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SINGLE SERVER RETRIAL QUEUES WITH SECOND OPTIONAL SERVICE UNDER ERLANG SERVICES

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ABSTRACT

Consider a single server retrial queueing system in which customers arrive in a Poisson process with arrival rate λ. *In this model the server provides two types of service namely Essential Service and Second Optional Service. The Essential service will be given to all customers through k-phases whereas the Second optional service is extended only to those optional customers as a single phase if they demand. The essential service time has Erlang-k distribution with service rate* $k\mu_1$ *for each phase. The second optional service has only one phase in it and the service time of second optional service follows an exponential distribution with parameter* μ_2. *If the server is free at the time of a primary call arrival, the arriving call begins to be served in Phase 1of the essential service phase immediately by the server then progresses through the remaining phases and must complete the last phase and after completion of this essential service, this customer either demands a second optional service with probability* **p** *or leaves the system with probability* **(1-p)** *before the next customer enters the first phase of the essential service. If the server is busy, then the arriving customer goes to orbit and becomes a source of repeated calls. We assume that the access from orbit to the service facility is governed by the classical retrial policy. This model is solved by using Matrix geometric Technique. Numerical studies have been done for Analysis of Mean number of customers in the orbit (MNCO),Truncation level (OCUT),Probability of server free and busy for various values of* λ, μ_1, μ_2, p, k *and* σ.

KEYWORDS: *Retrial queue - second optional service - classical retrial policy- Matrix Geometric Method*

2010 Mathematics Subject Classification*: 60K25, 65K30*

1. INTRODUCTION:

Consider a single server queueing model in which an arriving customer who finds the service facility busy is obliged to join a pool of unsatisfied customers called 'the orbit'. Each customer in orbit repeats his demand after a random amount of time until he finds the server free. Queueing systems with these characters are termed **Retrial Queues [1, 2, 3, 5, 6, 7]**. Queues in which retrials are allowed have been widely used to model problems in telephone, computer and communication systems. A complete description of situations in which retrials arise, the main results and methods of the theory of retrial queues are found in the monograph by Falin and Templeton **[6]**. In addition, a complete bibliography is given in Artalejo **[2]**. Analytic results are generally difficult to obtain due to the complexity of these retrial queueing models. There are a great number of numerical and approximations methods available, in this paper we will place more emphasis on the solutions by **Matrix geometric method [9, 12, 13].**

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Second optional service **[8]** plays a vital role in retrial queueing systems Kailash C. Madan **[11]** has studied an M/G/1 queue with second optional service using supplementary variable technique. The work of

Madan was generalized by Medhi who studied a single server Poisson input queue with a second optional channel. Gaudham Choudhury studied some aspects of M/G/1 queueing system with second optional service and derived the steady state queue size distribution at the stationary point of time for general second optional service.

The concept of second optional service is introduced in this paper in which the server provides two types of services namely an essential service and a second optional service. The essential service will be given all customers but the second optional service will be extended to only customers if they demand. Both the service times follow an exponential distribution.

2. MODEL DESCRIPTION:

Consider a single server retrial queueing system in which customers arrive in a Poisson process with arrival rate λ. These customers are identified as primary calls. Let k be the number of phases in the service station. Assume that the service time has Erlang-k distribution **[4]** with service rate $k\mu_1$

for each phase. Let the services in all phases be independent and identical and only one customer at a time is in the service mechanism. In this model the server provides two types of services that are Essential Service and Second Optional Service **[8, 10, and 11].** The essential service will be given to all customers in k phases with Erlang-k distribution service rate $k\mu_1$ at each phase whereas the other type of service namely a second optional service is extended to customers as a single phase who opt for it where the service time follows an exponential distribution with parameter μ_2. If the server is free at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leaves the system before the next customer enters the first phase. After completion of the essential service, this customer either demands a second optional service with probability p or leaves the system with probability (1-p). If the server is busy, then the arriving customer goes to orbit and becomes a source of repeated calls. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity σ. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while

the source which produced this repeated call disappears. Otherwise, the system state does not change.

2.1Retrial Policy:

Most of the queueing system with repeated attempts assume that each customer in the retrial group seeks service independently of each other after a random time exponentially distributed with rate σ so that the probability of repeated attempt during the interval $(t, t+\Delta t)$ given that there were n customers in orbit at time t is $n\sigma\,\Delta t + O(\Delta t)$. This discipline for access for the server from the retrial group is called **classical retrial rate policy.** The input flow of primary calls, interval between repetitions and service time in phases are mutually independent.

3. MATRIX GEOMETRIC SOLUTIONS:

Let N(t) be the random variable which represents the number of customers in orbit at time t and S(t) be the random variable which represents the phase in which customer is getting service and also the state of customer getting the second optional service at time t.

The random process is described as

$\{ \langle N(t), S(t)\rangle / N(t) = 0,1,2,3,\ldots ; S(t) = 0,1,2,3,\ldots, k+1\}$

S(t) = 0 means server is idle.

S(t) = i means server is busy with customer in the i^{th} phase of the essential service for i = 1, 2, 3,. . . , k

S(t) = k+1 for the server being busy with customer with second optional service

The possible state spaces are

$$\{(i, j) / i = 0, 1, 2, 3\ldots ;\ j = 0, 1, 2, 3\ldots,k+1 \}$$

The infinitesimal generator matrix **Q** is given below

$$Q = \begin{pmatrix} A_{00} & A_0 & O & O & O & \ldots \\ A_{10} & A_{11} & A_0 & O & O & \ldots \\ O & A_{21} & A_{22} & A_0 & O & \ldots \\ O & O & A_{32} & A_{33} & A_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Notations:

$S_1 = -(\lambda+k\mu_1)$ $S_2 = -(\lambda+\mu_2)$ $S_3 = (1-p)k\mu_1$

$S_4 = -(\lambda+n\sigma)$ $S_5 = -(\lambda+M\sigma)$

The matrices A_{00}, $A_{n\,n-1}$, $A_{n\,n}$ and $A_{n\,n+1}$ are square matrices of order k+2, where

$$A_{00} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \ldots & 0 & 0 \\ 0 & S_1 & k\mu_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & S_1 & k\mu_1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & S_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_3 & 0 & 0 & 0 & \ldots & S_1 & pk\mu_1 \\ \mu_2 & 0 & 0 & 0 & \ldots & 0 & S_2 \end{pmatrix}$$

$A_{nn-1} = (a_{ij})$ where

$a_{ij} = n\sigma$ if $i = 1, j = 2$

$= 0$ otherwise

$$A_{nn} = \begin{pmatrix} S_4 & \lambda & 0 & 0 & \ldots & 0 & 0 \\ 0 & S_1 & k\mu_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & S_1 & k\mu_1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & S_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_3 & 0 & 0 & 0 & \ldots & S_1 & pk\mu_1 \\ \mu_2 & 0 & 0 & 0 & \ldots & 0 & S_2 \end{pmatrix}$$

$A_{nn+1} = A_0 = (a_{ij})$ where

$a_{ij} = \lambda$ if $i = j$, $i = 2, 3, 4,\ldots,k+2$

$= 0$ otherwise

If the capacity of the orbit is finite say M, then A_{MM} is

$$\begin{pmatrix} S_5 & \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & -k\mu_1 & k\mu_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -k\mu_1 & k\mu_1 & \dots & 0 & 0 \\ 0 & 0 & 0 & -k\mu_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ S_3 & 0 & 0 & 0 & \dots & -k\mu_1 & pk\mu_1 \\ \mu_2 & 0 & 0 & 0 & \dots & 0 & -\mu_2 \end{pmatrix}$$

Let **X** be a steady-state probability vector of **Q** partitioned as **X** =(x(0),x(1),x(2), **...**) where **X** satisfies

$$\mathbf{XQ} = \mathbf{0} \quad \text{and} \quad \mathbf{Xe} = 1 \qquad (1)$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \dots, P_{ik+1})$ for i = 0,1,2,…

4. DIRECT TRUNCATION METHOD:

In this method one can truncate the system of equations in (1) for sufficiently large value of the number of customers in the orbit, say M. That is, the orbit size is restricted to M such that any arriving customer finding the orbit full is considered lost. The value of M can be chosen so that the loss probability is small. Due to the intrinsic nature of the system in (1), the only choice available for studying M is through algorithmic methods. While a number of approaches is available for determining the cut-off point, M , The one that seems to perform well (with respect to approximating the system performance measures) is to increase M until the largest individual change in the elements of **x** for successive values is less than Є a predetermined infinitesimal value.

ANALYSIS OF STEADY STATE PROBABILITIES:

Applying the Direct Truncation Method described in section 4, we find the steady state probability vector **X**. If M denotes the cut-off point or Truncation level, then the steady state probability vector $\mathbf{X}^{(M)}$ is partitioned as

$\mathbf{X}^{(M)} = (x(0), x(1), x(2), \dots, x(M))$, where $\mathbf{X}^{(M)}$ satisfies

$$\mathbf{X}^{(M)}\mathbf{Q} = \mathbf{0} \text{ and } \mathbf{X}^{(M)}\mathbf{e} = 1,$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \dots, P_{ik+1})$ for i = 0,1,2,…,M.

The above system of equations is solved exploiting the special structure of the co-efficient matrix. It is solved by Numerical methods. Since there is no clear cut choice for M, we may start the iterative process by taking, say M = 1 and increase it until the individual elements of **X** do not change significantly. That is, if M^* denotes the truncation point then

$\| x^{M^*}(i) - x^{M^*-1}(i) \|_\infty < \varepsilon$ where ε is an infinitesimal quantity.

5. STABILITY CONDITION:

Theorem

The inequality $\left(\frac{\lambda}{\mu_1} + \frac{\lambda p}{\mu_2}\right) < 1$ is the necessary and sufficient condition for system to be stable.

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial).The stationary probability vector **X** satisfies

$$\mathbf{XQ} = \mathbf{0} \text{ and } \mathbf{Xe} = \mathbf{1} \qquad (2)$$

Let R be the rate matrix and satisfying the equation

$$A_0 + RA_1 + R^2 A_2 = 0 \qquad (3)$$

The system is stable if sp(R) <1
We know that the Matrix R satisfies
sp(R) <1 if and only if

$$\Pi A_0 e < \Pi A_2 e \qquad (4)$$

where $\Pi = (\pi_1, \dots, \pi_k)$ and satisfies

$$\Pi A = 0 \text{ and } \Pi e = 1 \qquad (5)$$

And

$$A = A_0 + A_1 + A_2 \qquad (6)$$

Here A_0, A_1 **and** A_2 are square matrices of order **k+1** and $A_0 = \lambda I$, where I is the identity matrix of order k+1.

$$A_1 = \begin{pmatrix} S_1 & k\mu_1 & 0 & \dots & 0 & 0 \\ 0 & S_1 & k\mu_1 & \dots & 0 & 0 \\ 0 & 0 & S_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & S_1 & pk\mu_1 \\ 0 & 0 & 0 & \dots & 0 & S_2 \end{pmatrix}$$

The matrix A_2 is described as

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_3 & 0 & 0 & \dots & 0 & 0 \\ \mu_2 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

By substituting $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$ in equations (4) ,(5) and (6) ,we get

$$\left(\frac{\lambda}{\mu_1}+\frac{\lambda p}{\mu_2}\right)<1$$

The inequality $\left(\frac{\lambda}{\mu_1}+\frac{\lambda p}{\mu_2}\right)<1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n\geq 0\}$ is ergodic if $\left(\frac{\lambda}{\mu_1}+\frac{\lambda p}{\mu_2}\right)<1$. $\{Q_n, n\geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n\geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n\geq 0\}$ is positive recurrent if $|\psi_m|<\infty$ for all m and $\lim_{m\to\infty}\sup\psi_m<0$, where

$\psi_m = E((Q_{n+1} - Q_n) / Q_n = m$ for m=0,1,2,3,…

$$\psi_m = \left(\frac{\lambda}{\mu_1}+\frac{\lambda p}{\mu_2}\right)-\left(\frac{m\sigma}{\lambda+m\sigma}\right)$$

If $\left(\frac{\lambda}{\mu_1}+\frac{\lambda p}{\mu_2}\right)<1$, then $|\psi_m|<\infty$ for all m and $\lim_{m\to\infty}\sup\psi_m<0$

Therefore the embedded Markov chain $\{Q_n, n\geq 0\}$ is ergodic.

6. SPECIAL CASES:

1. If p = 0 then this model becomes the single server retrial queueing model with erlang k- type services
2. If k = 1 and p = 0 , then this model becomes the Single server retrial queueing model and our numerical results coincide with the following closed form of Number of customers in the orbit in the steady state

   Mean Number of Customers in the orbit = $\dfrac{\rho(\lambda+\rho\sigma)}{(1-\rho)\sigma}$

7. SYSTEM PERFORMANCE MEASURES:

In this section we list some important performance measures along with their formulae. These measures are used to bring out the qualitative behaviour of the queueing model under study. Numerical studies have been dealt to study the following measures.

P(n , 0) = Probability that there are n customers in the orbit and server is free

P(n , i) = Probability that there are n customers in the orbit and server is busy with customer in the i^{th} phase (i = 1,2,3, . . . ,k)

P(n , k+1) = Probability that there are n customers in the orbit and server is busy with second optional service.

1. The probability mass function of Server state

Let S(t) be the random variable which represents the phase in which customer is getting service at time t .

Prob (The server is idle) = $\sum_{i=0}^{\infty} p(i,0)$

Prob (The server busy with customer in the j th phase) =

$$\sum_{i=0}^{\infty} p(i,j)$$

Prob (The server busy with customer offering Second optional service) =

$$\sum_{i=0}^{\infty} p(i,k+1)$$

2. The probability mass function number of customers in the orbit

Let X(t) be the random variable representing the number of customers in the orbit.

Prob (No customers in the orbit) = $\sum_{j=0}^{k+1} p(0,j)$

Prob (i customers in the orbit) = $\sum_{j=0}^{k+1} p(i,j)$

3. The Mean number of customers in the orbit

MNCO = $\sum_{i=0}^{\infty} i(\sum_{j=0}^{k+1} p(i,j))$

4. **The probability that the orbiting customer is blocked**

Blocking Probability = $\sum_{i=1}^{\infty}\sum_{j=1}^{k+1} p(i,j)$

5. **The probability that an arriving customer enter into service immediately**

PSI = $\sum_{i=0}^{\infty} p(i,0)$

8. NUMERICAL STUDY:

The values of parameters λ, μ_1, μ_2, p, k and σ are chosen so that they satisfy the stability condition discussed in section 5.

The system performance measures of this model have been done and expressed in the form of tables which are shown below using the steady state probability vector **X** for various values of λ, μ_1, μ_2, p, k and σ.

For λ = 8, μ_1 = 10, μ_2 = 50, p = 0.5, k = 5 and σ = 100 , the steady state probability vector is **X** = (x(0), x(1), x(2), **. . .** ,x(M)), where

x [0] = [0.09651, 0.03003, 0.02589, 0.02232, 0.01924, 0.01659, 0.00715]

x [1] = [0.00970, 0.02225, 0.02275, 0.02269, 0.02221, 0.02144, 0.01023]
x[2] = [0.00486, 0.01861, 0.01918, 0.01966, 0.02002, 0.02021, 0.01012]

x[3] = [0.00287, 0.01565, 0.01614, 0.01662, 0.01709, 0.01752, 0.00895]

x[4] = [0.00184, 0.01307, 0.01349, 0.01392, 0.01436, 0.01479, 0.00761]

x[5] = [0.00124, 0.01085, 0.01121, 0.01159, 0.01197, 0.01236, 0.00638]

x[6] = [0.00086, 0.00897, 0.00928, 0.00960, 0.00993, 0.01026, 0.00530]

x[7] = [0.00061, 0.00740, 0.00766, 0.00793, 0.00820, 0.00849, 0.00439]

x[8] = [0.00044, 0.00609, 0.00631, 0.00653, 0.00676, 0.00700, 0.00362]

x[9] = [0.00032, 0.00500, 0.00518, 0.00537, 0.00556, 0.00576, 0.00298]

x[10] = [0.00024, 0.00410, 0.00425, 0.00440, 0.00456, 0.00473, 0.00245]

x[11] = [0.00018, 0.00336, 0.00348, 0.00361, 0.00374, 0.00388, 0.00201]

x[12] = [0.00013, 0.00275, 0.00285, 0.00296, 0.00306, 0.00318, 0.00165]

x[13] = [0.00010, 0.00225, 0.00233, 0.00242, 0.00251, 0.00260, 0.00135]

x[14] = [0.00008, 0.00184, 0.00191, 0.00198, 0.00205, 0.00213, 0.00110]

x[15] = [0.00006, 0.00150, 0.00156, 0.00161, 0.00167, 0.00174, 0.00090]

x[16] = [0.00004, 0.00122, 0.00127, 0.00132, 0.00137, 0.00142, 0.00074]

x[17] = [0.00003, 0.00100, 0.00104, 0.00108, 0.00112, 0.00116, 0.00060]

x[18] = [0.00003, 0.00081, 0.00085, 0.00088, 0.00091, 0.00094, 0.00049]

x[19] = [0.00002, 0.00066, 0.00069, 0.00072, 0.00074, 0.00077, 0.00040]

x[20] = [0.00002, 0.00054, 0.00056, 0.00058, 0.00060, 0.00063, 0.00033]

Similarly, we can find x (n) for $n \geq 20$ and it is noticed that $x(n) \rightarrow 0$ as $n \rightarrow \infty$. For the numerical parameters chosen above, $x(n) \rightarrow 0$ for $n \geq 42$ and the sum of the steady state probabilities becomes 0.99999999999. In the same manner, we can find steady state probability vector **X** for all values of λ, μ_1, μ_2, p, k and σ.

System Performance Measures for $\lambda = 8$, $\mu_1 = 10$, $\mu_2 = 50$, p =0.5, k = 5 and $\sigma = 100$

Probability that the server is idle = 0.12000

Probability that the server is busy with a customer in 1 phase = 0.160000

Probability that the server is busy with a customer in 2 phase = 0.160000

Probability that the server is busy with a customer in 3 phase = 0.160000

Probability that the server is busy with a customer in 4 phase = 0.160000

Probability that the server is busy with a customer in 5 phase = 0.160000

Probability that the server is busy with essential service = 0.800000

Probability that the server is busy with second optional service = 0.080000

Probability that the server is busy
= 0.880000

Probability mass function of number of customers in the orbit

No. of customers in the orbit	Probability
0	0.217292
1	0.131002
2	0.112441
3	0.094649
4	0.078922
5	0.065458
6	0.054096
7	0.044585
8	0.036670
9	0.030109
10	0.024689
11	0.020222
12	0.016547
13	0.013528
14	0.011053
15	0.009025
16	0.007365
17	0.006007
18	0.004898
19	0.003992
20	0.003252

Mean number of customers in the orbit
= 4.426436

Probability that the orbiting customer is blocked = 0.759028

Tables 1, 2 and 3 show the impact of λ (low, medium and high) and retrial rate σ over Mean number of customers in the orbit and we infer the following

- Mean number of customers in the orbit decreases as retrial rate σ increases.
- Mean number of customers in the orbit increases as λ increases.
- P_0 and P_1 are independent of retrial rate σ.
- This model becomes a classical queueing model with second optional service under erlang k- type service if σ is large.

Table 1: Low arrival rate Vs Mean number of customers in the orbit for $\lambda = 2$, $\mu_1 = 10$, $\mu_2 = 50$, $k = 5$, $p = 0.4$ and various values of σ

σ	OCUT	MNCO	P_0	P_1	P_2
1	10	0.5865	0.7840	0.2000	0.0160
2	8	0.3110	0.7840	0.2000	0.0160
3	8	0.2192	0.7840	0.2000	0.0160
4	8	0.1733	0.7840	0.2000	0.0160
5	7	0.1457	0.7840	0.2000	0.0160
6	7	0.1273	0.7840	0.2000	0.0160
7	7	0.1142	0.7840	0.2000	0.0160
8	7	0.1044	0.7840	0.2000	0.0160
9	7	0.0967	0.7840	0.2000	0.0160
10	7	0.0906	0.7840	0.2000	0.0160
20	7	0.0631	0.7840	0.2000	0.0160
30	7	0.0539	0.7840	0.2000	0.0160
40	7	0.0493	0.7840	0.2000	0.0160
50	7	0.0465	0.7840	0.2000	0.0160
60	7	0.0447	0.7840	0.2000	0.0160
70	7	0.0434	0.7840	0.2000	0.0160
80	6	0.0424	0.7840	0.2000	0.0160
90	6	0.0416	0.7840	0.2000	0.0160
100	6	0.0410	0.7840	0.2000	0.0160
200	6	0.0383	0.7840	0.2000	0.0160
300	6	0.0373	0.7840	0.2000	0.0160
400	6	0.0369	0.7840	0.2000	0.0160
500	6	0.0366	0.7840	0.2000	0.0160
600	6	0.0364	0.7840	0.2000	0.0160
700	6	0.0363	0.7840	0.2000	0.0160
800	6	0.0362	0.7840	0.2000	0.0160
900	6	0.0361	0.7840	0.2000	0.0160
1000	6	0.0361	0.7840	0.2000	0.0160
2000	6	0.0358	0.7840	0.2000	0.0160
3000	6	0.0357	0.7840	0.2000	0.0160
4000	6	0.0356	0.7840	0.2000	0.0160
5000	6	0.0356	0.7840	0.2000	0.0160
6000	6	0.0356	0.7840	0.2000	0.0160
7000	6	0.0356	0.7840	0.2000	0.0160
8000	6	0.0356	0.7840	0.2000	0.0160
9000	6	0.0356	0.7840	0.2000	0.0160
10000	6	0.0356	0.7840	0.2000	0.0160

Table 2: Medium arrival rate Vs Mean number of customers in the orbit $\lambda = 5$, $\mu_1 = 10$, $\mu_2 = 50$, $k = 5$, $p = 0.4$ and various values of σ

σ	OCUT	MNCO	P_0	P_1	P_2
1	32	6.2478	0.4600	0.5000	0.0400
2	25	3.3130	0.4600	0.5000	0.0400
3	22	2.3348	0.4600	0.5000	0.0400
4	21	1.8456	0.4600	0.5000	0.0400
5	20	1.5522	0.4600	0.5000	0.0400
6	19	1.3565	0.4600	0.5000	0.0400
7	18	1.2168	0.4600	0.5000	0.0400
8	18	1.1120	0.4600	0.5000	0.0400
9	18	1.0304	0.4600	0.5000	0.0400
10	17	0.9652	0.4600	0.5000	0.0400
20	16	0.6717	0.4600	0.5000	0.0400
30	16	0.5739	0.4600	0.5000	0.0400
40	15	0.5250	0.4600	0.5000	0.0400
50	15	0.4956	0.4600	0.5000	0.0400
60	15	0.4761	0.4600	0.5000	0.0400
70	15	0.4621	0.4600	0.5000	0.0400
80	15	0.4516	0.4600	0.5000	0.0400
90	15	0.4435	0.4600	0.5000	0.0400
100	15	0.4370	0.4600	0.5000	0.0400
200	15	0.4076	0.4600	0.5000	0.0400
300	15	0.3978	0.4600	0.5000	0.0400
400	15	0.3929	0.4600	0.5000	0.0400
500	15	0.3900	0.4600	0.5000	0.0400
600	15	0.3880	0.4600	0.5000	0.0400
700	15	0.3866	0.4600	0.5000	0.0400
800	15	0.3856	0.4600	0.5000	0.0400
900	15	0.3848	0.4600	0.5000	0.0400
1000	15	0.3841	0.4600	0.5000	0.0400
2000	15	0.3812	0.4600	0.5000	0.0400
3000	15	0.3802	0.4600	0.5000	0.0400
4000	15	0.3797	0.4600	0.5000	0.0400
5000	15	0.3794	0.4600	0.5000	0.0400
6000	15	0.3792	0.4600	0.5000	0.0400
7000	15	0.3791	0.4600	0.5000	0.0400
8000	15	0.3790	0.4600	0.5000	0.0400
9000	15	0.3789	0.4600	0.5000	0.0400
10000	15	0.3788	0.4600	0.5000	0.0400

Table 3: High arrival rate Vs Mean number of customers in the orbit $\lambda = 8$, $\mu_1 = 10$, $\mu_2 = 50$, $k = 5$, $p = 0.4$ and various values of σ

σ	OCUT	MNCO	P_0	P_1	P_2
1	167	54.0988	0.1360	0.8000	0.0640
2	121	28.6870	0.1360	0.8000	0.0640
3	103	20.2164	0.1360	0.8000	0.0640
4	94	15.9811	0.1360	0.8000	0.0640
5	88	13.4400	0.1360	0.8000	0.0640
6	84	11.7458	0.1360	0.8000	0.0640
7	80	10.5358	0.1360	0.8000	0.0640
8	78	9.6282	0.1360	0.8000	0.0640
9	76	8.9223	0.1360	0.8000	0.0640
10	75	8.3576	0.1360	0.8000	0.0640
20	67	5.8164	0.1360	0.8000	0.0640
30	64	4.9694	0.1360	0.8000	0.0640
40	63	4.5458	0.1360	0.8000	0.0640
50	62	4.2917	0.1360	0.8000	0.0640
60	62	4.1223	0.1360	0.8000	0.0640
70	61	4.0013	0.1360	0.8000	0.0640
80	61	3.9106	0.1360	0.8000	0.0640
90	61	3.8400	0.1360	0.8000	0.0640
100	60	3.7835	0.1360	0.8000	0.0640
200	59	3.5294	0.1360	0.8000	0.0640
300	59	3.4447	0.1360	0.8000	0.0640
400	59	3.4023	0.1360	0.8000	0.0640
500	59	3.3769	0.1360	0.8000	0.0640
600	59	3.3600	0.1360	0.8000	0.0640
700	59	3.3479	0.1360	0.8000	0.0640
800	59	3.3388	0.1360	0.8000	0.0640
900	59	3.3317	0.1360	0.8000	0.0640
1000	59	3.3261	0.1360	0.8000	0.0640
2000	59	3.3007	0.1360	0.8000	0.0640
3000	59	3.2922	0.1360	0.8000	0.0640
4000	59	3.2880	0.1360	0.8000	0.0640
5000	59	3.2854	0.1360	0.8000	0.0640
6000	59	3.2837	0.1360	0.8000	0.0640
7000	59	3.2825	0.1360	0.8000	0.0640
8000	59	3.2816	0.1360	0.8000	0.0640
9000	59	3.2809	0.1360	0.8000	0.0640
10000	59	3.2803	0.1360	0.8000	0.0640

Table 4 shows the effect of p over the mean number of customers in the orbit. We infer the following

- Mean number of customers in the orbit increases as p increases
- P_0 and P_2 are depend on p and P_1 is independent of p

Table 5 shows the effect of number of phases (k) over the mean number of customers in the orbit. We infer the following

- Mean number of customers in the orbit decreases as number phases increases

Table 4: p Vs Mean number of customers in the orbit $\lambda = 8$, $\mu_1 = 10$, $\mu_2 = 50$, $\sigma = 100$, $k = 5$ and various values of p

p	OCUT	MNCO	P_0	P_1	P_2
0.10	44	2.5252	0.1840	0.8000	0.0160
0.12	45	2.5883	0.1808	0.8000	0.0192
0.14	46	2.6537	0.1776	0.8000	0.0224
0.16	47	2.7214	0.1744	0.8000	0.0256
0.18	47	2.7917	0.1712	0.8000	0.0288
0.20	48	2.8647	0.1680	0.8000	0.0320
0.22	49	2.9406	0.1648	0.8000	0.0352
0.24	50	3.0194	0.1616	0.8000	0.0384
0.26	52	3.1014	0.1584	0.8000	0.0416
0.28	53	3.1868	0.1552	0.8000	0.0448
0.30	54	3.2758	0.1520	0.8000	0.0480
0.32	55	3.3686	0.1488	0.8000	0.0512
0.34	56	3.4655	0.1456	0.8000	0.0544
0.36	58	3.5667	0.1424	0.8000	0.0576
0.38	59	3.6726	0.1392	0.8000	0.0608
0.40	60	3.7835	0.1360	0.8000	0.0640
0.42	62	3.8997	0.1328	0.8000	0.0672
0.44	63	4.0217	0.1296	0.8000	0.0704
0.46	65	4.1498	0.1264	0.8000	0.0736
0.48	67	4.2846	0.1232	0.8000	0.0768
0.50	69	4.4266	0.1200	0.8000	0.0800
0.52	71	4.5764	0.1168	0.8000	0.0832
0.54	73	4.7346	0.1136	0.8000	0.0864
0.56	75	4.9020	0.1104	0.8000	0.0896
0.58	77	5.0794	0.1072	0.8000	0.0928
0.60	80	5.2676	0.1040	0.8000	0.0960
0.62	82	5.4679	0.1008	0.8000	0.0992
0.64	85	5.6813	0.0976	0.8000	0.1024
0.66	88	5.9091	0.0944	0.8000	0.1056
0.68	91	6.1529	0.0912	0.8000	0.1088
0.70	94	6.4145	0.0880	0.8000	0.1120

Table 5: k Vs Mean number of customers in the orbit $\lambda = 8$, $\mu_1=10, \mu_2 = 50, \sigma = 100$ p = 0.4 and various values of k

k	OCUT	MNCO	P_0	P_1	P_2
1	104	6.5599	0.1200	0.8000	0.0800
2	82	5.2266	0.1200	0.8000	0.0800
3	75	4.7822	0.1200	0.8000	0.0800
4	71	4.5600	0.1200	0.8000	0.0800
5	69	4.4266	0.1200	0.8000	0.0800
6	67	4.3377	0.1200	0.8000	0.0800
7	66	4.2742	0.1200	0.8000	0.0800
8	65	4.2266	0.1200	0.8000	0.0800
9	65	4.1896	0.1200	0.8000	0.0800
10	64	4.1600	0.1200	0.8000	0.0800
11	64	4.1357	0.1200	0.8000	0.0800
12	64	4.1155	0.1200	0.8000	0.0800
13	63	4.0984	0.1200	0.8000	0.0800
14	63	4.0838	0.1200	0.8000	0.0800
15	63	4.0711	0.1200	0.8000	0.0800

9. GRAPHICAL STUDY:

Figures 1, 2, 3 and 4 show the effect of retrial rate σ over the Mean number of customers in the orbit. The following figures show that the mean number of customers in the orbit decreases as retrial rate σ increases and this model becomes single server classical queueing system with second optional services under erlang-k type services if σ is large.

Figure 1: Impact of retrial rate over Mean number of customers in the orbit for $\lambda = 5$, $\mu_1 = 10, \mu_2 = 50$, k = 5, p = 0.4 and σ varies from 1 to 10

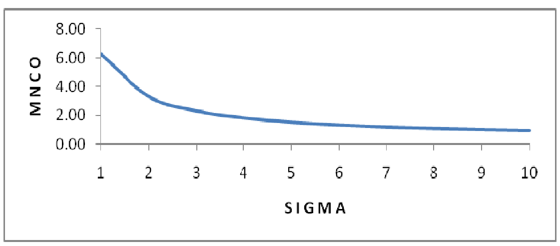


Figure 2: **I**mpact of retrial rate over Mean number of customers in the orbit for $\lambda = 5$, $\mu_1 = 10, \mu_2 = 50$, k = 5, p = 0.4 and σ various from 10 to 100

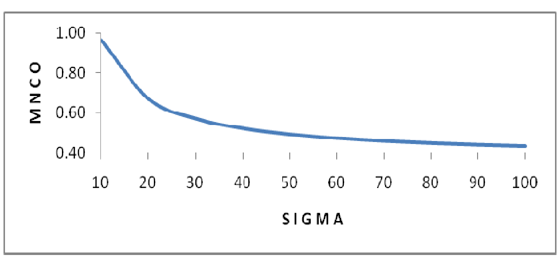


Figure 3: Impact of retrial rate over Mean number of customers in the orbit for $\lambda = 5$, $\mu_1 = 10, \mu_2 = 50$, k = 5, p = 0.4 and σ varies from 100 to 1000

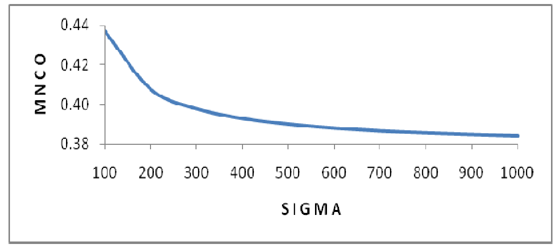


Figure 4: Impact of retrial rate over Mean number of customers in the orbit for $\lambda=5$, $\mu_1 = 10, \mu_2 = 50$, k = 5, p = 0.4 and σ varies from 1000 to 10000

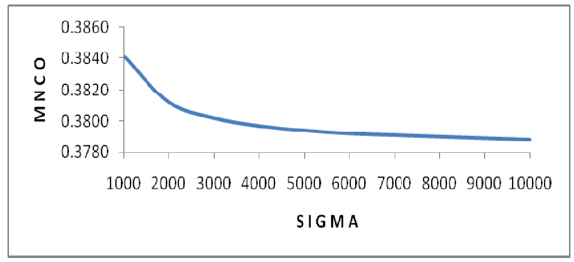


Figure 5 shows the effect of p over the Mean number of customers in the orbit. The following figure shows that the mean number of customers in the orbit increases as p increases and this model becomes single server classical queueing system with erlang-k type services if $p \to 0$.

Figure 5: Impact of p over Mean number of customers in the orbit for $\lambda = 8$, $\mu_1 = 10$, $\mu_2 = 50$, k = 5, $\sigma = 100$ and various values of p

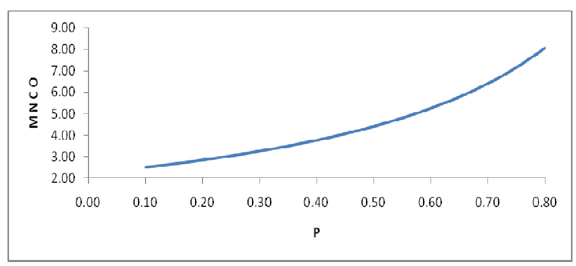
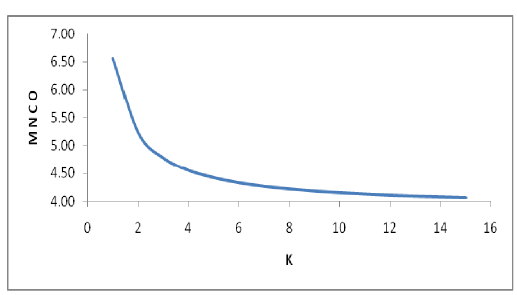


Figure 6 shows the effect of k over the Mean number of customers in the orbit. The following figure shows that the mean number of customers in the orbit decreases as k increases

Figure 6: Impact of k over Mean number of customers in the orbit for $\lambda = 8$, $\mu_1 = 10$, $\mu_2 = 50$, p = 0.4, $\sigma = 100$ and various values of k



10. CONCLUSION:

It is observed from numerical and graphical studies that Mean number of customers in the orbit decreases as the retrial rate increases and the probabilities for the server being idle, busy are independent over retrial rate. Mean number of customers in the orbit increases as the probability of accepting the second optional service increases As the number of phases k increases, Mean number of customers in the orbit decreases Moreover the various special cases discussed in section 7 are particular cases of this research work. This research work can further be extended by introducing various parameters like negative arrival, vacation policies etc.

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