International Journal of Mathematical Archive-3(3), 2012, Page: 1211-1221
Available online through www.ijma.info ISSN 2229 – 5046

GOODMAN-RØNNING TYPE CLASS OF HARMONIC UNIVALENT FUNCTIONS INVOLVING CONVOLUTIONAL OPERATORS

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(Received on: 28-02-12; Accepted on: 21-03-12)

ABSTRACT

In this paper, we study a convolutional approach of harmonic univalent functions. For this purpose we introduce a Goodman-Rønning type class $\mathcal{S}_H(\rho, i, j, \phi, \psi; \alpha)$ *of harmonic univalent functions involving convolutional operators. A sufficient coefficient condition for the normalized harmonic functions to be in this class is obtained. It is also shown that this coefficient condition is necessary for the functions in its subclass* $\mathcal{TS}_H(\rho, i, j, \phi, \psi; \alpha)$. *We further obtain extreme points, bounds and a covering result for the class* $\mathcal{TS}_H(\rho, i, j, \phi, \psi; \alpha)$ *and show that this class is closed under convolutions and convex combinations. Conditions on the coefficients of* ϕ *and* ψ *lead various well-known results proved earlier as well as to generate number of new results.*

AMS Classification [2000]*: 30C45, 30C50.*

Keywords: *Harmonic functions, Univalent functions, Convolution.*

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1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D}$, we can write $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathbb{D}$ (see [5]).
Denote by $\mathcal{S}_H$ the class of function $f = h + \overline{g}$ which are harmonic, univalent and sense-preserving in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{S}_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Note that the class $\mathcal{S}_H$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of f is zero i.e. $g \equiv 0$. For this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

For basic results on harmonic functions one may refer to [1], [14], [15] and the references therein. Harmonic functions, associated with Hohlov operators defined by convolutions, have been investigated in the work of Ahuja [2]. Harmonic functions, associated with more generalized linear operators defined by convolutions, are also considered in [13]. Also, subclasses of $\mathcal{S}_H$, associated with Sălăgean operators are studied in [11], [22] etc. Harmonic functions, associated with convolutional operators are studied recently by Dixit et al. in [6]. Further, Goodman-Rønning type harmonic univalent functions are studied in [16] (see also [3], [12], [21]).

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Let $\rho \geq 0,\ i, j \in \mathbb{N}_0, 0 \leq \alpha < 1,$ and functions $\phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k$ and $\psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$ be analytic in $\mathbb{U}$ with non-decreasing sequences $\{\lambda_k\}, \{\mu_k\}$ satisfying the condition $\lambda_k \geq k\,(k \geq 1), \mu_k < \lambda_k\ (k \geq 2).$ In this paper, involving convolutions, we define a Goodman-Rønning type class $\mathcal{S}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ of functions $f = h + \overline{g} \in \mathcal{S}_H$ such that $D_\psi^j f(z) \neq 0\ \left(z \in \mathbb{U}\right)$ satisfying

$$\operatorname{Re}\left(\frac{D_\phi^i f(z)}{D_\psi^j f(z)}\right) > \rho\left|\frac{D_\phi^i f(z)}{D_\psi^j f(z)} - 1\right| + \alpha, \tag{1.3}$$

or, equivalently

$$\operatorname{Re}\left\{\left(1 + \rho e^{i\theta}\right)\left(\frac{D_\phi^i f(z)}{D_\psi^j f(z)}\right) - \rho e^{i\theta}\right\} > \alpha, \theta \in \mathbb{R}, \tag{1.4}$$

where $D_\phi^i : \mathcal{S}_H \to \mathcal{S}_H$ is a convolutional operator and is defined by

$$D_\phi^i f(z) = h(z) * \phi(z) + (-1)^i \overline{g(z) * \phi(z)},$$

and

$$D_\psi^j f(z) = h(z) * \psi(z) + (-1)^j \overline{g(z) * \psi(z)}.$$

The operator "$*$" stands for the Hadamard product or convolution of two power series.

We further denote by $\mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$, a subclass of $\mathcal{S}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_H$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |\,a_k\,|\, z^k,\ g(z) = (-1)^{i-1} \sum_{k=1}^{\infty} |\,b_k\,|\, z^k,\ |\,b_1\,| < 1. \tag{1.5}$$

Note that the class $\mathcal{S}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ is of generalized nature, which includes most of earlier defined subclasses of $\mathcal{S}_H$ as well as many new classes.

For instance, for a well known Koebe function: $\mathcal{K}(z) = \dfrac{z}{(1-z)^2} = z\dfrac{d}{dz}\left(\dfrac{1}{1-z}\right) =: \mathcal{K}_1(z) \in \mathcal{S},$ denote

$\mathcal{K}_n(z) = z\left(\mathcal{K}_{n-1}(z)\right)', n \geq 2$ and $\mathcal{K}_0(z) = \displaystyle\int_0^z \frac{\mathcal{K}(t)}{t} dt = \frac{z}{1-z}, z \in \mathbb{U}.$ The series expansion of $\mathcal{K}_n(z), n \in \mathbb{N}_0 = \{0, 1, 2, ..\}$ is given by

$$\mathcal{K}_n(z) = z + \sum_{k=2}^{\infty} k^n\ z^k.$$

Thus, we note that for $h \in \mathcal{S}$

$$h(z) * \mathcal{K}_n(z) =: \mathcal{D}^n h(z),$$

defines a well studied Salagean operator $\mathcal{D}^n,\ n \in \mathbb{N}_0$ (see [17]).

For $\alpha_i \in \mathbb{C}$ $\left(\alpha_i, \dfrac{\alpha_i}{A_i} \neq 0, -1, -2, ...,\ A_i > 0; i = 1, 2, ..., p\right)$ and $\beta_i \in \mathbb{C}$ $\left(\beta_i, \dfrac{\beta_i}{B_i} \neq 0, -1, -2, ..., B_i > 0; i = 1, 2, ..., q\right)$ such that $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0,$ we denote the Wright's generalized hypergeometric function [20]: ${}_p\psi_q\left[\begin{matrix}(\alpha_i, A_i)_{1,p}\\(\beta_i, B_i)_{1,q}\end{matrix}; z\right] =: {}_p\psi_q[z]$, its normalized form:

$z\dfrac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\ {}_p\psi_q[z] =: \widetilde{\psi}(z)$, and $z\widetilde{\psi}'(z) =: \widetilde{\psi}_1$. The series expansion of $\widetilde{\psi}(z)$ is given by

$$\widetilde{\psi}(z) = z + \sum_{k=2}^{\infty}\sigma_k[\alpha_1]\, z^k$$

where

$$\sigma_k[\alpha_1] = \frac{\prod_{i=1}^{p}\dfrac{\Gamma(\alpha_i + (k-1)A_i)}{\Gamma(\alpha_i)}}{\prod_{i=1}^{q}\dfrac{\Gamma(\beta_i + (k-1)B_i)}{\Gamma(\beta_i)}(k-1)!}. \qquad (1.6)$$

For $h \in \mathcal{S}$

$$h(z) * \widetilde{\psi}(z) =: W_q^p[\alpha_1]h(z),$$

defines Dzoik-Raina operator [7]. As special cases, operator $W_q^p[\alpha_1]$ contains such further linear operators as Dzoik-Srivastava operator [8] (when $A_i = B_i = 1$) which include Hovlov operator (when $p = 2, q = 1$), Carlson-Shaffer operator, Ruscheweyh derivative operator, generalized Bernardi-Libera-Livingston operator and the fractional derivative operator. Details and references about these operators can be found in [7] and [8].

Few subclasses studied earlier of $\mathcal{S}_H$, may obtained by specializing the parameters, are as follows.

- $\mathcal{S}_H(0,1,0,\widetilde{\psi}_1,\widetilde{\psi};\alpha) \equiv \mathcal{W}_H([\alpha_1],\alpha)$ and $\mathcal{TS}_H(0,1,0,\widetilde{\psi}_1,\widetilde{\psi};\alpha) \equiv \mathcal{TW}_H([\alpha_1],\alpha)$ studied by Murugusundaramoorthy and Raina [13].
- $\mathcal{S}_H(0,m,n,\mathcal{K}_m,\mathcal{K}_n;\alpha) \equiv \mathcal{S}_H(m,n;\alpha)$ and $\mathcal{TS}_H(0\ m,n,\mathcal{K}_m,\mathcal{K}_n;\alpha) \equiv \mathcal{TS}_H(m,n;\alpha)$ studied by Yalcin [22].
- $\mathcal{S}_H(1,n+1,n,\mathcal{K}_{n+1},\mathcal{K}_n;\alpha) \equiv \mathcal{RS}_H(n,\alpha)$ and $\mathcal{TS}_H(1,n+1,n,\mathcal{K}_{n+1},\mathcal{K}_n;\alpha) \equiv \overline{\mathcal{RS}}_H(n,\alpha)$ studied by Yalcin et al. [21].
- $\mathcal{S}_H(0,n+1,n,\mathcal{K}_{n+1},\mathcal{K}_n;\alpha) \equiv \mathcal{S}_H(n;\alpha)$ and $\mathcal{TS}_H(0,n+1,n,\mathcal{K}_{n+1},\mathcal{K}_n;\alpha) \equiv \mathcal{TS}_H(n;\alpha)$ studied by Jahangiri et al. [11].
- $\mathcal{S}_H(01,1,0,\mathcal{K}_1,\mathcal{K}_0;\alpha) \equiv \mathcal{G}_H(\alpha)$ and $\mathcal{TS}_H(1,1,0,\mathcal{K}_1,\mathcal{K}_0;\alpha) \equiv \mathcal{TG}_H(\alpha)$ studied by Rosy et al. [16].
- $\mathcal{S}_H(0,1,0,\mathcal{K}_1,\mathcal{K}_0;\alpha) \equiv \mathcal{S}_H^*(\alpha)$ and $\mathcal{TS}_H(0,1,0,\mathcal{K}_1,\mathcal{K}_0;\alpha) \equiv \mathcal{TS}_H^*(\alpha)$ studied by Jahangiri [10].
- $\mathcal{S}_H(0,2,1,\mathcal{K}_2,\mathcal{K}_1;\alpha) \equiv \mathcal{K}_H(\alpha)$ and $\mathcal{TS}_H(0,2,1,\mathcal{K}_2,\mathcal{K}_1;\alpha) \equiv \mathcal{TK}_H(\alpha)$ studied by Jahangiri [10].
- $\mathcal{S}_H(0,1,0,\phi,\mathcal{K}_0;\alpha) \equiv \mathcal{S}_H(\phi,\phi;\alpha)$ and $\mathcal{TS}_H(0,1,0,\phi,\mathcal{K}_0;\alpha) \equiv \mathcal{TS}_H(\phi,\phi;\alpha)$ studied by Frasin [9].
- $\mathcal{S}_H(0,2,1,\mathcal{K}_2,\mathcal{K}_1;0) \equiv \mathcal{K}_H$ and $\mathcal{TS}_H(0,2,1,\mathcal{K}_2,\mathcal{K}_1;0) \equiv \mathcal{TK}_H$, $\mathcal{S}_H(0,1,0,\mathcal{K}_1,\mathcal{K}_0;0) \equiv \mathcal{S}_H^*$ and $\mathcal{TS}_H(0,1,0,\mathcal{K}_1,\mathcal{K}_0;0) \equiv \mathcal{TS}_H^*$ studied by Silverman [18], Silverman and Silvia [19] (see also [4]).

We prove results including, coefficient inequality, bounds, extreme points, convolution and convex combination for functions in $\mathcal{TS}_H(\rho,i,j,\phi,\psi;\alpha)$.

2. MAIN RESULTS:

We begin with a sufficient coefficient condition for functions to be in class $\mathcal{S}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$.

Theorem: 1 Let for $i, j \in \mathbb{N}_0, \rho \geq 0, 0 \leq \alpha < 1, \lambda_k \geq k\left(k \geq 1\right), \mu_k < \lambda_k\ (k \geq 2)$ a function $f = h + \overline{g},$ where h and g are of the form (1.1), satisfies

$$\sum_{k=2}^{\infty} \frac{\left(1+\rho\right)\lambda_k - \left(\alpha+\rho\right)\mu_k}{1-\alpha} | a_k | + \sum_{k=1}^{\infty} \frac{\left(1+\rho\right)\lambda_k - \left(-1\right)^{i-j}\left(\alpha+\rho\right)\mu_k}{1-\alpha} | b_k | \leq 1, \tag{2.1}$$

then f is sense-preserving, harmonic univalent in $\mathbb{U}$ and $f \in \mathcal{S}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$.

Proof: Under the given hypothesis and for $\lambda_1 = 1 = \mu_1$, we note that for $k \geq 1$,

$$k \leq \frac{\left(1+\rho\right)\lambda_k - \left(\alpha+\rho\right)\mu_k}{1-\alpha}, \ k \leq \frac{\left(1+\rho\right)\lambda_k - \left(-1\right)^{i-j}\left(\alpha+\rho\right)\mu_k}{1-\alpha}. \tag{2.2}$$

Hence, for $f = h + \overline{g},$ where h and g are of the form (1.1), we get that

$$\begin{aligned} | h^{'}(z) | &\geq 1 - \sum_{k=2}^{\infty} k | a_k | r^{k-1} > 1 - \sum_{k=2}^{\infty} k | a_k | > 1 - \sum_{k=2}^{\infty} \frac{\left(1+\rho\right)\lambda_k - \left(\alpha+\rho\right)\mu_k}{1-\alpha} | a_k | \\ &\geq \sum_{k=1}^{\infty} \frac{\left(1+\rho\right)\lambda_k - \left(-1\right)^{i-j}\left(\alpha+\rho\right)\mu_k}{1-\alpha} | b_k | \geq \sum_{k=1}^{\infty} k | b_k | > \sum_{k=1}^{\infty} k | b_k | r^{k-1} \geq | g^{'}(z) |, \end{aligned}$$

which proves that f is sense-preserving in $\mathbb{U}.$ To show that f is univalent in $\mathbb{U}$, suppose $z_1, z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$, then

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right|$$

$$> 1 - \left| \frac{\sum_{k=1}^{\infty} k | b_k |}{1 - \sum_{k=2}^{\infty} k | a_k |} \right| \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\left(1+\rho\right)\lambda_k - \left(-1\right)^{i-j}\left(\alpha+\rho\right)\mu_k}{1-\alpha} | b_k |}{1 - \sum_{k=2}^{\infty} \frac{\left(1+\rho\right)\lambda_k - \left(\alpha+\rho\right)\mu_k}{1-\alpha} | a_k |} \geq 0,$$

which proves the univalence. Now, to show $f \in \mathcal{S}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$, we need to show (1.3), that is for $\theta \in \mathbb{R},$

$$\operatorname{Re}\left(\frac{D_\phi^i f(z) - \alpha D_\psi^j f(z) - \rho e^{i\theta} \left| D_\phi^i f(z) - D_\psi^j f(z) \right|}{D_\psi^j f(z)} \right) \geq 0, \tag{2.3}$$

or
$$\operatorname{Re}\left(\frac{\left(1-\alpha\right) z + \sum_{k=2}^{\infty} \left(\lambda_k - \alpha\mu_k\right) a_k z^k + (-1)^i \sum_{k=1}^{\infty} \left(\lambda_k - \left(-1\right)^{i-j} \alpha\mu_k\right) \overline{b_k z^k} - \rho e^{i\theta} K}{z + \sum_{k=2}^{\infty} \mu_k a_k z^k + (-1)^j \sum_{k=1}^{\infty} \mu_k \overline{b_k z^k}} \right) \geq 0, \tag{2.4}$$

where $K=\left|\sum_{k=2}^{\infty}\left(\lambda_k-\mu_k\right)a_k z^k+(-1)^i\sum_{k=1}^{\infty}\left(\lambda_k-\left(-1\right)^{i-j}\mu_k\right)\overline{b_k z^k}\right|.$ At $z=0,$ the condition (2.4) is trivial. For $0\neq z\in\mathbb{U}$, the left hand side of (2.4) can be expressed as

$$\operatorname{Re}\left(\frac{1-\alpha+A(z)}{1+B(z)}\right)=\left(1-\alpha\right)\frac{1+w(z)}{1-w(z)}, \tag{2.5}$$

where

$$A(z)=\sum_{k=2}^{\infty}\left(\lambda_k-\alpha\mu_k\right)a_k z^{k-1}+(-1)^i\sum_{k=1}^{\infty}\left(\lambda_k-\left(-1\right)^{i-j}\alpha\mu_k\right)\overline{b_k z^k}z^{-1}-\rho e^{i\theta}Kz^{-1}$$

and

$$B(z)=\sum_{k=2}^{\infty}\mu_k a_k z^{k-1}+(-1)^j\sum_{k=1}^{\infty}\mu_k\overline{b_k z^k}z^{-1}.$$

We only need to show $\left|w(z)\right|\leq 1.$ From (2.5), we get for $0\neq\left|z\right|=r<1,$

$$\left|w(z)\right|=\left|\frac{A(z)-\left(1-\alpha\right)B(z)}{2\left(1-\alpha\right)+A(z)+\left(1-\alpha\right)B(z)}\right|$$

$$\leq\frac{\left(1+\rho\right)\left[\sum_{k=2}^{\infty}\left(\lambda_k-\mu_k\right)\left|a_k\right|+\sum_{k=1}^{\infty}\left(\lambda_k-\left(-1\right)^{i-j}\mu_k\right)\left|b_k\right|\right]}{2\left(1-\alpha\right)-\sum_{k=2}^{\infty}\left(\left(1+\rho\right)\lambda_k-\left(\rho+2\alpha-1\right)\mu_k\right)\left|a_k\right|-\sum_{k=1}^{\infty}\left(\left(1+\rho\right)\lambda_k-\left(-1\right)^{i-j}\left(\rho+2\alpha-1\right)\mu_k\right)\left|b_k\right|}$$

$$\leq 1,$$

if (2.1) holds. This proves Theorem 1.

We next show that the above sufficient coefficient condition is also necessary for functions in the class $\mathcal{TS}_H\left(\rho,i,j,\phi,\psi;\alpha\right)$.

Theorem: 2 Let the function $f=h+\overline{g}$ be such that h and g are given by (1.5). Then, $f\in\mathcal{TS}_H\left(\rho,i,j,\phi,\psi;\alpha\right)$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{\left(1+\rho\right)\lambda_k-\left(\alpha+\rho\right)\mu_k}{1-\alpha}\left|a_k\right|+\frac{\left(1+\rho\right)\lambda_k-\left(-1\right)^{i-j}\left(\alpha+\rho\right)\mu_k}{1-\alpha}\left|b_k\right|\right)\leq 2, \tag{2.6}$$

where $a_1=1,\ i,j\in\mathbb{N}_0,\ \rho\geq 0,0\leq\alpha<1,\ \lambda_k\geq k\left(k\geq 1\right),\mu_k<\lambda_k\ (k\geq 2)$. The result is sharp.

Proof: The if part, follows from Theorem 1. To prove the "only if" part, let $f\in\mathcal{TS}_H\left(\rho,i,j,\phi,\psi;\alpha\right),$ then from the equivalent condition (1.4), we have

$$\operatorname{Re}\left\{\left(1+\rho e^{i\theta}\right)\left(\frac{h(z)*\phi(z)+(-1)^i\overline{g(z)*\phi(z)}}{h(z)*\psi(z)+(-1)^j\overline{g(z)*\psi(z)}}\right)-\rho e^{i\theta}-\alpha\right\}\geq 0,\ z\in\mathbb{U},$$

or

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}\left\{\left(1+\rho e^{i\theta}\right)\lambda_k-\left(\alpha+\rho e^{i\theta}\right)\mu_k\right\}|a_k|z^k+(-1)^{2i-1}\sum_{k=1}^{\infty}\left\{\left(1+\rho e^{i\theta}\right)\lambda_k-\left(-1\right)^{i-j}\left(\alpha+\rho e^{i\theta}\right)\mu_k\right\}|b_k|\overline{z^k}}{z-\sum_{k=2}^{\infty}\mu_k|a_k|z^k+(-1)^{i+j-1}\sum_{k=1}^{\infty}\mu_k|b_k|\overline{z^k}}\right\}\geq 0,$$

which can be written as

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}\{\lambda_k-\alpha\mu_k\}|a_k|z^k+(-1)^{2i-1}\sum_{k=1}^{\infty}\left\{\lambda_k-(-1)^{i-j}\alpha\mu_k\right\}|b_k|\overline{z^k}}{z-\sum_{k=2}^{\infty}\mu_k|a_k|z^k+(-1)^{i+j-1}\sum_{k=1}^{\infty}\mu_k|b_k|\overline{z^k}}\right.$$

$$\left.-\rho e^{i\theta}\frac{\sum_{k=2}^{\infty}\{\lambda_k-\mu_k\}|a_k|z^k+\sum_{k=1}^{\infty}\left\{\lambda_k-(-1)^{i-j}\mu_k\right\}|b_k|\overline{z^k}}{z-\sum_{k=2}^{\infty}\mu_k|a_k|z^k+(-1)^{i+j-1}\sum_{k=1}^{\infty}\mu_k|b_k|\overline{z^k}}\right\}\geq 0.$$

If we choose z to be real and $z\to 1^-$, and using the fact that $Re\left(e^{i\theta}\right)\leq\left|e^{i\theta}\right|=1,$ we obtain

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left\{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k\right\}|a_k|-\sum_{k=1}^{\infty}\left\{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k\right\}|b_k|}{1-\sum_{k=2}^{\infty}\mu_k|a_k|+(-1)^{i+j-1}\sum_{k=1}^{\infty}\mu_k|b_k|}\geq 0,$$

or, equivalently,

$$\sum_{k=2}^{\infty}\left\{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k\right\}|a_k|+\sum_{k=1}^{\infty}\left\{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k\right\}|b_k|\leq 1-\alpha,$$

which is the required condition (2.6). Sharpness of (2.6) can be seen by the function

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k}|x_k|z^k+(-1)^{i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k}|y_k|\overline{z^k},$$

where $i,j\in\mathbb{N}_0,\rho\geq 0,0\leq\alpha<1,\lambda_k\geq k\ (k\geq 1),\mu_k<\lambda_k(k\geq 2)$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$

For the classes $\mathcal{W}_H([\alpha_1],\alpha),\mathcal{TS}_H(m,n;\alpha),\overline{\mathcal{RS}}_H(n,\alpha)$ and $\mathcal{TS}_H(\phi,\phi;\alpha)$ mentioned in Section 1, Theorem 2 yields following results which include the results for other known classes discussed in Section 1.

Corollary: 1 [13] Let the function $f=h+\overline{g}$ be such that h and g are given by (1.5). Then, $f\in\mathcal{TW}_H([\alpha_1],\alpha)$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{k+\alpha}{1-\alpha}|a_k|+\frac{k+\alpha}{1-\alpha}|b_k|\right)\left|\sigma_k[\alpha_1]\right|\leq 2, \tag{2.7}$$

where $a_1=1,0\leq\alpha<1$ *and* $\sigma_k[\alpha_1]$ is given by (1.6).

Corollary: 2 [22] Let the function $f=h+\overline{g}$ be such that h and g are given by (1.5). Then, $f\in\mathcal{TS}_H(m,n;\alpha)$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{k^m-\alpha k^n}{1-\alpha}|a_k|+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|\right)\leq 2, \tag{2.8}$$

where $a_1=1,m\in\mathbb{N},n\in\mathbb{N}_0,m>n,0\leq\alpha<1.$

Corollary: 3 [21] Let the function $f = h + \overline{g}$ be such that h and g are given by (1.5). Then, $f \in \overline{\mathcal{RS}}_H(n,\alpha)$ if and only if

$$\sum_{k=1}^{\infty} k^n \left(\frac{2k-1-\alpha}{1-\alpha} |a_k| + \frac{2k+1+\alpha}{1-\alpha} |b_k| \right) \leq 2, \tag{2.9}$$

where $a_1 = 1, 0 \leq \alpha < 1.$

Corollary: 4 Let the function $f = h + \overline{g}$ be such that h and g are given by (1.5). Then, $f \in \mathcal{TS}_H(\phi,\phi;\alpha)$ if and only if

$$\sum_{k=1}^{\infty} \left(\frac{\lambda_k - \alpha}{1-\alpha} |a_k| + \frac{\lambda_k + \alpha}{1-\alpha} |b_k| \right) \leq 2, \tag{2.10}$$

where $a_1 = 1, \lambda_k \geq k \ (k \geq 1), 0 \leq \alpha < 1.$

3. BOUNDS:

Our next theorem provides the bounds for the functions in $\mathcal{TS}_H(\rho,i,j,\phi,\psi;\alpha)$ which is followed by a covering result for this class.

Theorem: 3 Let $f = h + \overline{g}$ with h and g are of the form (1.5) belongs to the class $\mathcal{TS}_H(\rho,i,j,\phi,\psi;\alpha)$, then for $|z| = r < 1$,

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\alpha}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} - \frac{(1+\rho)-(-1)^{i-j}(\alpha+\rho)}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} |b_1| \right) r^2, \tag{3.1}$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\alpha}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} - \frac{(1+\rho)-(-1)^{i-j}(\alpha+\rho)}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} |b_1| \right) r^2. \tag{3.2}$$

The result is sharp.

Proof: We only prove the result for upper bound. The result for the lower bound can similarly be obtained.

Let $f \in \mathcal{TS}_H(\rho,i,j,\phi,\psi;\alpha)$, then on taking the absolute value of f, we get for $|z| = r < 1$,

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \\ &\leq (1+|b_1|)r + \frac{(1-\alpha)r^2}{(1+\rho)\lambda_2-(\alpha+\rho)\mu_2}\sum_{k=2}^{\infty}\left(\frac{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k}{1-\alpha}|a_k| \right. \\ &\qquad \left. + \frac{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_k| \right) \\ &\leq (1+|b_1|)r + \left(\frac{1-\alpha}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} - \frac{(1+\rho)-(-1)^{i-j}(\alpha+\rho)}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} |b_1| \right) r^2, \end{aligned}$$

by (2.6). The bounds (3.1) and (3.2) are sharp for the function given by

$$f(z) = z + (-1)^{i-1}|b_1|\overline{z} + (-1)^{i-1}\left(\frac{1-\alpha}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} - \frac{(1+\rho)-(-1)^{i-j}(\alpha+\rho)}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} |b_1| \right) \overline{z^2} \tag{3.3}$$

.

for $i, j \in \mathbb{N}_0, \rho \geq 0, 0 \leq \alpha < 1, \lambda_2 \geq 2, \mu_2 < \lambda_2$ and $|b_1| < (1-\alpha)/(1+\rho)-(-1)^{i-j}(\alpha+\rho)$.

A covering result follows from (3.2).

Corollary: 5 Let $f \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$, then for $i, j \in \mathbb{N}_0$, $\rho \ge 0$, $0 \le \alpha < 1$, $\lambda_2 \ge 2$, $\mu_2 < \lambda_2$

$$\left\{\omega : |\omega| < \left(1 - \frac{1-\alpha}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2}\right) + \left(\frac{1+\rho-(-1)^{i-j}(\alpha+\rho)}{(1+\rho)\lambda_2 - (\alpha+\rho)\mu_2} - 1\right)|b_1|\right\} \subset f(\mathbb{U}).$$

Remark: 1 Theorem 3 verifies the results for the classes $\mathcal{W}_H([\alpha_1], \alpha), \mathcal{TS}_H(m, n; \alpha), \overline{\mathcal{RS}}_H(n, \alpha)$ and $\mathcal{TS}_H(\phi, \phi; \alpha)$ mentioned in Section 1.

4. EXTREME POINTS:

In this section we determine the extreme points of $\mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$.

Theorem: 4 Let $h_1(z) = z, h_k(z) = z - \dfrac{1-\alpha}{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k} z^k$ $(k \ge 2)$ and

$g_k(z) = z + \dfrac{(-1)^{i-1}(1-\alpha)}{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}\overline{z^k}$ $(k \ge 1)$. Then $f \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$, if and only if it can be expressed as

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)), \tag{4.1}$$

where $x_k \ge 0, y_k \ge 0$ and $\sum_{k=1}^{\infty}(x_k + y_k) = 1$. In particular, the extreme points of $\mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ are $\{h_k\}$ and $\{g_k\}$.

Proof: Suppose that

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)).$$

Then,

$$f(z) = \sum_{k=1}^{\infty}(x_k + y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}x_k z^k + (-1)^{i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}y_k\overline{z^k}$$

$$= z - \sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}x_k z^k + (-1)^{i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}y_k\overline{z^k}$$

$$\in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right).$$

Since,

$$\sum_{k=2}^{\infty}\frac{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}{1-\alpha}\,\frac{1-\alpha}{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}x_k$$

$$+\sum_{k=1}^{\infty}\frac{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}\,\frac{1-\alpha}{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}y_k$$

$$= \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k = 1 - x_1 \le 1.$$

Conversely, if $f \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$, then $|a_k| \leq \dfrac{1-\alpha}{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}$, $k \geq 2$ and $|b_k| \leq \dfrac{1-\alpha}{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}$, $k \geq 1$. Setting $x_k = \dfrac{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}{1-\alpha}|a_k|,\ k \geq 2$ and $y_k = \dfrac{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_k|,\ k \geq 1.$ Then, by Theorem 2, $\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leq 1.$ We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \geq 0$. Consequently, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof of Theorem 4.

5. CONVOLUTION AND CONVEX COMBINATIONS:

In this section, we show that the class $\mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ is invariant under convolution and convex combinations of its members.

Let the function $f = h + \overline{g}$, where h and g are of the form (1.5) and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1\ ^{i-1}\sum_{k=1}^{\infty} |B_k| \overline{z^k}. \tag{5.1}$$

We define the convolution

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1\ ^{i})^{1}\sum_{k=1}^{\infty} |b_k B_k| \overline{z^k}.$$

Theorem: 5 If $f \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ and $F \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ then $f * F \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$.

Proof: Let $f = h + \overline{g}$, where h and g are of the form (1.5) and F be of the form (5.1). Let both f and F be in $\mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$. Then by Theorem 2, we have

$$\sum_{k=2}^{\infty} \frac{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_k| \leq 1, \tag{5.2}$$

and

$$\sum_{k=2}^{\infty} \frac{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}{1-\alpha}|A_k| + \sum_{k=1}^{\infty} \frac{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|B_k| \leq 1 \tag{5.3}$$

From (5.3), we conclude that $\left|A_k\right| \leq 1, k = 2, 3, \ldots$ and $\left|B_k\right| \leq 1, k = 1, 2, \ldots$

So, for $f * F$, we may write

$$\sum_{k=2}^{\infty} \frac{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}{1-\alpha}|a_k A_k| + \sum_{k=1}^{\infty} \frac{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_k B_k|$$
$$\leq \sum_{k=2}^{\infty} \frac{(1+\rho)\lambda_k - (\alpha+\rho)\mu_k}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{(1+\rho)\lambda_k - (-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_k|$$
$$\leq 1.$$

Thus $f * F \in \mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$.

Finally, we prove that $\mathcal{TS}_H\left(\rho, i, j, \phi, \psi; \alpha\right)$ is closed under convex combination of its members.

Theorem: 6 The class $\mathcal{TS}_H\left(\rho,i,j,\phi,\psi;\alpha\right)$ is closed under convex combination.

Proof: For $m=1,2,...$ suppose that $f_m \in \mathcal{TS}_H\left(\rho,i,j,\phi,\psi;\alpha\right)$ where $f_m(z)$ is given by

$$f_m(z) = z-\sum_{k=2}^{\infty}|a_{m,k}|z^k+(-1)^{i-1}\sum_{k=1}^{\infty}|b_{m,k}|z^k.$$

Then, by Theorem 2, we have

$$\sum_{k=1}^{\infty}\left(\frac{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k}{1-\alpha}|a_{m,k}|+\frac{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_{m,k}|\right)\le 2. \tag{5.4}$$

For $\sum_{m=1}^{\infty}t_m=1, 0\le t_m\le 1$, the convex combination of $f_m(z)$ may be written as

$$\sum_{m=1}^{\infty}t_m f_m(z) = z-\sum_{k=2}^{\infty}\sum_{m=1}^{\infty}t_m|a_{m,k}|z^k+(-1)^{i-1}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}t_m|b_{m,k}|z^k.$$

Now

$$\sum_{k=1}^{\infty}\left(\frac{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k}{1-\alpha}\sum_{m=1}^{\infty}t_m|a_{m,k}|+\frac{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}\sum_{m=1}^{\infty}t_m|b_{m,k}|\right)$$

$$=\sum_{m=1}^{\infty}t_m\sum_{k=1}^{\infty}\left(\frac{(1+\rho)\lambda_k-(\alpha+\rho)\mu_k}{1-\alpha}|a_{m,k}|+\frac{(1+\rho)\lambda_k-(-1)^{i-j}(\alpha+\rho)\mu_k}{1-\alpha}|b_{m,k}|\right)$$

$$\le 2\sum_{m=1}^{\infty}t_m=2$$

and so by Theorem 2, we have $\sum_{m=1}^{\infty}t_m f_m(z)\in\mathcal{TS}_H\left(\rho,i,j,\phi,\psi;\alpha\right)$.

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