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Minimal *pg*-open sets and Maximal *pg*-closed sets

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ABSTRACT

The object of the present paper is to study the notions of minimal pg-closed set, maximal pg-open set, minimal pg-open set and maximal pg-closed set and their basic properties are studied.

Keywords: *pg-closed set and minimal pg-closed set, maximal pg-open set, minimal pg-open set and maximal pg-closed set*

1. Introduction:

Norman Levine introduced the concept of generalized closed sets in topological spaces. After him many authors concentrated in this direction and defined more than 25 types of generalized closed sets. Nakaoka and Oda have introduced minimal open sets and maximal open sets, which are subclasses of open sets. A. Vadivel and K. Vairamanickam introduced minimal rgα-open sets and maximal rgα-open sets in topological spaces. S. Balasubramanian and P.A.S. Vyjayanthi introduced minimal v-open sets and maximal v-open sets; minimal v-closed sets and maximal v-closed sets in topological spaces. Recently S. Balasubramanian introduced minimal vg-open sets and maximal vg-open sets; minimal vg-closed sets and maximal vg-closed sets in topological spaces. Inspired with these developments we further study a new type of closed and open sets namely minimal pg-closed sets, maximal pg-open sets, minimal pg-open sets and maximal pg-closed sets. Throughout the paper a space X means a topological space (X, τ). The class of pg-closed sets is denoted by PGC(X). For any subset A of X its complement, interior, closure, pg-interior, pg-closure are denoted respectively by the symbols A^c, A^o, A^-, $pg(A)^0$ *and* $pg(A)^-$.

2. Preliminaries:

Definition 2.1: A$\subset$ X is called
(i) closed if its complement is open.
(ii) rα-open[*v*-open] if $\exists$ U$\in$ αO(X)[RO(X)] such that U$\subset$ A$\subset$ αcl(U)[U$\subset$ A$\subset$cl(U)].
(iii) semi-θ-open if it is the union of semi-regular sets and its complement is semi-θ-closed.
(iv) r-closed[α-closed; pre-closed; β—closed] if A = cl(A^o)[$(cl(A^o))^o \subseteq$ A; cl(A^o) $\subseteq$ A; $cl((cl(A))^o)\subseteq$ A].
(v) Semi closed[*v*-closed] if its complement if semi open[*v*-open].
(vi) g-closed[rg-closed] if cl A$\subseteq$ U whenever A$\subseteq$ U and U is open[r-open] in X.
(vii) pg-closed[gp-closed] if pcl(A) $\subseteq$U whenever A$\subseteq$U and U is pre-open[open] in X.

Definition 2.02: Let A$\subset$X.
(i) A point x$\in$A is the *pg*-interior point of A iff $\exists$ G$\in$*PGO*(X, τ) such that x$\in$G$\subset$A.
(ii) A point $x \in X$ is said to be an *pg*-limit point of A iff for each U$\in$*PGO(X)*, U$\cap$(A~{x}) ≠ ϕ.
(iii) A point x$\in$A is said to be *pg*-isolated point of A if $\exists$ U$\in$*PGO(X)* such that U$\cap$A = {x}.

Definition 2.03: Let A$\subset$X.
(i) Then A is said to be *pg*-discrete if each point of A is *pg*-isolated point of A. The set of all *pg*-isolated points of A is denoted by I_{pg}(A).

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(ii) For any $A\subset X$, the intersection of all *pg*-closed sets containing A is called the *pg*-closure of A and is denoted by $pg(A)^-$.
(iii) For any $A\subset X$, $A \sim pg(A)^0$ is said to be *pg*-border or *pg*-boundary of A and is denoted by $B_{pg}(A)$.
(iv) For any $A\subset X$, $pg\,[pg(X\sim A)^-]^0$ is said to be the *pg*-exterior $A\subset X$ and is denoted by $pg(A)^e$.

Definition 2.04: The set of all *pg*-interior points A is said to be *pg*-interior of A and is denoted by $pg(A)^0$.

Theorem 2.01: (i) Let $A\subseteq Y\subseteq X$ and Y is regularly open subspace of X then $A\in PGO(Y, \tau_{/Y})$ iff Y is *pg*-open in X
(ii) Let $Y\subseteq X$ and A is a *pg*-neighborhood of x in Y. Then A is *pg*-neighborhood of x in Y iff Y is *pg*-open in X.

Theorem 2.02: Arbitrary intersection of *pg*-closed sets is *pg*-closed. More Precisely, Let $\{A_i: i \in I\}$ be a collection of *pg*-closed sets, then $\cap_{i\in I}A_i$ is again *pg*-closed.

Note 2: Finite union and finite intersection of *pg*-closed sets is not *pg*-closed in general.

Theorem 2.03: Let $X = X_1\times X_2$. Let $A_1\in PGC(X_1)$ and $A_2\in PGC(X_2)$, then $A_1\times A_2\in PGC(X_1\times X_2)$.

3. Minimal *pg*-open Sets and Maximal *pg*-closed Sets:

We now introduce minimal *pg*-open sets and maximal *pg*-closed sets in topological spaces as follows.

Definition 3.1: A proper nonempty *pg*-open subset U of X is said to be a **minimal *pg*-open set** if any *pg*-open set contained in U is ϕ or U.

Example 1: Let X = {a, b, c, d}; $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. {a} and {b} are both Minimal open set and Minimal *pg*-open set.

Remark 1: Minimal open set and minimal *pg*-open set are independent to each other:

Example 2: Let X = {a, b, c, d}; $\tau = \{\phi, \{a, b\}, \{a, b, c\}, X\}$. {a, b} is Minimal open but not Minimal *pg*-open and {a}, {b} are Minimal *pg*-open but not Minimal open.

Remark 2: From the above example and known results we have the following implications

Theorem 3.1:
(i) Let U be a minimal *pg*-open set and W be a *pg*-open set. Then $U \cap W = \phi$ or $U\subset W$.
(ii) Let U and V be minimal *pg*-open sets. Then $U \cap V = \phi$ or U = V.

Proof:
(i) Let U be a minimal *pg*-open set and W be a *pg*-open set. If $U \cap W = \phi$, then there is nothing to prove.
If $U \cap W\neq \phi$. Then $U \cap W \subset U$. Since U is a minimal *pg*-open set, we have $U \cap W = U$. Therefore $U \subset W$.

(ii) Let U and V be minimal *pg*-open sets. If $U\cap V \neq \phi$, then $U\subset V$ and $V\subset U$ by (i). Therefore U = V.

Theorem 3.2: Let U be a minimal *pg*-open set. If $x\in U$, then $U\subset W$ for any regular open neighborhood W of x.

Proof: Let U be a minimal *pg*-open set and x be an element of U. Suppose $\exists$ a regular open neighborhood W of x such that $U \not\subset W$. Then $U \cap W$ is a *pg*-open set such that $U \cap W \subset U$ and $U \cap W \neq \phi$. Since U is a minimal *pg*-open set, we have $U\cap W = U$. That is $U \subset W$, which is a contradiction for $U \not\subset W$. Therefore $U\subset W$ for any regular open neighborhood W of x.

Theorem 3.3: Let U be a minimal *pg*-open set. If $x\in U$, then $U\subset W$ for some *pg*-open set W containing x.

Theorem 3.4: Let U be a minimal *pg*-open set. Then $U = \cap\{W: W\in PGO(X, x)\}$ for any element x of U.

Proof: By theorem [3.3] and U is *pg*-open set containing x, we have $U\subset\cap\{ W: W\in PGO(X, x)\}\subset U$.

Theorem 3.5: Let U be a nonempty *pg*-open set. Then the following three conditions are equivalent.
(i) U is a minimal *pg*-open set
(ii) $U\subset pg(S)^-$ for any nonempty subset S of U
(iii) $pg(U)^- = pg(S)^-$ for any nonempty subset S of U.

Proof: (i) $\Rightarrow$ (ii) Let $x\in U$; U be minimal *pg*-open set and $S(\neq \phi)\subset U$. By theorem [3.3], for any *pg*-open set W containing x, $S\subset U\subset W \Rightarrow S\subset W$. Now $S = S\cap U\subset S\cap W$. Since $S\neq \phi$, $S\cap W\neq \phi$. Since W is any *pg*-open set containing x, by theorem[5.03], $x\in pg(S)^-$. That is $x\in U \Rightarrow x\in pg(S)^- \Rightarrow U\subset pg(S)^-$ for any nonempty subset S of U.

(ii) $\Rightarrow$ (iii) Let S be a nonempty subset of U. That is $S\subset U \Rightarrow pg(S)^-\subset pg(U)^- \rightarrow$ (1). Again from (ii) $U\subset pg(S)^-$ for any $S(\neq \phi)\subset U \Rightarrow pg(U)^-\subset pg(pg(S)^-)^- = pg(S)^-$. That is $pg(U)^-\subset pg(S)^- \rightarrow$ (2).

From (1) and (2), we have $pg(U)^- = pg(S)^-$ for any nonempty subset S of U.

(iii) $\Rightarrow$ (i) From (3) we have $pg(U)^- = pg(S)^-$ for any nonempty subset S of U. Suppose U is not a minimal *pg*-open set.

Then $\exists$ a nonempty *pg*-open set V such that $V \subset U$ and $V \neq U$. Now $\exists$ an element a in U such that $a\notin V \Rightarrow a\in V^c$. That is $pg(\{a\})^-\subset pg(V^c)^- = V^c$, as V^c is *pg*-closed set in X. It follows that $pg(\{a\})^- \neq pg(U)^-$. This is a contradiction for $pg(\{a\})^- = pg(U)^-$ for any $\{a\}(\neq \phi)\subset U$. Therefore U is a minimal *pg*-open set.

Theorem 3.6: Let V be a nonempty finite *pg*-open set. Then $\exists$ at least one (finite) minimal *pg*-open set U such that $U\subset V$.

Proof: Let V be a nonempty finite *pg*-open set. If V is a minimal *pg*-open set, we may set U = V. If V is not a minimal *pg*-open set, then $\exists$ (finite) *pg*-open set V_1 such that $\phi \neq V_1\subset V$. If V_1 is a minimal *pg*-open set, we may set $U = V_1$. If V_1 is not a minimal *pg*-open set, then $\exists$ (finite) *pg*-open set V_2 such that $\phi \neq V_2 \subset V_1$. Continuing this process, we have a sequence of *pg*-open sets $V \supset V_1 \supset V_2 \supset V_3 \supset ..... \supset V_k \supset ......$ Since V is a finite set, this process repeats only finitely. Then finally we get a minimal *pg*-open set $U = V_n$ for some positive integer n.

[A topological space X is said to be locally finite space if each of its elements is contained in a finite open set.]

Corollary 3.1: Let X be a locally finite space and V be a nonempty *pg*-open set. Then $\exists$ at least one (finite) minimal *pg*-open set U such that $U \subset V$.

Proof: Let X be a locally finite space and V be a nonempty *pg*-open set. Let x in V. Since X is locally finite space, we have a finite open set V_x such that x in V_x. Then $V\cap V_x$ is a finite *pg*-open set. By Theorem 3.6 $\exists$ at least one (finite) minimal *pg*-open set U such that $U \subset V\cap V_x$. That is $U\subset V\cap V_x\subset V$. Hence $\exists$ at least one (finite) minimal *pg*-open set U such that $U\subset V$.

Corollary 3.2: Let V be a finite minimal open set. Then $\exists$ at least one (finite) minimal *pg*-open set U such that $U\subset V$.

Proof: Let V be a finite minimal open set. Then V is a nonempty finite *pg*-open set. By Theorem 3.6, $\exists$ at least one (finite) minimal *pg*-open set U such that $U\subset V$.

Theorem 3.7: Let U; U_λ be minimal *pg*-open sets for any element$\lambda\in\Gamma$. If $U\subset \cup_{\lambda\in\Gamma}U_\lambda$, then $\exists$ an element $\lambda \in\Gamma$ such that $U = U_\lambda$.

Proof: Let $U\subset \cup_{\lambda\in\Gamma}U_\lambda$. Then $U \cap(\cup_{\lambda\in\Gamma}U_\lambda) = U$. That is $\cup_{\lambda\in\Gamma}(U \cap U_\lambda) = U$. Also by theorem [3.1] (ii), $U \cap U_\lambda = \phi$ or $U = U_\lambda$ for any $\lambda\in\Gamma$. It follows that $\exists$ an element $\lambda\in\Gamma$ such that $U = U_\lambda$.

Theorem 3.8: Let U; U_λ be minimal *pg*-open sets for any $\lambda\in\Gamma$. If $U = U_\lambda$ for any $\lambda\in\Gamma$, then $(\cup_{\lambda\in\Gamma}U_\lambda) \cap U = \phi$.

Proof: Suppose that $(\cup_{\lambda\in\Gamma}U_\lambda) \cap U\neq \phi$. That is $\cup_{\lambda\in\Gamma}(U_\lambda \cap U) \neq \phi$. Then $\exists$ an element $\lambda\in\Gamma$ such that $U \cap U_\lambda \neq \phi$. By theorem 3.1(ii), we have $U = U_\lambda$, which contradicts the fact that $U \neq U_\lambda$ for any $\lambda\in\Gamma$. Hence $(\cup_{\lambda\in\Gamma}U_\lambda)\cap U = \phi$.

We now introduce maximal *pg*-closed sets in topological spaces as follows.

Definition 3.2: A proper nonempty *pg*-closed $F\subset X$ is said to be **maximal *pg*-closed set** if any *pg*-closed set containing F is either X or F.

Example 3: In Example 1, {b, c, d} is Maximal closed and Maximal *pg*-closed.

Remark 3: Maximal closed set and maximal *pg*-closed set are independent to each other:

Example 4: In Example 2, {c, d} is Maximal closed but not Maximal *pg*-closed and {a, c, d}, {b, c, d} are Maximal *pg*-closed but not Maximal closed.

Remark 4: From the known results and by the above example we have the following implications:

Theorem 3.9: A proper nonempty subset F of X is maximal *pg*-closed set iff X-F is a minimal *pg*-open set.

Proof: Let F be a maximal *pg*-closed set. Suppose X-F is not a minimal *pg*-open set. Then $\exists$ *pg*-open set $U \neq X\text{-}F$ such that $\phi \neq U \subset X\text{-}F$. That is $F \subset X\text{-}U$ and X-U is a *pg*-closed set which is a contradiction for F is a minimal *pg*-open set.

Conversely let X-F be a minimal *pg*-open set. Suppose F is not a maximal *pg*-closed set. Then $\exists$ *pg*-closed set $E \neq F$ such that $F \subset E \neq X$. That is $\phi \neq X\text{-}E \subset X\text{-}F$ and X-E is a *pg*-open set which is a contradiction for X-F is a minimal *pg*-open set. Therefore F is a maximal *pg*-closed set.

Theorem 3.10:
(i) Let F be a maximal *pg*-closed set and W be a *pg*-closed set. Then $F \cup W = X$ or $W \subset F$.
(ii) Let F and S be maximal *pg*-closed sets. Then $F \cup S = X$ or $F = S$.

Proof: (i) Let F be a maximal *pg*-closed set and W be a *pg*-closed set. If $F \cup W = X$, then there is nothing to prove. Suppose $F \cup W \neq X$. Then $F \subset F \cup W$. Therefore $F \cup W = F \Rightarrow W \subset F$.
(ii) Let F and S be maximal *pg*-closed sets. If $F \cup S \neq X$, then we have $F \subset S$ and $S \subset F$ by (i). Therefore $F = S$.

Theorem 3.11: Let F be a maximal *pg*-closed set. If x is an element of F, then for any *pg*-closed set S containing x, $F \cup S = X$ or $S \subset F$.

Proof: Let F be a maximal *pg*-closed set and x is an element of F. Suppose $\exists$ *pg*-closed set S containing x such that $F \cup S \neq X$. Then $F \subset F \cup S$ and $F \cup S$ is a *pg*-closed set, as the finite union of *pg*-closed sets is a *pg*-closed set. Since F is a *pg*-closed set, we have $F \cup S = F$. Therefore $S \subset F$.

Theorem 3.12: Let F_α, F_β, F_δ be maximal *pg*-closed sets such that $F_\alpha \neq F_\beta$. If $F_\alpha \cap F_\beta \subset F_\delta$, then either $F_\alpha = F_\delta$ or $F_\beta = F_\delta$

Proof: Given that $F_\alpha \cap F_\beta \subset F_\delta$. If $F_\alpha = F_\delta$ then there is nothing to prove.
If $F_\alpha \neq F_\delta$ then we have to prove $F_\beta = F_\delta$. Now $F_\beta \cap F_\delta = F_\beta \cap (F_\delta \cap X) = F_\beta \cap (F_\delta \cap (F_\alpha \cup F_\beta)$(by theorem 3.10 (ii)) $F_\beta \cap ((F_\delta \cap F_\alpha) \cup (F_\delta \cap F_\beta)) = (F_\beta \cap F_\delta \cap F_\alpha) \cup (F_\beta \cap F_\delta \cap F_\beta) = (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta)$ (by $F_\alpha \cap F_\beta \subset F_\delta$) $= (F_\alpha \cup F_\delta) \cap F_\beta = X \cap F_\beta$ (Since F_α and F_δ are maximal *pg*-closed sets by theorem[3.10](ii), $F_\alpha \cup F_\delta = X$) $= F_\beta$.
That is $F_\beta \cap F_\delta = F_\beta \Rightarrow F_\beta \subset F_\delta$ Since F_β and F_δ are maximal *pg*-closed sets, we have $F_\beta = F_\delta$ Therefore $F_\beta = F_\delta$

Theorem 3.13: Let F_α, F_β and F_δ be different maximal *pg*-closed sets to each other. Then $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.

Proof: Let $(F_\alpha \cap F_\beta) \subset (F_\alpha \cap F_\delta) \Rightarrow (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta) \subset (F_\alpha \cap F_\delta) \cup (F_\delta \cap F_\beta) \Rightarrow (F_\alpha \cup F_\delta) \cap F_\beta \subset F_\delta \cap (F_\alpha \cup F_\beta)$. Since by theorem 3.10(ii), $F_\alpha \cup F_\delta = X$ and $F_\alpha \cup F_\beta = X \Rightarrow X \cap F_\beta \subset F_\delta \cap X \Rightarrow F_\beta \subset F_\delta$ From the definition of maximal *pg*-closed set it follows that $F_\beta = F_\delta$, which is a contradiction to the fact that F_α, F_β and F_δ are different to each other. Therefore $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.

Theorem 3.14: Let F be a maximal *pg*-closed set and x be an element of F. Then $F = \cup$ { S: S is a *pg*-closed set containing x such that $F \cup S \neq X$}.

Proof: By theorem 3.12 and fact that F is a *pg*-closed set containing x, we have $F \subset \cup$ {S: S is a *pg*-closed set containing x such that $F \cup S \neq X$} – F. Therefore we have the result.

Theorem 3.15: Let F be a proper nonempty cofinite *pg*-closed set. Then $\exists$ (cofinite) maximal *pg*-closed set E such that $F \subset E$.

Proof: If F is maximal *pg*-closed set, we may set E = F. If F is not a maximal *pg*-closed set, then $\exists$ (cofinite) *pg*-closed set F_1 such that $F \subset F_1 \neq X$. If F_1 is a maximal *pg*-closed set, we may set $E = F_1$. If F_1 is not a maximal *pg*-closed set, then $\exists$ a (cofinite) *pg*-closed set F_2 such that $F \subset F_1 \subset F_2 \neq X$. Continuing this process, we have a sequence of *pg*-closed, $F \subset F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots$. Since F is a cofinite set, this process repeats only finitely. Then, finally we get a maximal *pg*-closed set $E = E_n$ for some positive integer n.

Theorem 3.16: Let F be a maximal *pg*-closed set. If x is an element of X-F. Then $X\text{-}F \subset E$ for any *pg*-closed set E containing x.

Proof: Let F be a maximal *pg*-closed set and x in X-F. E $\not\subset$ F for any *pg*-closed set E containing x. Then E $\cup$ F = X by theorem 3.10(ii). Therefore X-F $\subset$ E.

4. Minimal *pg*-Closed set and Maximal *pg*-open set:

We now introduce minimal *pg*-closed sets and maximal *pg*-open sets in topological spaces as follows.

Definition 4.1: A proper nonempty *pg*-closed subset F of X is said to be a **minimal *pg*-closed set** if any *pg*-closed set contained in F is ϕ or F.

Example 5: In Example 2**,** {d} is both Minimal closed set and Minimal *pg*-closed set, {a}, {b}, {c} are Minimal *pg*-closed set but not Minimal closed set.

Remark 5: Minimal closed and minimal *pg*-closed set are independent to each other:

Example 6: In Example 1, {c, d} is Minimal closed but not Minimal *pg*-closed set and {c} and {d} are Minimal *pg*-closed but not Minimal closed.

Definition 4.2: A proper nonempty *pg*-open U$\subset$X is said to be a **maximal *pg*-open set** if any *pg*-open set containing U is either X or U.

Example 7: In Example 2, {a, b, c} is Maximal open set and maximal *pg*-open set but {a, b, d}, {a, c, d} and {b, c, d} are Maximal *pg*-open but not maximal open.

Remark 6: Maximal open set and maximal *pg*-open set are independent to each other.

Example 8: In Example 1, {a, b} is Maximal open set but not maximal *pg*-open set and {a, b, c}, {a, b, d} are Maximal *pg*-open but not maximal open.

Theorem 4.1: A proper nonempty subset U of X is maximal *pg*-open set iff X-U is a minimal *pg*-closed set.

Proof: Let U be a maximal *pg*-open set. Suppose X-U is not a minimal *pg*-closed set. Then $\exists$ *pg*-closed set V $\neq$ X-U such that $\phi \neq$ V$\subset$ X-U. That is U$\subset$ X-V and X-V is a *pg*-open set which is a contradiction for U is a minimal *pg*-closed set. Conversely let X-U be a minimal *pg*-closed set. Suppose U is not a maximal *pg*-open set. Then $\exists$ *pg*-open set E $\neq$ U such that U $\subset$ E $\neq$ X. That is $\phi \neq$ X-E $\subset$ X-U and X-E is a *pg*-closed set which is a contradiction for X-U is a minimal *pg*-closed set. Therefore U is a maximal *pg*-closed set.

Lemma 4.1:
(i) Let U be a minimal *pg*-closed set and W be a *pg*- closed set. Then U $\cap$ W = ϕ or U subset W.
(ii) Let U and V be minimal *pg*- closed sets. Then U $\cap$ V = ϕ or U = V.

Proof:
(i) Let U be a minimal *pg*-closed set and W be a *pg*-closed set. If U $\cap$ W =ϕ, then there is nothing to prove.

If U $\cap$ W $\neq \phi$. Then U $\cap$W $\subset$U. Since U is a minimal *pg*-closed set, we have U $\cap$ W = U. Therefore U$\subset$W.

(ii) Let U and V be minimal *pg*-closed sets. If U$\cap$V $\neq \phi$, then U$\subset$ V and V$\subset$ U by (i). Therefore U = V.

Theorem 4.2: Let U be a minimal *pg*-closed set. If x$\in$U, then U$\subset$W for any regular open neighborhood W of x.

Proof: Let U be a minimal *pg*-closed set and x be an element of U. Suppose $\exists$ an regular open neighborhood W of x such that U $\not\subset$ W. Then U $\cap$ W is a *pg*-closed set such that U $\cap$ W $\subset$ U and U $\cap$ W $\neq \phi$. Since U is a minimal *pg*-closed set, we have U$\cap$ W = U. That is U $\subset$ W, which is a contradiction for U $\not\subset$ W. Therefore U$\subset$W for any regular open neighborhood W of x.

Theorem 4.3: Let U be a minimal *pg*-closed set. If x$\in$U, then U$\subset$W for some *pg*-closed set W containing x.

Theorem 4.4: Let U be a minimal *pg*-closed set. Then U = $\cap${W: W$\in$*PGO*(X, x)} for any element x of U.

Proof: By theorem [4.3] and U is *pg*-closed set containing x, we have U$\subset\cap${ W: W$\in$*PGO*(X, x)}$\subset$U.

Theorem 4.5: Let U be a nonempty *pg*-closed set. Then the following three conditions are equivalent.
(i) U is a minimal *pg*-closed set
(ii) $U \subset pg(S)^-$ for any nonempty subset S of U
(iii) $pg(U)^- = pg(S)^-$ for any nonempty subset S of U.

Proof: (i) $\Rightarrow$ (ii) Let $x \in U$; U be minimal *pg*-closed set and $S(\neq \phi) \subset U$. By theorem[4.3], for any *pg*-closed set W containing x, $S \subset U \subset W \Rightarrow S \subset W$. Now $S = S \cap U \subset S \cap W$. Since $S \neq \phi$, $S \cap W \neq \phi$. Since W is any *pg*-closed set containing x, by theorem[4.3], $x \in pg(S)^-$. That is $x \in U \Rightarrow x \in pg(S)^- \Rightarrow U \subset pg(S)^-$ for any nonempty subset S of U.

(ii) $\Rightarrow$ (iii) Let S be a nonempty subset of U. That is $S \subset U \Rightarrow pg(S)^- \subset pg(U)^- \rightarrow (1)$. Again from (ii) $U \subset pg(S)^-$ for any $S(\neq \phi) \subset U \Rightarrow pg(U)^- \subset pg(pg(S)^-)^- = pg(S)^-$. That is $pg(U)^- \subset pg(S)^- \rightarrow (2)$.

From (1) and (2), we have $pg(U)^- = pg(S)^-$ for any nonempty subset S of U.

(iii) $\Rightarrow$ (i) From (3) we have $pg(U)^- = pg(S)^-$ for any nonempty subset S of U. Suppose U is not a minimal *pg*-closed set. Then $\exists$ a nonempty *pg*-closed set V such that $V \subset U$ and $V \neq U$. Now $\exists$ an element a in U such that $a \notin V \Rightarrow a \in V^c$. That is $pg(\{a\})^- \subset pg(V^c)^- = V^c$, as V^c is *pg*-closed set in X. It follows that $pg(\{a\})^- \neq pg(U)^-$. This is a contradiction for $pg(\{a\})^- = pg(U)^-$ for any $\{a\}(\neq \phi) \subset U$. Therefore U is a minimal *pg*-closed set.

Theorem 4.6: Let V be a nonempty finite *pg*-closed set. Then $\exists$ at least one (finite) minimal *pg*-closed set U such that $U \subset V$.

Proof: Let V be a nonempty finite *pg*-closed set. If V is a minimal *pg*-closed set, we may set U = V. If V is not a minimal *pg*-closed set, then $\exists$ (finite) *pg*-closed set V_1 such that $\phi \neq V_1 \subset V$. If V_1 is a minimal *pg*-closed set, we may set $U = V_1$. If V_1 is not a minimal *pg*-closed set, then $\exists$ (finite) *pg*-closed set V_2 such that $\phi \neq V_2 \subset V_1$. Continuing this process, we have a sequence of *pg*-closed sets $V \supset V_1 \supset V_2 \supset V_3 \supset \ldots \supset V_k \supset \ldots$ Since V is a finite set, this process repeats only finitely. Then finally we get a minimal *pg*-closed set $U = V_n$ for some positive integer n.

Corollary 4.1: Let X be a locally finite space and V be a nonempty *pg*-closed set. Then $\exists$ at least one (finite) minimal *pg*-closed set U such that $U \subset V$.

Proof: Let X be a locally finite space and V be a nonempty *pg*-closed set. Let x in V. Since X is locally finite space, we have a finite open set V_x such that x in V_x. Then $V \cap V_x$ is a finite *pg*-closed set. By Theorem 4.6 $\exists$ at least one (finite) minimal *pg*-closed set U such that $U \subset V \cap V_x$. That is $U \subset V \cap V_x \subset V$. Hence $\exists$ at least one (finite) minimal *pg*-closed set U such that $U \subset V$.

Corollary 4.2: Let V be a finite minimal open set. Then $\exists$ at least one (finite) minimal *pg*-closed set U such that $U \subset V$.

Proof: Let V be a finite minimal open set. Then V is a nonempty finite *pg*-closed set. By Theorem 4.6, $\exists$ at least one (finite) minimal *pg*-closed set U such that $U \subset V$.

Theorem 4.7: Let U; U_λ be minimal *pg*-closed sets for any element $\lambda \in \Gamma$. If $U \subset \cup_{\lambda \in \Gamma} U_\lambda$, then $\exists$ an element $\lambda \in \Gamma$ such that $U = U_\lambda$.

Proof: Let $U \subset \cup_{\lambda \in \Gamma} U_\lambda$. Then $U \cap (\cup_{\lambda \in \Gamma} U_\lambda) = U$. That is $\cup_{\lambda \in \Gamma}(U \cap U_\lambda) = U$. Also by lemma[4.1] (ii), $U \cap U_\lambda = \phi$ or $U = U_\lambda$ for any $\lambda \in \Gamma$. It follows that $\exists$ an element $\lambda \in \Gamma$ such that $U = U_\lambda$.

Theorem 4.8: Let U; U_λ be minimal *pg*-closed sets for any $\lambda \in \Gamma$. If $U = U_\lambda$ for any $\lambda \in \Gamma$, then $(\cup_{\lambda \in \Gamma} U_\lambda) \cap U = \phi$.
Proof: Suppose that $(\cup_{\lambda \in \Gamma} U_\lambda) \cap U \neq \phi$. That is $\cup_{\lambda \in \Gamma}(U_\lambda \cap U) \neq \phi$. Then $\exists$ an element $\lambda \in \Gamma$ such that $U \cap U_\lambda \neq \phi$. By lemma[4.1](ii), we have $U = U_\lambda$, which contradicts the fact that $U \neq U_\lambda$ for any $\lambda \in \Gamma$. Hence $(\cup_{\lambda \in \Gamma} U_\lambda) \cap U = \phi$.

Theorem 4.9: A proper nonempty subset F of X is maximal *pg*-open set iff X-F is a minimal *pg*-closed set.

Proof: Let F be a maximal *pg*-open set. Suppose X-F is not a minimal *pg*-open set. Then $\exists$ *pg*-open set $U \neq X\text{-}F$ such that $\phi \neq U \subset X\text{-}F$. That is $F \subset X\text{-}U$ and X-U is a *pg*-open set which is a contradiction for F is a minimal *pg*-closed set.

Conversely let X-F be a minimal *pg*-open set. Suppose F is not a maximal *pg*-open set. Then $\exists$ *pg*-open set $E \neq F$ such that $F \subset E \neq X$. That is $\phi \neq X\text{-}E \subset X\text{-}F$ and X-E is a *pg*-open set which is a contradiction for X-F is a minimal *pg*-closed set. Therefore F is a maximal *pg*-open set.

Theorem 4.10:
(i) Let F be a maximal *pg*-open set and W be a *pg*-open set. Then $F \cup W = X$ or $W \subset F$.
(ii) Let F and S be maximal *pg*-open sets. Then $F \cup S = X$ or $F = S$.

Proof: (i) Let F be a maximal *pg*-open set and W be a *pg*-open set. If $F \cup W = X$, then there is nothing to prove. Suppose $F \cup W \neq X$. Then $F \subset F \cup W$. Therefore $F \cup W = F \Rightarrow W \subset F$.
(ii) Let F and S be maximal *pg*-open sets. If $F \cup S \neq X$, then we have $F \subset S$ and $S \subset F$ by (i). Therefore $F = S$.

Theorem 4.11: Let F be a maximal *pg*-open set. If x is an element of F, then for any *pg*-open set S containing x, $F \cup S = X$ or $S \subset F$.

Proof: Let F be a maximal *pg*-open set and x is an element of F. Suppose $\exists$ *pg*-open set S containing x such that $F \cup S \neq X$. Then $F \subset F \cup S$ and $F \cup S$ is a *pg*-open set, as the finite union of *pg*-open sets is a *pg*-open set. Since F is a *pg*-open set, we have $F \cup S = F$. Therefore $S \subset F$.

Theorem 4.12: Let F_α, F_β, F_δ be maximal *pg*-open sets such that $F_\alpha \neq F_\beta$. If $F_\alpha \cap F_\beta \subset F_\delta$, then either $F_\alpha = F_\delta$ or $F_\beta = F_\delta$

Proof: Given that $F_\alpha \cap F_\beta \subset F_\delta$. If $F_\alpha = F_\delta$ then there is nothing to prove.
If $F_\alpha \neq F_\delta$ then we have to prove $F_\beta = F_\delta$. Now $F_\beta \cap F_\delta = F_\beta \cap (F_\delta \cap X) = F_\beta \cap (F_\delta \cap (F_\alpha \cup F_\beta)$(by thm. 4.10 (ii)) $= F_\beta \cap ((F_\delta \cap F_\alpha) \cup (F_\delta \cap F_\beta)) = (F_\beta \cap F_\delta \cap F_\alpha) \cup (F_\beta \cap F_\delta \cap F_\beta) = (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta)$ (by $F_\alpha \cap F_\beta \subset F_\delta$) $= (F_\alpha \cup F_\delta) \cap F_\beta = X \cap F_\beta$ (Since F_α and F_δ are maximal *pg*-open sets by theorem[4.10](ii), $F_\alpha \cup F_\delta = X$) $= F_\beta$. That is $F_\beta \cap F_\delta = F_\beta \Rightarrow F_\beta \subset F_\delta$ Since F_β and F_δ are maximal *pg*-open sets, we have $F_\beta = F_\delta$ Therefore $F_\beta = F_\delta$

Theorem 4.13: Let F_α, F_β and F_δ be different maximal *pg*-open sets to each other. Then $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.

Proof: Let $(F_\alpha \cap F_\beta) \subset (F_\alpha \cap F_\delta) \Rightarrow (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta) \subset (F_\alpha \cap F_\delta) \cup (F_\delta \cap F_\beta) \Rightarrow (F_\alpha \cup F_\delta) \cap F_\beta \subset F_\delta \cap (F_\alpha \cup F_\beta)$. Since by theorem 4.10(ii), $F_\alpha \cup F_\delta = X$ and $F_\alpha \cup F_\beta = X \Rightarrow X \cap F_\beta \subset F_\delta \cap X \Rightarrow F_\beta \subset F_\delta$ From the definition of maximal *pg*-open set it follows that $F_\beta = F_\delta$, which is a contradiction to the fact that F_α, F_β and F_δ are different to each other. Therefore $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.

Theorem 4.14: Let F be a maximal *pg*-open set and x be an element of F. Then $F = \cup\{S$: S is a *pg*-open set containing x such that $F \cup S \neq X\}$.

Proof: By theorem 4.12 and fact that F is a *pg*-open set containing x, we have $F \subset \cup\{S$: S is a *pg*-open set containing x such that $F \cup S \neq X\}$ – F. Therefore we have the result.

Theorem 4.15: Let F be a proper nonempty cofinite *pg*-open set. Then $\exists$ (cofinite) maximal *pg*-open set E such that $F \subset E$.

Proof: If F is maximal *pg*-open set, we may set E = F. If F is not a maximal *pg*-open set, then $\exists$ (cofinite) *pg*-open set F_1 such that $F \subset F_1 \neq X$. If F_1 is a maximal *pg*-open set, we may set $E = F_1$. If F_1 is not a maximal *pg*-open set, then $\exists$ a (cofinite) *pg*-open set F_2 such that $F \subset F_1 \subset F_2 \neq X$. Continuing this process, we have a sequence of *pg*-open, $F \subset F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots$. Since F is a cofinite set, this process repeats only finitely. Then, finally we get a maximal *pg*-open set $E = E_n$ for some positive integer n.

Theorem 4.16: Let F be a maximal *pg*-open set. If x is an element of X-F. Then $X\text{-}F \subset E$ for any *pg*-open set E containing x.

Proof: Let F be a maximal *pg*-open set and x in X-F. $E \not\subset F$ for any *pg*-open set E containing x. Then $E \cup F = X$ by theorem 4.10(ii). Therefore $X\text{-}F \subset E$.

Conclusion:

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