<1.
(ii) $A(X) \subset B(X)$.

If B(X) or A(X) is complete subspace of X, then A and B have a unique common fixed point.

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