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A METHOD TO IMPROVE THE NUMERICAL SOLUTION OF FUZZY INITIAL VALUE PROBLEMS

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ABSTRACT

This paper presents a solution for first order fuzzy differential equation by rational extrapolation method based on standard Euler method and modified midpoint method that increases the order of accuracy of the solution. This method is discussed in detail and is followed by a complete error analysis. The accuracy and efficiency of the proposed method is illustrated by solving some fuzzy initial value problem.

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1. INTRODUCTION

Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. The concept of fuzzy derivative was first introduced by S.L.Change and L.A.Zadeh in [5].D.Dubois and Prade in [7] discussed differentiation with fuzzy features. M.L.Puri and D.A.Ralesec in [14] and R.Goetschel and W.Voxman in [8] contributed towards the differential of fuzzy functions. The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva in [9,10] and by S.Seikkala in [15].Recently many research papers are focused on numerical solution of fuzzy initial value problems (FIVPS).Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M. Friedman, A. Kandel in [12] through Euler method and authors in [2,13]by Runge – Kutta methods.S.Abbasbandy and T.Allah Viranloo tried to improve the solution of fuzzy initial value problems by Polynomial Extrapolation in[1].

This paper is organised as follows: In section 2, some basic results on fuzzy numbers and definition of fuzzy derivative are given. The Rational Extrapolation method and Modified Midpoint method are discussed in section3. Section 4 contains fuzzy Cauchy problem whose numerical solution is the main interest of this paper. The proposed method is illustrated by some solved numerical example in section 5 and compared with Euler's method ,Runge Kutta method and with Polynomial Extrapolation. The conclusion is in section 6.

2. PRELIMINARIES

Definition 2.1: A fuzzy number u is a fuzzy subset of R (ie) $u: R \rightarrow [0,1]$ satisfying the following conditions:
1.u is normal (ie) $\exists\ x_0 \in R$ with $u(x_0) = 1$.
2.u is convex fuzzy set (ie) $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1], x, y \in R$.
3.u is upper semi continuous on R. 4.$\overline{\{x \in R, u(x) > 0\}}$ is compact.

Let E be the class of all fuzzy subsets of R. Then E is called the space of fuzzy numbers [9].

Clearly, $R \subset E$ and $R \subset E$ is understood as$R = \{\aleph_x : \aleph\ is\ usual\ real\ number\}$.

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An arbitrary fuzzy number is represented by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right), 0 \leq r \leq 1$ that satisfies the following requirements.
1. $\underline{u}(r)$is a bounded left continous non-decreasing function over [0,1],with respect to any 'r'.
2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0,1] with respect to any 'r'.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Then the r-level set is $[u]_r = \{x \ \backslash u(x) \geq r\}, 0 < r \leq 1$ is a closed and bounded interval,

denoted by$[u]_r = [\underline{u}(r), \overline{u}(r)]$.

And clearly, $[u]_0 = \{x \ \backslash u(x) > 0\}$ is compact.

Definition 2.2: A triangular fuzzy number u is a fuzzy set in E that is characterised by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l \leq u_c \leq u_r$ such that $[u]_0 = [u_l; u_r]$ and$[u]_1 = \{u_c\}$.

The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \frac{x-u_l}{u_c-u_l}, \ u_l \leq x \leq u_c \\ 1, x = u_c \\ \frac{u_r-x}{u_r-u_c}, u_c \leq x \leq u_r \end{cases}.$$

We will write (i) $u > 0, if\ u_l > 0$. (ii) $u \geq 0, if\ u_l \geq 0$. (iii) $u < 0, if\ u_c < 0$. (iv) $u \leq 0, if\ u_c \leq 0$.

Let I be a real interval.A mapping $y: I \to E$ is called a fuzzy process, and its $\alpha - level$ set is denoted by

$[y(t)]_\alpha = \left[\underline{y}(t,y), \overline{y}(t,y)\right], t \in I, 0 < \alpha \leq 1.$

The seikkala derivative $y^{'}(t)$ of a fuzzy process is defined by

$[y^{'}(t)]_\alpha = \left[\underline{y}^{'}(t,y), \overline{y}^{'}(t,y)\right], t \in I, 0 < \alpha \leq 1$, provided that this equation defines a fuzzy number, as in [15].

Lemma 2.1: Let $u, v \in E$ and s scalar, then for $r \in (0,1]$

$$\begin{aligned} &[u+v]_r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)] \\ &[u-v]_r = [\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)] \\ &[u.v]_r = [\min\{\underline{u}(r).\underline{v}(r), \underline{u}(r).\overline{v}(r), \overline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r)\}, \\ &\qquad\qquad \max\{\underline{u}(r).\underline{v}(r), \underline{u}(r).\overline{v}(r), \overline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r)\}], \\ &[su]_r = s\,[u]_r. \end{aligned}$$

3. RATIONAL EXTRAPOLATION METHOD:

Consider the initial value problem

$$\begin{cases} y^{'}(t) = f\left(t, y(t)\right); t_0 \leq t \leq b \\ \qquad y(t_0) = y_0 \end{cases} \tag{3.1}$$

We assume that
1. $f\left(t, y(t)\right)$ is defined and continuous in the strip$t_0 \leq t \leq b$,$-\infty < y < \infty$ with t_0 and b finite.

2. There exists a constant L such that for any t in $[t_0, b]$ and any two numbers$y\ and\ y^*$,
$|f(t,y) - f(t,y^*)| \leq L|y - y^*|$. These conditions are sufficient to prove that there exists on$[t_0, b]$, a unique continuous, differentiable function $y(t)$ satisfying (3.1)

In many situations in numerical analysis we wish to evaluate a number A_0,but are able to compute only an approximation $A(h)$,where h is a positive discretization parameter (typically step length) and where $A(h) \to A_0$ as$h \to 0$.

Let us suppose that, for any fixed N, $A(h)$ possesses an asymptotic expansion of the form

$A(h) = \mathrm{A}_0 + \mathrm{A}_1\mathrm{h} + \mathrm{A}_2\mathrm{h}^2 + \cdots\cdots\cdots\cdots\cdots + \mathrm{A}_N\mathrm{h}^N + \mathrm{R}_N(\mathrm{h}),\ \mathrm{R}_N(\mathrm{h}) = \mathrm{O}(\mathrm{h}^{N+1})$ as $h \to 0$.

Where the co-efficients $\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2, \cdots\cdots\cdots\cdots\cdots, \mathrm{A}_N$ are independent of h.

(i.e) $A(h) \sim \mathrm{A}_0 + \mathrm{A}_1\mathrm{h} + \mathrm{A}_2\mathrm{h}^2 + \cdots\cdots\cdots\cdots\cdots$ (3.2)

From a general sequence $h_0, h_1, h_2, \cdots\cdots\cdots\cdots\cdots$ of values of h,

where $h_0 > h_1 > h_2 > \cdots\cdots\cdots\cdots\cdots > h_s > 0$........................... (3.3)

for each value of h_s ,we compute $A(h_s)$ and denote the result by $a_s^{(0)}$.The rational extrapolation algorithm is defined by the following tableau

h_0	$a_0^{(0)}$			
h_1	$a_1^{(0)}$	$a_0^{(1)}$		
h_2	$a_2^{(0)}$	$a_1^{(1)}$	$a_0^{(2)}$	
h_3	$a_3^{(0)}$	$a_2^{(1)}$	$a_1^{(2)}$	$a_0^{(3)}$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (3.4)

Where $a_s^{(0)} = A(h_s),\ a_s^{(-1)} = 0,$

$$a_s^{(m)} = a_{s+1}^{(m-1)} + \frac{a_{s+1}^{(m-1)} - a_s^{(m-1)}}{\left(\frac{h_s}{h_{m+s}}\right)^2 \left[1 - \frac{\left(a_{s+1}^{(m-1)} - a_s^{(m-1)}\right)}{\left(a_{s+1}^{(m-1)} - a_{s+1}^{(m-2)}\right)}\right] - 1},$$

$$m = 1,2,\cdots\cdots\cdots\cdots\cdots,\ s = 0,1,2,\cdots\cdots\cdots \quad (3.5)$$

Then $a_s^{(m)} = A_0 + O(h_s^{2m+2})$.

3.1 Application to initial value problem in ordinary differential equations:

Let $y(t;h)$ be the approximation at t, given by the numerical method with step length h, to the theoretical solution $y(t)$ of the initial value problem (3.1).

We intend to use Rational Extrapolation to furnish approximations to $y(t)$ at the basic points $t_0 + jH, j = 0,1,2,\cdots$, where H is the basic step length.

We first choose a step length $h_0 = \frac{H}{N_0}$, where N_0 is a positive integer, and apply the numerical method N_0 times starting from $t = t_0$ to obtain an approximation $y(t_0 + H; h_0)$to the theoretical solution $y(t_0 + H)$.A second step length $h_1 = \frac{H}{N_1}$, N_1- a positive integer greater thanN_0, is chosen, and the method applied N_1 times, again starting from t_0,to yield the approximation $y(t_0 + H; h_1)$. Proceeding in this fashion for the sequence of step lengths$\{h_s\}$,

where $h_s = H/N_s$,$\{N_s\ / s = 0,1,2, \ldots\ldots\ldots\ldots\ldots\ldots..S$, beging an increasing sequence of +ve integers$\}$,we obtain the sequence of approximations $\{y(t_0 + H; h_s)\ /\ s = 0,1,2,3 \ldots\ldots.S\}$ to $y(t_0 + H)$.

Provided that there exists, for the given numerical method, an asymptotic expansion of the form

$A(h) \sim \mathrm{y(t)} + \mathrm{A}_1\mathrm{h} + \mathrm{A}_2\mathrm{h}^2 + \cdots\cdots\cdots\cdots\cdots$ (3.1.1)

(or)$A(h) \sim \mathrm{y(t)} + \mathrm{A}_2\mathrm{h}^2 + \mathrm{A}_4\mathrm{h}^4 + \cdots\cdots\cdots\cdots\cdots$, (3.1.2)

Then we can set $a_s^{(0)} = y(t_0 + H; h_s)$ in (3.4) and apply the process of repeated Rational Extrapolation using (3.5).

We then take the last entry in the main diagonal of the tableau given in (3.4) as our final approximation to $y(t_0 + H)$, and denote it by $y^*(t_0 + H; H)$.

To obtain a numerical solution at the next basic point $t_0 + 2H$, we apply the whole of the above procedure to the new initial value problem

$y' = f(t, y), y(t_0 + H) = y^*(t_0 + H; H)$.

3.2 Modified Midpoint Method:

Let $h_s = \frac{H}{N_s}, N_s$ is even (3.2.1)

and let $y_0 = y(t_0)$. (3.2.2)

By Euler's method we find the first approximation as

$y_1 = y_0 + h_s f(t_0, y_0)$. (3.2.3)

The other approximations are given by

$y_{m+2} = y_m + 2h_s f(t_{m+1}, y_{m+1}) m = 0,1,2,3 \cdots\cdots\cdots\cdots\cdots\cdots\cdots N_s - 1,$ (3.2.4)

And the end point correction is given by

$y(x_0 + H; h_s) = \frac{1}{4} y_{N_s+1} + \frac{1}{2} y_{N_s} + \frac{1}{4} y_{N_s-1}$ (3.2.5)

This is called the Modified Midpoint method.

If (3.2.1) to (3.2.5) is repeated for an increasing sequence N_s, $s = 0,1, \cdots\cdots\cdots\cdots\cdots\cdots\cdots S$, of even integers, then Rational extrapolation, using (3.4) and (3.5), can be applied as described in section (3).

4. A FUZZY CAUCHY PROBLEM

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f\big(t, y(t)\big), t \in I = [0, T] \ . \\ \qquad y(0) = y_0. \end{cases} \tag{4.1}$$

Where f is a continuous mapping from $R_+ \, X \, R$ into R and $y_0 \in E$ with r-level sets.

$$[y_0]_r = \left[\underline{y}(0, r), \overline{y}(0, r)\right], r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t, y)$ when $y = y(t)$ is a fuzzy number,

$f(t, y)(s) = \sup\{y(\tau) \backslash s = f(t, y)\}, s \in R.$

It follows that $[f(t, y)]_r = \left[\underline{f}(t, y; r), \overline{f}(t, y; r)\right], r \in (0,1],$

Where $\underline{f}(t, y; r) = \min\left\{f(t, u) \backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\},$

$\overline{f}(t, y; r) = \max\left\{f(t, u) \backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}.$

Theorem 4.1: Let f satisfy $\left|f(t, v) - f(t, \overline{v})\right| \leq g(t, |v - \overline{v}|), t \geq 0, v, \overline{v} \in R,$ (4.2)

where $g: R_+ X \, R_+$ is a continuous mapping such that $r \to g(t, r)$ is nondecreasing,the initial value problem

$u'(t) = g\big(t, u(t)\big), u(0) = u_0,$ (4.3) has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (4.3) for $u_0 \equiv 0$. Then the fuzzy initial value problem (3.1) has a unique fuzzy solution.

Proof: see [15].

In this paper we suppose (4.1) satisfies the hypothesis of theorem4.1, also.

Let the exact solution $[Y(t)]_r = \left[\underline{Y}(t;r), \overline{Y}(t;r)\right]$ is approximated by some

$[y(t)]_r = \left[\underline{y}(t;r), \overline{y}(t;r)\right]$.

From (3.2.1) to (3.2.5), we have

$\underline{y_0}(r) = \underline{y}(t_0;r)$,

$\underline{y_1}(r) = \underline{y_0}(r) + h_s \underline{f}(t_0, y_0(r))$,

$$\underline{y_{m+2}}(r) = \underline{y_m}(r) + 2h_s \underline{f}(t_{m+1}, y_{m+1}(r)), m = 0,1,2,3 \cdots\cdots\cdots\cdots\cdots\cdots\cdots N_s - 1, \tag{4.4}$$

$$\underline{y}(t_0 + H; h_s; r) = \frac{1}{4}\underline{y}_{N_s+1}(r) + \frac{1}{2}\underline{y}_{N_s}(r) + \frac{1}{4}\underline{y}_{N_s-1}(r) \tag{4.5}$$

$\overline{y_0}(r) = \overline{y}(t_0;r)$

$\overline{y_1}(r) = \overline{y_0}(r) + h_s \overline{f}(t_0, y_0(r))$,

$$\overline{y_{m+2}}(r) = \overline{y_m}(r) + 2h_s \overline{f}(t_{m+1}, y_{m+1}(r)), m = 0,1,2,3 \cdots\cdots\cdots\cdots\cdots\cdots\cdots N_s - 1, \tag{4.6}$$

$$\overline{y}(t_0 + H; h_s; r) = \frac{1}{4}\overline{y}_{N_s+1}(r) + \frac{1}{2}\overline{y}_{N_s}(r) + \frac{1}{4}\overline{y}_{N_s-1}(r) \tag{4.7}$$

The exact and approximate solutions at $t_n, 0 \le n \le N_s - 1$ are denoted by

$[Y(t_n)]_r = \left[\underline{Y}(t_n;r), \overline{Y}(t_n;r)\right]$ and $[y(t_n)]_r = \left[\underline{y}(t_n;r), \overline{y}(t_n;r)\right]$,respectively.

$$\text{Define,} F[t, y(t,r)] = \underline{f}(t, y(r)), \tag{4.8}$$

$$G[t, y(t,r)] = \overline{f}(t, y(r)), \tag{4.9}$$

$\therefore$ By (4.4),(4.6),(4.8)&(4.9),

$$\underline{Y}(t_{m+1};r) = \underline{Y}(t_m;r) + 2h_s F[t_m, Y_m(r)] \tag{4.10}$$

$$\overline{Y}(t_{m+1};r) = \overline{Y}(t_m;r) + 2h_s G[t_m, Y_m(r)] \tag{4.11}$$

We define,

$$\underline{y}(t_{m+1};r) = \underline{y}(t_m;r) + 2h_s F[t_m, y_m(r)] \tag{4.12}$$

$$\overline{y}(t_{m+1};r) = y(m;r) + 2h_s G[t_m, y_m(r)] \tag{4.13}$$

The following lemmas will be applied to show the convergences of theses approximates.

i.e., $\lim_{h\to 0} \underline{y}(t,r) = \underline{Y}(t,r)$ and, $\lim_{h\to 0} \overline{y}(t,r) = \overline{Y}(t,r)$.

Lemma: 4.1 Let the sequence of numbers $\{W_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le A|W_n| + B$, $0 \le n \le N - 1$, for some given positive constants A and B, then

$$|W_n| \le A^n|W_0| + B\frac{A^n - 1}{A - 1}, 0 \le n \le N.$$

Proof: see [12]

Lemma: 4.2 Let the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \leq |W_n| + A max\{|W_n|, |V_n|\} + B,\ |V_{n+1}| \leq |V_n| + A max\{|W_n|, |V_n|\} + B$, for some given positive constants A and B, and denote $U_n = |W_n| + |V_n|, 0 \leq n \leq N$.

Then $U_n \leq \overline{A^n} U_0 + \overline{B}\frac{\overline{A^n}-1}{\overline{A}-1}, 0 \leq n \leq N$, where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

Proof: see [12]

By replacing $y(t,r) = [u,v]$ in Eq. (4.8) and (4.9), we have

$$F[t,u,v] = \underline{f}(t,u,v), \tag{4.14}$$
$$G[t,u,v] = \overline{f}(t,u,v), \tag{4.15}$$

∴ The domain where F and G are defind is given by $K = \{(t,u,v)/\ 0 \leq t \leq T, -\infty < u \leq v\}$

Theorem 4.1: let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^2(K)$ and let the partial derivatives of F and G be bounded over K.Then, for arbitrary fixed $r, 0 \leq r \leq 1$, the approximate solutions (4.12 & 4.13) converge to the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

Proof: see [12].

Now, in a general sequence $h_0 > h_1 > h_2 > \cdots\cdots\cdots\cdots > h_s > 0$.........., for each value of h_s, we compute $\underline{A}(h_s)$ and denote the result by$\underline{a_s^{(0)}}$.

The Rational Extrapolation algorithm is defined by the following tableau

h_0	$\underline{a_0^{(0)}}$			
h_1	$\underline{a_1^{(0)}}$	$\underline{a_0^{(1)}}$		
h_2	$\underline{a_2^{(0)}}$	$\underline{a_1^{(1)}}$	$\underline{a_0^{(2)}}$	
h_3	$\underline{a_3^{(0)}}$	$\underline{a_2^{(1)}}$	$a_1^{(2)}$	$\underline{a_0^{(3)}}$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

Where $\underline{a_s^{(0)}} = \underline{A}(h_s), \underline{a_s^{(-1)}} = 0,$

$$\underline{a_s^{(m)}} = \underline{a_{s+1}^{(m-1)}} + \frac{\underline{a_{s+1}^{(m-1)}} - \underline{a_s^{(m-1)}}}{\left(\frac{h_s}{h_{m+s}}\right)^2 \left[1 - \frac{\left(\underline{a_{s+1}^{(m-1)}} - \underline{a_s^{(m-1)}}\right)}{\left(\underline{a_{s+1}^{(m-1)}} - \underline{a_{s+1}^{(m-2)}}\right)}\right] - 1},$$

$$m = 1,2,\cdots\cdots\cdots\cdots\cdots, s = 0,1,2,\cdots\cdots\cdots \tag{4.6}$$

Then $\underline{a_s^{(m)}} = \underline{A_0} + O(h_s^{2m+2})$.

Now, for each value of h_s , we compute $\overline{A}(h_s)$ and denote the result by$\overline{a_s^{(0)}}$.

The Rational Extrapolation algorithm is defined by the following tableau

h_0	$\overline{a_0^{(0)}}$			
h_1	$\overline{a_1^{(0)}}$	$\overline{a_0^{(1)}}$		
h_2	$\overline{a_2^{(0)}}$	$\overline{a_1^{(1)}}$	$\overline{a_0^{(2)}}$	
h_3	$\overline{a_3^{(0)}}$	$\overline{a_2^{(1)}}$	$\overline{a_1^{(2)}}$	$\overline{a_0^{(3)}}$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

Where $\overline{a_s^{(0)}} = \overline{A}(h_s), \overline{a_s^{(-1)}} = 0,$

$$\overline{a_s^{(m)}} = \overline{a_{s+1}^{(m-1)}} + \frac{\overline{a_{s+1}^{(m-1)}} - \overline{a_s^{(m-1)}}}{\left(\frac{h_s}{h_{m+s}}\right)^2 \left[1 - \frac{\left(\overline{a_{s+1}^{(m-1)}} - \overline{a_s^{(m-1)}}\right)}{\left(\overline{a_{s+1}^{(m-1)}} - \overline{a_{s+1}^{(m-2)}}\right)}\right] - 1},$$

$$m = 1,2, \cdots\cdots\cdots\cdots\cdots, s = 0,1,2, \cdots\cdots\cdots \tag{4.7}$$

Then $\overline{a_s^{(m)}} = \overline{A_0} + O(h_s^{2m+2})$.

5. NUMERICAL EXAMPLE

Example 1: Consider the fuzzy initial value problem,

$$\begin{cases} y'(t) = y(t), t \in [0,1], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), 0 < r \leq 1. \end{cases}$$

The exact solution is given by $\underline{Y}(t;r) = \underline{y}(t;r)e^t$, $\overline{Y}(t;r) = \overline{y}(t;r)e^t$

Which at $t = 1, Y(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], 0 < r \leq 1.$

By the repeated application of (4.4),(4.5),(4.6)&(4.7), for different values of 'h' we have Tables 1 to 12.Here, RA EXT represents the approximations by Rational Extrapolations method and POL EXT represents the approximations by Polynomial Extrapolations given in[1].

From Table:1 to Table:12,we may find the superiority of the Rational Extrapolation method over the Polynomial Extrapolation method.

Table: 1

h	$\underline{y}(t;0)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	1.96875								
1/4	2.0185547	2.0357210	2.0351563						
1/8	2.0334724	2.0384941	2.0384450	2.0387112	2.0386642				
1/16	2.0373885	2.0386972	2.0386939	2.0387113	2.0387105	2.0387113	2.0387112		
1/32	2.0383798	2.0387104	2.0387102	2.0387113	2.0387113	2.0387113	2.0387113	2.0387113	2.0387113

Table: 2

h	$\underline{y}(t;0.2)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.1								
1/4	2.153125	2.1714357	2.1708333						
1/8	2.1690372	2.1743937	2.1743413	2.1746253	2.1745752				
1/16	2.1732144	2.1746104	2.1746068	2.1746255	2.1746245	2.1746255	2.1746253		
1/32	2.1742718	2.1746245	2.1746243	2.1746255	2.1746255	2.1746255	2.1746255	2.1746255	2.1746255

Table: 3

h	$\underline{y}(t;0.4)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.23125								
1/4	2.2876953	2.3071504	2.3065104						
1/8	2.3046021	2.3102934	2.3102377	2.3105394	2.3104862				
1/16	2.3090403	2.3105235	2.3105197	2.3105395	2.3105385	2.3105395	2.3105393		
1/32	2.3101638	2.3105385	2.3105383	2.3105395	2.3105395	2.3105395	2.3105395	2.3105395	2.3105395

Table: 4

h	$\underline{y}(t;0.6)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.3625								
1/4	2.4222656	2.4428651	2.4421875						
1/8	2.4401669	2.4461929	2.4461340	2.4464534	2.4463971				
1/16	2.4448662	2.4464367	2.4464326	2.4464537	2.4464525	2.4464537	2.4464534		
1/32	2.4460558	2.4464526	2.4464523	2.4464537	2.4464536	2.4464537	2.4464536	2.4464537	2.4464536

Table: 5

h	$\underline{y}(t;0.8)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.49375								
1/4	2.5568359	2.5785799	2.5778876						
1/8	2.5757317	2.5820925	2.5820303	2.5823675	2.5823065				
1/16	2.5806921	2.5823499	2.5823456	2.5823678	2.5823456	2.5823678	2.5823462		
1/32	2.5819478	2.5823666	2.5823664	2.5823677	2.5823678	2.5823677	2.5823682	2.5823677	2.5823683

Table: 6

h	$\underline{y}(t;1)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.625								
1/4	2.6914062	2.7142946	2.7135416						
1/8	2.7112966	2.7179922	2.7179267	2.7182816	2.7182190				
1/16	2.716518	2.7182629	2.7182585	2.7182817	2.7182806	2.7182817	2.7182816		
1/32	2.7178398	2.7182807	2.7182804	2.7182819	2.7182819	2.7182819	2.7182819	2.7182819	2.7182819

Table: 7

h	$\overline{y}(t;0)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.953125								
1/4	3.027832	3.0535814	3.0527343						
1/8	3.0502086	3.0577411	3.0576675	3.0580667	3.0579964				
1/16	3.0560828	3.0580459	3.0580409	3.0580671	3.0580658	3.0580671	3.0580669		
1/32	3.0575697	3.0580657	3.0580653	3.0580670	3.0580669	3.0580670	3.0580669	3.0580670	3.0580669

Table: 8

h	$\overline{y}(t;0.2)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.8875								
1/4	2.9605469	2.9857241	2.9848959						
1/8	2.9824262	2.9897913	2.9897193	2.9901097	2.9900409				
1/16	2.9881698	2.9900893	2.9900843	2.9901100	2.9901086	2.9901100	2.9901097		
1/32	2.9896237	2.9901086	2.9901083	2.9901099	2.9901099	2.9901099	2.9901099	2.9901099	2.9901099

Table: 9

h	$\overline{y}(t;0.4)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.821875								
1/4	2.8932617	2.9178667	2.9170573						
1/8	2.9146438	2.9218416	2.9217712	2.9221528	2.9220855				
1/16	2.9202569	2.9221327	2.9221279	2.9221529	2.9221517	2.9221529	2.9221528		
1/32	2.9216778	2.9221517	2.9221514	2.9221530	2.9221530	2.9221530	2.9221530	2.9221530	2.9221530

Table: 10

h	$\overline{y}(t;0.6)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.75625								
1/4	2.8259766	2.8500094	2.8492188						
1/8	2.8468614	2.8538918	2.8538230	2.8541957	2.8541299				
1/16	2.8523439	2.8541761	2.8541714	2.8541959	2.8541946	2.8541959	2.8541956		
1/32	2.8537318	2.8541947	2.8541944	2.8541960	2.8541959	2.8541960	2.8541959	2.8541960	2.8541959

Table: 11

h	$\overline{y}(t;0.8)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.690625								
1/4	2.7586914	2.7821520	2.7813802						
1/8	2.779079	2.7859420	2.7858749	2.7862387	2.7861745				
1/16	2.784431	2.7862196	2.7862150	2.7862389	2.7862377	2.7862389	2.7862387		
1/32	2.7857858	2.7862377	2.7862374	2.7862389	2.7862389	2.7862389	2.7862389	2.7862389	2.7862389

Table: 12

h	$\overline{y}(t;1)$	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT	RA EXT	POL EXT
1/2	2.625								
1/4	2.6914062	2.7142946	2.7135416						
1/8	2.7112966	2.7179922	2.7179267	2.7182816	2.7182190				
1/16	2.716518	2.7182629	2.7182585	2.7182817	2.7182806	2.7182817	2.7182816		
1/32	2.7178398	2.7182807	2.7182804	2.7182819	2.7182819	2.7182819	2.7182819	2.7182819	2.7182819

The Exact and Approximate solutions by Rational Extrapolation, Polynomial Extrapolation, Runge-Kutta method of order 4 with h=0.01and by Euler's Approximation with h=0.01 are given in Table: 13.

Table: 13

R	Exact solution	Rational Extrapolation	Polynomial Extrapolation	RungeKutta method of order 4 with h=0.01	Euler's Approximation h=0.01
0	2.0387113,3.0580670	2.0387113,3.0580670	2.0387113,3.0580669	2.0370216,3.0555324	2.0286104,3.0429156

0.2	2.1746254,2.9901100	2.1746255,2.9901099	2.1746255,2.9901099	2.172823,2.9876314	2.1638511,2.9752952
0.4	2.3105395,2.9221529	2.3105395,2.9221530	2.3105395,2.9221530	2.3086245,2.9197309	2.2990918,2.9076749
06	2.4464536,2.8541959	2.4464537,2.8541960	2.4464536,2.8541959	2.4444259,2.8518302	2.4343324,2.8400545
0.8	2.5823677,2.7862388	2.5823677,2.7862389	2.5823683,2.7862389	2.5802273,2.7839295	2.5695731,2.7724342
1	*2.7182818,2.7182818*	*2.7182819,2.7182819*	*2.7182819,2.7182819*	*2.7160288,2.7160288*	*2.7048138,2.7048138*

6. CONCLUSIONS

In this work, we have used the proposed rational extrapolation method to find a numerical solution of fuzzy differential equations. Taking into account the convergence order of the Euler method is $O(h)$ and that of Runge-Kutta method of order 4 as $O(h^4)$ a higher order of convergence $O(h^{2m+2})$is obtained by the proposed method. Comparison of the solutions of example 5.1 shows that the proposed method gives a better solution than the Euler method and the Runge-Kutta method of order 4.The comparison also shows the efficiency of the Rational Extrapolation method over Polynomial Extrapolation method.

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