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INCLINE RELATIONAL EQUATIONS

AR. Meenakshi. & P. Shakila Banu*

Department of Mathematics, Karpagam University, Coimbatore –21, India
E-mail: arm_meenakshi@yahoo.co.in, shakimeeran10@gmail.com

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ABSTRACT

Inclines are additively idempotent semirings in which products are less than (or) equal to either factor. In this paper, we discuss the consistency of incline relational equations, that is, equations of the form xA=b where A is a matrix and b is a vector over an incline. We apply our results for incline relational equations involving matrices over special types of inclines such as incline whose elements are all linearly ordered, incline whose idempotent elements are all linearly ordered, a regular incline whose elements are all linearly ordered and a distributive lattice whose elements are all linearly ordered. We deduce the solution set of a fuzzy relational equation as a special case.

Keywords: *Incline, regular incline, distributive lattice.*

MS Classification: *15A24, 15B33.*

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1. INTRODUCTION

Inclines are a generalization of Boolean and Fuzzy algebra. The notion of inclines and their applications were described in Cao, Kim and Roush [1]. Kim and Roush have surveyed and outlined algebraic properties of inclines and incline matrices [4]. Recently in [6], it is proved that an element in an incline is regular if and only if it is idempotent, further some characterization of regular elements in an incline are discussed and exhibited that every commutative regular incline is a distributive lattice.

Sanchez [7] initiated the study on fuzzy relational equations of the form xA=b, based on the max-min composition. A method of determining minimum solutions of fuzzy relational equations are provided in [3]. In [2], Cho has proved that xA=b is consistent when A is regular, that is, the matrix equation AXA = A has a solution.

In this paper, we discuss the consistency of the equation xA=b where A is a matrix and b is a vector over an incline £. We have determined the existence of the maximum solution of xA=b under the condition that each column of A is comparable with the corresponding component of the vector b in £. This leads to the structure of the solution set Ω(A,b), where A is a matrix over special types of inclines such as incline whose elements are all linearly ordered, incline whose idempotent elements are all linearly ordered, a regular incline whose elements are all linearly ordered and a distributive lattice whose elements are all linearly ordered. This includes the result found in [7] as a special case for fuzzy relational equations. In section 2, we present the basic definitions, notations and required results on inclines. In section 3, the consistency of incline relational equations are discussed. The results in the present paper are the generalization of the results on fuzzy relational equation available in the literature [2, 3,7].

2. PRELIMINARIES

In this section, we present some basic definitions, notations and required results on inclines.

Definition 2.1: An incline is a nonempty set £ with binary operations addition and multiplication denoted as (+,·) (We usually suppress the 'dot' in a·b and write as ab) satisfying the following axioms: for a, b, c ε £

(i) a + b = b + a
(ii) a+ (b + c) = (a + b) + c: a(bc) = (ab) c
(iii) a(b + c) = ab +ac: (b +c)a = ba + ca
(iv) a + a =a
(v) a +ac =a: c +ac = c

Corresponding author: P. Shakila Banu*, *E-mail: shakimeeran10@gmail.com

Throughout we consider an incline (£, +, ·) with the order relation ≤" defined as $x \le y \Leftrightarrow x + y = y$. This incline order relation has the following properties:

$x + y \ge x$ and $x + y \ge y$ for any x, y ε £ (2.1)

$xy \le x$ and $xy \le y$ for any x, y ε £ (2.2)

Definition 2.2: S is said to be a comparable subset of an incline £ if and only if x and y are comparable elements, that is either $x \le y$ (or) $y \le x$ for each x, y ε S.

Definition 2.3: a ε £ is said to be regular if there exists an elements x ε £ such that a x a = a. Then x is called a g-inverse of a and a{1} denotes the set of all g-inverses of a.

An incline £ is regular if and only if each element of £ is regular.

Definition 2.4: [6] An element a ε £ is regular if and only if a is idempotent.

Proposition 2.1: [6] A commutative incline £ is regular <=> £ is a distributive lattice.

Let $£_{mn}$ and $£^n$ denotes the set of all m x n matrices and the set of all n vectors over £ respectively. Let DL be the set of all idempotent elements in £. DL_{mn} and DL^n be the set of all m x n matrices and the set of all vectors in DL. For A ε $£_{mn}$, A_{i*} and A_{*j} denotes the ith row and jth column of A respectively. Throughout the matrix operations in $£_{mn}$ are induced by the incline operations in £.

Definition 2.5: Let A= (a_{ik}) ε $£_{mn}$ and B= (b_{kj}) ϵ $£_{nl}$ then AB =R is defined as $\sum_{k=1}^{n} a_{ik} b_{kj} = r_{ij}$ for all i and j. Where 'Σ' denotes the addition operation on £.

Definition 2.6: Let A= (a_{ij}) ε $£_{mn}$ and B= (b_{ij}) ϵ $£_{mn}$. We write $A \ge B$ if $a_{ij} \ge b_{ij}$ for all i and j.

3. INCLINE RELATIONAL EQUATIONS

In this section, we discuss the consistency of the equation of the form

$$x A = b \qquad (3.1)$$

with $x = [x_j / j \varepsilon N_m]$, $b = [b_k / k \varepsilon N_n]$ and A= (a_{ij}) ε $£_{mn}$; where N_r denotes the set of all positive integers 1 to r. Let Ω (A,b) denotes the set of all solutions of (3.1).

Definition 3.1: If Ω (A, b) is a comparable set then any element $\hat{x}$ of Ω (A, b) is called a maximal solution of x A=b if for all xε Ω (a,b), $x \ge \hat{x}$ implies $x = \hat{x}$.

Definition 3.2: If Ω (A,b) is a comparable set then any element $\check{x}$ of Ω (A,b) is called a minimal solution of x A=b if for all x ε Ω (A, b), $x \le \check{x}$ implies $x = \check{x}$.

Lemma 3.1: Let x A=b be as in equation (3.1). If $\sum_j (a_{jk}) < b_k$ for some k ε N_n, then Ω (A, b) = φ.

Proof: If $\sum_j (a_{jk}) < b_k$, then by using incline properties (2.2) and (2.1), we have

$$x_j a_{jk} \le a_{jk} \le \sum_j (a_{jk})$$

$$\sum_j (x_j a_{jk}) \le \sum_j (a_{jk}) < b_k$$

Hence no values of x_j ε £ satisfies the equation x A=b. Therefore Ω (A,b) = φ.

Remark 3.1: The condition $\sum_j (a_{jk}) < b_k$ in Lemma (3.1) is essential. This is illustrated in the following:

Illustration 3.1: £ = ([0,1], sup (x, y), ×), '×' denotes the usual multiplication. £ is an incline.

Let us consider the equation x A=b,

where $A = \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$ ε $£_{22}$ and $b = (0.9 \quad 0.1)$ ε $£^2$ are given.

Since 0.6 < 0.9, the condition $\sum_j (a_{jk}) < b_k$ holds for the 1st column and $0.7 \nless 0.1$, the condition $\sum_j (a_{jk}) < b_k$ fails for the 2nd column.

Hence Ω (A, b) = φ.

Theorem 3.1: Let xA =b be as in (3.1), such that A_{*k} and b_k are comparable for each k. Then Ω (A,b) ≠ φ if and only if $\hat{x} = [\hat{x}_j \ / j \varepsilon \ N_m]$ defined as

$$\hat{x}_j = \min \sigma (a_{jk}, b_k) \tag{3.2}$$

Where $\sigma (a_{jk}, b_k) = \begin{Bmatrix} b_k & if & a_{jk} > b_k \\ 1 & otherwise & \end{Bmatrix}$ is the maximum solution.

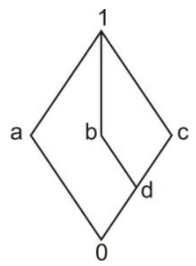
Proof: If Ω (A, b) ≠ φ then $\hat{x}$ is a solution of equation (3.2). For if $\hat{x}$ is not a solution, then $\hat{x}$ A≠ b. Hence $\sum_j \hat{x}_j a_{jk0} \neq b_{ko}$ for atleast one $k_o \varepsilon N_n$. By definition of $\hat{x}$, since $\hat{x}_j \leq b_k$ for each k, $\hat{x}_j \leq b_{ko}$. By our assumption,$\sum_j (a_{jk\,0}) < b_{ko}$ for some $k_o \varepsilon N_n$ and by Lemma (3.1), Ω (A,b) = φ which is a contradiction. Hence $\hat{x}$ is a solution of equation (3.2). Next, let us to prove that $\hat{x}$ is the maximum solution. If possible let us assume that y be a solution of equation (3.2) such that y > $\hat{x}$, by using definition (2.6), $y_j > \hat{x}_j$ for each j ε N_m, that is $y_{jo} > \hat{x}_{jo}$ for j_o ε N_m. Since A_{*k} is comparable with b_k, for each k ε N_n, σ ($a_{jk,}b_k$) can be determined. Therefore, by definition of $\hat{x}$, we have $y_{jo} > \hat{x}_{j0} = \min \sigma (a_{jk,}b_k)$.
Since Ω (A,b) ≠ φ by Lemma (3.1), $\sum_j (a_{jk}) > b_k$ for each k ε N_n. Hence $b_{ko} \neq \Sigma(y_j \, a_{jko})$, for k_0 ε N_n which contradicts our assumption y ε Ω (A,b). Therefore $\hat{x}$ is the maximum solution of equation (3.2). Converse is trivial.

Remark 3.2: In the above Theorem (3.1), the condition that the kth column of A and the kth component of b to be comparable is essential in determining the maximum solution. This is illustrated in the following example.

Example 3.2: Let us consider the incline £ = {0, a, b, c, d, 1}, lattice ordered by the following Hasse graph.

Define £x£ → £ as follows

$$xy = \begin{Bmatrix} d & if \ x, y \varepsilon \ \{1, b, c, d\} \\ 0 & otherwise \end{Bmatrix}$$



Let us consider the equation xA=b,

where $A = \begin{pmatrix} 1 & c \\ b & 0 \end{pmatrix} \varepsilon £_{22}$ and b = [d 0] ε $£^2$ are given.

Since 1 > d, the condition $\sum_j (a_{jk}) < b_k$ in Lemma (3.1) fails for the 1st column. Since c > 0, the condition $\sum_j (a_{jk}) < b_k$ in Lemma (3.1) fails for the 2nd column. Therefore, Ω (A, b) ≠ φ. Next to determine the solution set Ω (A,b).

$$[x_1 \ x_2] \begin{pmatrix} 1 & c \\ b & 0 \end{pmatrix} = [d \ 0]$$

$x_1 1 + x_2 \, b = d$ and $x_1 \, c = 0$. Since $x_1 \, c = 0$, for every $x_1 \varepsilon \ \{0, a\}$ and $x_1 1 + \ x_2 \, b = d, \ for \ every$ $x_1, x_2 \varepsilon \ \{1, b, c, d\}$.

Therefore Ω (A, b) = { (0,1) (0,b) (0,c), (0,d), (a,1) (a, b) (a,c) (a, d)}.

In Ω (A,b), the elements (a, b) and (a, c) are not comparable. ThereforeΩ (A, b) is not a comparable set. Hence by definition (3.1), it has no maximum element. Thus xA=b has no maximum solution. However, Ω (A, b) ≠ φ.

Thus Theorem (3.1) fails.

Remark 3.3: If the elements of £ are linearly ordered then the comparability of A_{*k} and $b_{k,}$ for k ε N_n automatically holds. Hence Theorem (3.1) reduces to the following:

Corollary 3.1: Let £ be an incline whose elements are all linearly ordered and the equation xA=b be as in (3.1). Then Ω (A,b) ≠ φ if and only if $\hat{x}$ defined as in (3.2) is the maximum solution.

Theorem 3.2: Let £ be an incline whose idempotent elements are all linearly ordered and the equation xA=b with x=[x_j /j ε N_m], b=[b_k/kεN_n] εDL^n and A ε DL_{mn}. Then Ω (A, b) ≠ φ if and only if $\hat{x}$ defined as in (3.2), is the maximum solution.

Proof: Since idempotent elements are all linearly ordered, for A ε DL_{mn}, b ε DL^n, the comparability of A_{*k} and b_k automatically holds for each k ε N_n. Then the theorem can be proved in a similar manner as that of Theorem (3.1).

Corollary 3.2: Let £ be a regular incline whose elements are all linearly ordered and the equation xA=b be as in (3.1). Then Ω (A,b) ≠φ if and only if $\hat{x}$ defined as in (3.2), is the maximum solution.

Proof: Since £ is a regular incline by definition (2.4) each element of £ is idempotent and therefore DL=£. Then the rest follows from Theorem (3.2).

Remark 3.4: Since by Proposition (2.1), a commutative regular incline is a distributive lattice and DL=£, Theorem (3.2) reduces to the following:

Corollary 3.3: Let £ be a distributive lattice whose elements are all linearly ordered and the equation xA=b be as in (3.1). Then Ω (A,b) ≠ φ if and only if $\hat{x}$ defined as in (3.2), is the maximum solution.

Remark 3.5: The condition that A ε DL_{mn} in Theorem (3.2) is essential. The condition that the elements are to be linearly ordered cannot be relaxed in the above Corollaries (3.1), (3.2) and (3.3). These are illustrated in the following.

Example 3.3: Let us consider the incline £ in Example (3.2), whose idempotent elements 0 and d are linearly ordered. For the equation xA=b,

Where A= $\begin{pmatrix} 1 & c \\ d & 0 \end{pmatrix}$ ε DL_{22} and b = [d 0] ∉ DL^2.

In Example (3.2), we have seen that Ω (A, b) ≠ φ but xA=b has no maximum solution.

Thus Theorem (3.2) fails.

Example 3.4: Let us consider the set D = {a, b, c} and the incline £ = (P (D),∪, ∩), where P(D) is the power set of D with set inclusion '⊆' as the order relation '≤'. Here, £ is a commutative regular incline hence by proposition (2.1), £ is a distributive lattice whose elements are all idempotent but not linearly ordered. For instance, {a}, {b} and {c} are not comparable.

Let us consider the equation x A=b,

where A= $\begin{pmatrix} \{a, c\} & \{c\} \\ \{a\} & \{a, b, c\} \end{pmatrix}$ ε $£_{22}$ and b = [{c} {b, c}] ε$£^2$ are given.

Here the condition $\sum_j (a_{jk}) < b_k$ fails for both the columns. Therefore by Lemma (3.1), Ω (A, b) ≠ φ.

Next to determine the solution set Ω (A, b)

$$[x_1\ x_2] \begin{pmatrix} \{a, c\} & \{c\} \\ \{a\} & \{a, b, c\} \end{pmatrix} = [\{c\}\,\{b, c\}]$$

On computation we get,

x_1 ε {{c} {b, c}} and x_2 ε {{b} {b, c}}

Therefore Ω (A, b) = {({c}, {b}) ({c}, {b, c}) ({b, c}, {b}) ({b, c}, {b, c})}.

In Ω (A, b), the elements ({c}, {b, c}) and ({b, c}, {b}) are not comparable. Hence by definition (3.1), xA=b has no maximum solution. However Ω (A, b) ≠ φ.

Thus corollaries (3.1), (3.2) and (3.3) fail.

Remarks 3.6: It is well known that (p.2 [1]), every Fuzzy algebra is an incline. However, the converse not true. This can be seen from the incline

£ = {[0, 1], sup(x, y),×} where '×' is the ordinary multiplication. £ is an incline whose elements are all linearly ordered. Here, for any x, y ε£ if $x \le y$ then $x + y = y$ but $xy \neq x$. Therefore £ is not the max-min fuzzy algebra.

If the equation (3.1) is a fuzzy relational equation that is, A is a matrix over the max-min fuzzy algebra then Theorem (3.1) reduces to the following result of Sanchez [7] (Quoted in [5] p.70).

Corollary 3.4: Let xA=b be the fuzzy relational equation as in (3.1). Then Ω (A, b) $\neq \varphi$ if and only if $\hat{x}$ defined as in (3.2), is the maximum solution.

CONCLUSION

In this paper, we have discussed the consistency of the equation of the form xA=b. We have determined the condition for the existence of the maximum solution of xA=b. Further, in xA=b, if A is regular then as in [2] it can be seen that Xb is a solution for all X satisfying the equation AXA=A. Thus the main results in the present paper are the generalization of the results shown in the references [2], [3] and [7].

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