International Journal of Mathematical Archive-3(3), 2012, Page: 1098-1210
Available online through www.ijma.info ISSN 2229 – 5046

SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN GENERALIZED B-METRIC SPACES

Renu Chugh, Vivek kumar* and Tamanna Kadian
Department of Mathematics, M. D. University, Rohtak-124001(INDIA)
E-mail: ratheevivek15@yahoo.com

(Received on: 12-03-12; Accepted on: 28-03-12)

ABSTRACT

The aim of this paper is two fold, first we define the concept of generalized b-metric spaces and then we prove the existence of fixed points for multivalued contraction mappings in generalized b-metric spaces using Picard iteration and also Jungck iteration. Our results extend, improve and unify a multitude of classical results in fixed point theory of single and multivalued contraction mappings. We obtain more general results than those of Nadler[23],Berinde and Berinde[10], M.O. Olatinwo and C.O. Imoru[24] and Daffer and Kaneko[16].

***Keywords**: Multivalued weak contraction, fixed point, b-metric space.*

***(2000) Mathematics Subject Classification:** 47H06, 47H10*

1. INTRODUCTION

Metric fixed point theory is an important mathematical discipline because of its applications in areas such as variational and linear inequalities, optimization and approximation theory.

The concept of b-metric space appeared in some works, such as N. Bourbaki, I. A. Bakhtin , S. Czerwik , J. Heinonen, ect. Several papers deal with the fixed point theory for singlevalued and multivalued operators in b-metric spaces (see[3],[12],[13]). Generalizations of metric spaces were proposed by Gahler[31],(called 2-metric spaces) and Dhage[2],(called D-metric spaces). Unfortunately, it was shown that certain theorems involving Dhage's D-metric spaces are flawed, and most of the results claimed by Dhage and others are invalid. In 2005, Mustafa and Sims[35] introduced a new structure of generalized metric spaces, which are called G-metric spaces as generalization of metric space (X, d), to develop and introduce a new fixed point theory for various mappings in this new structure. The study of fixed point theorems for multivalued mappings has been initiated by Markin[21] and Nadler[23]. We introduce the concept of generalized b-metric spaces in the sequel. Presently, let (X, G) be a generalized metric space and CB(X) denote the family of all non-empty closed and bounded subsets of X. For A, B, C $\subset$ X, define the distance between A, B and C by $D_G(A,B,C) = \inf\{G(a, b, c) : a\in A, b\in B, c\in C\}$, the diameter of A, B and C by $\delta_G(A ,B ,C) = \sup\{G(a, b, c) : a\in A, b\in B, c\in C\}$ and the Hausdorff-Pompeiu metric on CB(X) by

$$H_G(A, B, C) = \max\{\sup\{G(a, b, C) : a\in A, b\in B\}, \sup \{G(b, c, A) :b\in B, c\in C\}, \sup\{G(c, a, B) : c\in C, a\in A\}\}$$

H_G (A, B, C) is induced by G .

Let P(X) be the family of all non-empty subsets of X and T: X $\to$ P(X) a multivalued mapping. Then an element $x\in X$ such that $x \in T(x)$ is called a fixed point of T. Denote the set of all fixed point of T by Fix(T), that is,

$$Fix(T) = \{x\in X : x\in T(x)\}.$$

The following definitions shall be required in the sequel.

Definition 1.1: Let (X, d) be a metric space and T: X$\to$P(X) a multivalued operator. T is said to be a multivalued weakly Picard -operator iff for each $x\in X$ and any $y\in T(x)$, their exist a sequence $\{x_n\}_{n=0}^{\infty}$ such that

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1}\in T(x_n)$ for all n = 0,1,…….;
(iii) the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent and its limit is a fixed point of T

Corresponding author: Vivek kumar*, *E-mail: ratheevivek15@yahoo.com

Definition 1.2: Let (X, d) be a metric space and S, T: X→P(X) multivalued operator . The pair (S, T) will be called multivalued weakly Jungck operator iff for each $x \in X$ and any $y \in T(x)$, their exist a sequence $\{Sx_n\}_{n=0}^{\infty} \subset P(X)$ such that

(iv) $Sx_0=x$, $Sx_1= y$;
(v) $Sx_{n+1} \in T(x_n)$ for all n = 0,1,…….;
(vi) the sequence $\{Sx_n\}_{n=0}^{\infty}$ converges to Sz for some $z \in X$ and $Sz \in Tz$,that is , S and T have a coincidence at z.

Let C(S, T) be the set of coincidence points of S and T.

Definition 1.3: A function $\phi: R_+ \to R_+$ is called (c)-comparison if it satisfies
(i) ϕ is monotonic increasing ;
(ii) $\phi^n (t) \to 0$ as $n\to\infty$, $\forall\ t > 0$ (ϕ^n stands for the nth iterate of ϕ) ;
(iii) $\sum_{n=0}^{\infty} \phi^n (t) < \infty$ for all $t > 0$.

We say that ϕ is a comparison function if it satisfies (i) and (ii) only. See [6] and [30] for detail.

Remark 1.3: Every comparison function $\phi : R_+ \to R_+$ satisfies $\phi(t) < t$.

Theorem 1.1[23]: Let (X, d) be a complete metric space and T:X→CB(X) a set valued α-contraction ,that is, a mapping for which there exist a constant $\alpha\in(0,1)$, such that

$$H (Tx, Ty) \le \alpha d(x, y)$$

Theorem 1.2: (Berinde and Berinde [10]**):-**Let (X, d) be a complete metric space and T:X→CB(X) a generalized multivalued (θ,L) – contraction. Then,
(i) Fix (T) $\neq \phi$
(ii) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point u of T for which the following estimates hold:

$$d(x_n,u) \le \frac{h^n}{1-h} d(x_1, x_0) , n = 0, 1, 2, 3, \ldots$$

$$d(x_n,u) \le \frac{h}{1-h} d(x_n ,x_{n-1}), \ n =1, 2,3\ldots$$

for a certain constant h <1.

Theorem 1.3: (Berinde and Berinde[10]**):-** Let (X, d) be a complete metric space and T: X→CB(X) a generalized multivalued (α,L) – weak contraction. that is, a mapping for which there exist a function

$\alpha: [0, \infty) \to [0, 1)$ satisfying $\lim_{r\to t^+} \sup \alpha(r) < 1$, for every $t\in[0, \infty)$, such that

$$H(Tx,Ty) \le \alpha(d(x, y))d(x,y) + LD(y,Tx)\ \forall\ x, y \in X.$$

Then T has a fixed point.

The following definitions shall be required in the sequel .

Definition 1.4: Let X be a nonempty set and $s \ge 1$ a real number. A function $G : X \times X \times X \to R^+ \cup \{0\}$ is said to be a generalized b-metric space if it satisfy the following properties :

(G1) $G(x, y, z) = 0$ iff $x = y = z$
(G2) $0 < G(x, x, y)\ \forall\ x, y \in X$, with $x \neq y$.
(G3) $G(x, x, y) \le G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) =\ldots\ldots$, (symmetry in all the three variables)

(G5) $G(x, y, z) \le s\,[G(x, a, a) + G(a, y, z)]\ \forall\ x, y, z \in X,\ a \in X$ and $s \ge 1$ (rectangle inequality)

The pair (X, G) is called a generalized b-metric space.

Example of definition 1.4: Let $X = \{x_1, x_2, x_3, x_4\}$, $d(x_1, x_2) = k \ge 2$ and $d(x_1, x_3) = d(x_1, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = 1$,

$d(x_i, x_j) = d(x_j, x_i)$ for all i, j = 1, 2, 3, 4

and

$d(x_i, x_i) = 0$, i = 1, 2, 3, 4.

If we define generalized metric by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ then

$$G(x, y, z) \le \frac{k}{2}\,[G(x, a, a) + G(a, y, z)]\ \forall\ x, y, z, a \in X$$

So, (X, G) will be a generalized b-metric space.

Definition 1.5: Let (X, G) be a generalized b-metric space and $T: X \to P(X)$ a multivalued operator. T is said to be a generalized multivalued (ψ, ϕ) weak contraction iff there exists a continuous monotonic increasing function $\phi: R_+ \to R_+$ with $\phi(0) = 0$ and a continuous comparison function $\psi : R_+ \to R_+$ such that

$$H_G(Tx, Tx, Ty) \le q^{-1}\,[\psi(G(x, x, y)) + \phi(D_G(y, Tx, Tx))],\ \ q > 1,\ \forall\ x, y \in X \tag{*}$$

Definition 1.6: We say that T is a generalized multivalued ϕ-weak contraction iff there exists a function $\alpha : [0, \infty) \to [0, 1)$ and two continuous monotonic increasing functions $\phi_1, \phi_2 : R_+ \to R_+$ with $\phi_1(0) = 1$ and $\phi_2(0) = 0$ such that

$$H_G(Tx, Tx, Ty) \le [\alpha(G(x, x, y))\, G(x, x, y)\,]^{\varphi_1(D_G(y,Tx,Tx))} + \phi_2(D_G(y, Tx, Tx)),\ \forall\ x, y \in X \tag{**}$$

where $\lim\limits_{r \to t^+} \sup \alpha(r) < 1$, for every $t \in [0, \infty)$.

Definition 1.7: Let (X, G) be a generalized b-metric space and $S,T : X \to P(X)$ multivalued operators. Then the pair (S,T) will be called a multivalued (θ, ϕ) weak J-contraction iff there exists a constant $\theta \in (0,1)$ and a continuous monotonic increasing function $\phi : R_+ \to R_+$ with $\phi(0) = 0$ such that

$$H_G(Tx, Tx, Ty) \le \theta\, G(Sx, Sx, Sy) + \phi(D_G(Sy,Tx, Tx))\quad q > 1,\ \forall\ x, y \in X \tag{***}$$

The contractive condition (***) can be modified to the following form: The pair (S,T) will be called a generalized multi-valued (α, ϕ) –weak J- contraction iff there exist a function $\alpha : [0, \infty) \to [0, 1)$ and a continuous monotonic increasing function $\phi : R_+ \to R_+$ with $\phi(0) = 0$ such that

$$H_G(Tx, Tx, Ty) \le \alpha(G(Sx, Sx, Sy))\, G(Sx, Sx, Sy) + \phi(D_G(Sy, Tx, Tx))]\ q > 1,\ \forall\ x, y \in X \tag{****}$$

where $\lim\limits_{r \to t^+} \sup \alpha(r) < 1$, for every $t \in [0, \infty)$.

We shall require the following lemmas in the sequel.

Lemma 1.1: Let (X, G) be a generalized metric space. Let $A, B \subset X$ and $q > 1$. Then for every $a \in A$, there exists $b \in B$ such that

$$G(a, a, b) \le q\, H_G(A, A, B) \tag{1.1}$$

Proof: If $H_G(A, A, B) = 0$ then $a \in B$ and (1.1) holds for $b = a$.

If $H_G(A, A, B) > 0$, then let us denote

$$\in\, = (h^{-1} - 1)\, H_G(A, A, B) > 0 \tag{1.2}$$

Using the definition of $D_G(a, a, B)$ and $H_G(A, A, B)$, it follows that, for any $\in\, > 0$, there exists $b \in B$ such that

$$G(a, a, b) \le D_G(a, a, B) + \in\ \le H_G(A, A, B) + \in \tag{1.3}$$

Now, by inserting (1.2) in (1.3), we get

$$\begin{aligned} G(a, a, b) &\le H_G(A, A, B) + h^{-1} H_G(A, A, B) - H_G(A, A, B) \\ &\le \frac{1}{h} H_G(A, A, B) \\ &\le q\, H_G(A, A, B), \text{ where } \frac{1}{h} = q. \end{aligned}$$

Lemma 1.2: Let A, B $\subseteq$ CB(X) and let $a \in A$. Then, there exists $b \in B$ such that

$$G(a, a, b) \le H_G(A, A, B) + \eta, \text{ where } \eta > 0.$$

Lemma 1.2 is a simple consequence of the definition of H_G (A, B, C).

2. MAIN RESULTS

Theorem 2.1: Let (X,G) be a complete generalized b-metric space with continuous generalized b-metric and T: $X \to CB(X)$ a generalized multivalued (ψ,ϕ) – weak contraction. Suppose that $\psi : R_+ \to R_+$ is continuous (c)–comparison function and $\phi : R_+ \to R_+$ is a continuous monotonic increasing function such that $\phi(0) = 0$. Then,

(i) Fix T $\neq \phi$

(ii) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point x* of T

(iii) the a priori and a posteriori error estimates are given by

$$G(x_n, x_n, x^*) \le s \sum_{k=0}^{\infty} \psi^{k+n} (G(x_0, x_0, x_1)), s \ge 1, n = 1, 2, 3, \ldots \quad \ldots \tag{2.1.1}$$

$$G(x_n, x_n, x^*) \le s \sum_{k=0}^{\infty} \psi^{k} (G(x_{n-1}, x_{n-1}, x_n)), s \ge 1, n = 1, 2, 3 \ldots \ldots \tag{2.1.2}$$

respectively.

Proof: Let $x_0 \in X$ and $x_1 \in Tx_0$. If $H_G(Tx_0, Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, that is $x_1 \in Tx_1$, which implies Fix T $\neq \phi$.

Let $H_G(Tx_0, Tx_0, Tx_1) \neq 0$ .Then, we have by lemma 1.1 that there exists $x_2 \in Tx_1$ such that

$$G(x_1, x_1, x_2) \le q\, H_G(Tx_0, Tx_0, Tx_1), \; q > 1$$

so that by (∗) we have

$$\begin{aligned} G(x_1, x_1, x_2) &\le q\, q^{-1} [\psi(G(x_0, x_0, x_1)) + \phi(D_G(x_1, Tx_0, Tx_0))] \\ &= \psi(G(x_0, x_0, x_1)) + \phi(D_G(x_1, x_1, x_1)) \\ &= \psi(G(x_0, x_0, x_1)) \end{aligned}$$

If $H_G(Tx_1, Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$, that is $x_2 \in Tx_2$.

Let $H_G(Tx_1, Tx_1, Tx_2) \neq 0$. Again by lemma 1.1, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} G(x_2, x_2, x_3) &\le q\, H_G(Tx_1, Tx_1, Tx_2) \\ &\le qq^{-1}[\psi(G(x_1, x_1, x_2)) + \phi(D_G(x_2, Tx_1, Tx_1))] \quad (2.1.3) \\ &= \psi(G(x_1, x_1, x_2)) + \phi(D_G(x_2, x_2, x_2)) \\ &= \psi(G(x_1, x_1, x_2)) \le \psi^2(G(x_0, x_0, x_1). \end{aligned}$$

By induction, we obtain

$$G(x_n, x_n, x_{n+1}) \le \psi^n(G(x_0, x_0, x_1)) \tag{2.1.4}$$

Therefore by the property (G5) of definition 1.4, we have

$$G(x_n, x_n, x_{n+p}) \le s[G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots\ldots + G(x_{n+p-1}, x_{n+p-1}, x_{n+p})]$$

$$\le s[\psi^n(G(x_0, x_0, x_1)) + \psi^{n+1}(G(x_0, x_0, x_1)) + \ldots\ldots + \psi^{n+p-1}(G(x_0, x_0, x_1))] \tag{2.1.5}$$

$$G(x_n, x_n, x_{n+p}) \le s \sum_{k=n}^{n+p-1} \psi^k (G(x_0, x_0, x_1)) \tag{2.1.6}$$

From (2.1.6), we have

$$G(x_n, x_n, x_{n+p}) \le s \sum_{k=n}^{n+p-1} \psi^k (G(x_0, x_0, x_1))$$

$$= s\left[\sum_{k=0}^{n+p-1} \psi^k (G(x_0, x_0, x_1) - \sum_{k=0}^{n-1} \psi^k (G(x_0, x_0, x_1))\right] \to 0 \text{ as } n\to\infty. \tag{2.1.7}$$

We therefore have from (2.1.7), that for any $x_0 \in X$, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X, G) is a complete generalized b-metric space, then $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$. that is

$$\lim_{n\to\infty} x_n = x^* \tag{2.1.8}$$

Therefore by (∗) we have that

$$D_G(x^*, x^*, Tx^*) \le s[G(x^*, x^*, x_{n+1}) + G(x_{n+1}, x_{n+1}, Tx^*)]$$

$$\le s[G(x^*, x^*, x_{n+1}) + H_G(Tx_n, Tx_n, Tx^*)]$$

$$\le s\, G(x^*, x^*, x_{n+1}) + sq^{-1} [\psi(G(x_n, x_n, x^*)) + \phi(D_G(x^*, Tx_n, Tx_n))] \tag{2.1.9}$$

By using (2.1.8), the continuity of the functions ψ, ϕ and the fact that $x_{n+1} \in Tx_n$, then $\phi(D_G(x^*, Tx_n\ Tx_n)) \to 0$ as $n\to\infty$ and $\psi(G(x_n, x_n, x^*)) \to 0$ as $n\to\infty$.

It follows from (2.1.9) that $D_G(x^*, x^*, Tx^*) = 0$ as $n\to\infty$. Since Tx^* is closed then $x^* \in Tx^*$.

To prove a priori error estimate in (2.1.1), we have from (2.1.6) that

$$G(x_n, x_n, x_{n+p}) \le s \sum_{k=n}^{n+p-1} \psi^k (G(x_0, x_0, x_1))$$

$$= s \sum_{k=0}^{p-1} \psi^{n+k} (G(x_0, x_0, x_1))$$

from which it follows by the continuity of the generalized b-metric that

$$G(x_n, x_n, x^*) = \lim_{p\to\infty} G(x_n, x_n, x_{n+p})$$

$$\le s \sum_{k=0}^{\infty} \psi^{n+k} (G(x_0, x_0, x_1))$$

which gives the result in (2.1.1).

To prove result in (2.1.2), we get by condition (∗) and lemma 1.1 that

$$G(x_n, x_n, x_{n+1}) \le q\, H_G(Tx_{n-1}, Tx_{n-1}, Tx_n)$$

$$\leq qq^{-1}[\psi(G(x_{n-1}, x_{n-1}, x_n)) + \phi(D_G(x_n, Tx_{n-1}, Tx_{n-1}))]$$

$$= \psi(G(x_{n-1}, x_{n-1}, x_n)) + \phi(D_G(x_n, x_n, x_n))$$

$$= \psi(G(x_{n-1}, x_{n-1}, x_n))$$

Also, we have

$$G(x_{n+1}, x_{n+1}, x_{n+2}) \leq \psi(G(x_n, x_n, x_{n+1}))$$

$$\leq \psi^2(G(x_{n-1}, x_{n-1}, x_n))$$

so that in general we obtain

$$G(x_{n+k}, x_{n+k}, x_{n+k+1}) \leq \psi^{k+1}(G(x_{n-1}, x_{n-1}, x_n)),\ k = 0, 1, 2, \ldots. \tag{2.1.10}$$

Using (2.1.10) in (2.1.5) yields

$$G(x_n, x_n, x_{n+p}) \leq s[\psi(G(x_{n-1}, x_{n-1}, x_n)) + \psi^2(G(x_{n-1}, x_{n-1}, x_n)) + \ldots + \psi^{p-1}(G(x_{n-1}, x_{n-1}, x_n))]$$

$$= s\sum_{k=0}^{p-1}\psi^k(G(x_{n-1}, x_{n-1}, x_n)) \tag{2.1.11}$$

Again taking limit in (2.1.11) as $p\to\infty$ and using the continuity of the generalized b-metric, we have

$$G(x_n, x_n, x^*) = \lim_{p\to\infty} G(x_n, x_n, x_{n+p})$$

$$\leq s\sum_{k=0}^{\infty}\psi^k(G(x_{n-1}, x_{n-1}, x_n)),$$ giving the result in (2.1.2).

Remark 2.1: Theorem 2.1 is a generalization of theorem 1.2 as well as theorem 5 of Nadler [29] .

Theorem 2.2:- Let (X, G) be a complete generalized b-metric space with continuous generalized b-metric and $T : X\to CB(X)$ a generalized multi-valued ϕ-weak contraction. Suppose that there exists a function $\alpha :[0, \infty) \to [0, 1)$ satisfying $\lim_{r\to t^+} \sup \alpha(r) < 1$, for every $t\in[0, \infty)$ and two continuous monotone increasing functions ϕ_1 and $\phi_2 : R_+ \to R_+$ such that $\phi_1(0) = 1$ and $\phi_2(0) = 0$. Then, T has at least one fixed point.

Proof: Suppose $x_0\in X$ and $x_1\in Tx_0$. We choose a positive integer N_1 such that

$$\alpha^{N_1}(G(x_0, x_0, x_1)) \leq [1-\alpha(G(x_0, x_0, x_1)]\, G(x_0, x_0, x_1) \tag{2.2.1}$$

By lemma 1.2, there exists $x_2 \in Tx_1$ such that

$$G(x_1, x_1, x_2) \leq H_G(Tx_0, Tx_0, Tx_1) + \alpha^{N_1}(G(x_0, x_0, x_1)) \tag{2.2.2}$$

Using (**) and (2.2.1) in (2.2.2), then we have

$$G(x_1, x_1, x_2) \leq [\alpha(G(x_0, x_0, x_1))G(x_0, x_0, x_1)\ ]^{\phi_1(D_G(x_1, Tx_0, Tx_0))} + \phi_2(D_G(x_1, Tx_0, Tx_0)) + \alpha^{N_1}(G(x_0, x_0, x_1))$$

$$= \alpha(G(x_0, x_0, x_1))\, G(x_0, x_0, x_1) + \alpha^{N_1}(G(x_0, x_0, x_1)) \leq G(x_0, x_0, x_1)$$

Now, we choose again a positive integer N_2, $N_2 > N_1$ such that

$$\alpha^{N_2}(G(x_1, x_1, x_2)) \leq [1-\alpha(G(x_1, x_1, x_2))]\, G(x_1, x_1, x_2) \tag{2.2.3}$$

Since $Tx_2 \in CB(X)$, by lemma 1.2 again, we can select $x_3 \in Tx_2$ such that

$$G(x_2, x_2, x_3) \leq H_G(Tx_1, Tx_1, Tx_2) + \alpha^{N_2}(G(x_1, x_1, x_2)) \tag{2.2.4}$$

Again using (**) and (2.2.3) in (2.2.4), then we get

$$G(x_2, x_2, x_3) \leq [\alpha(G(x_1, x_1, x_2))\, G(x_1, x_1, x_2)\ ]^{\phi_1(D_G(x_2, Tx_1, Tx_1))} + \phi_2(D_G(x_2, Tx_1, Tx_1)) + \alpha^{N_2}(G(x_1, x_1, x_2))$$

By induction, since $Tx_k \in CB(X)$, for each k, we may choose a positive integer N_k such that

$$\alpha^{N_k}(G(x_{k-1}, x_{k-1}, x_k)) \leq [1-\alpha(G(x_{k-1}, x_{k-1}, x_k))]\, G(x_{k-1}, x_{k-1}, x_k) \tag{2.2.5}$$

By selecting $x_{k+1} \in Tx_k$ such that

$$G(x_k, x_k, x_{k+1}) \leq H_G(Tx_{k-1}, Tx_{k-1}, Tx_k) + \alpha^{N_k}(G(x_{k-1}, x_{k-1}, x_k)) \tag{2.2.6}$$

so that using (∗∗) and (2.2.5) in (2.2.6) yield

$$G(x_k, x_k, x_{k+1}) \leq G(x_{k-1}, x_{k-1}, x_k) \tag{2.2.7}$$

Let $G_k = G(x_{k-1}, x_{k-1}, x_k)$, k = 1, 2,…..

The inequality relation (2.2.7) shows that the sequence $\{G_k\}$ of non-negative numbers is decreasing. Therefore, $\lim\limits_{k\to\infty} G_k$ exists. Thus, set $\lim\limits_{k\to\infty} G_k = c \geq 0$.

We now prove that the Picard iteration or orbit$\{x_k\} \subset X$ so generated is a Cauchy sequence. By condition on α, for t = c we have

$$\lim_{t\to c^+} \alpha(t) < 1.$$

For $k \geq k_0$, let $\alpha(G_k) < h$, where $\lim\limits_{t\to c^+} \sup \alpha(t) < h < 1$.

Using (2.2.6), we have by deduction that $\{G_k\}$ satisfies the recurrence inequality:

$$G_{k+1} \leq G_k\, \alpha(G_k) + \alpha^{N_k}(G_k),\ k = 1, 2, \ldots. \tag{2.2.8}$$

Using induction in (2.2.8) leads to

$$G_{k+1} \leq \prod_{j=1}^{k} \alpha(G_j) G_1 + \sum_{m=1}^{k-1} \prod_{J=m+1}^{k} \alpha(G_j) \alpha^{N_m}(G_m) + \alpha^{N_k}(G_k),\ k \geq 1 \tag{2.2.9}$$

We now find a suitable upper bound for the right hand side of (2.2.9), using the fact that $\alpha < 1$ as follows:

$$G_{k+1} \leq \prod_{j=1}^{k} \alpha(G_j) G_1 + \sum_{m=1}^{k-1} \prod_{J=m+1}^{k} \alpha(G_j)\, \alpha^{N_m}(G_m) + \alpha^{N_k}(G_k)$$

$$< G_1 h^k + \sum_{m=1}^{k-1} h^{k-m} h^{N_m} + h^{N_k} = G_1 h^k + h^k \sum_{m=1}^{k-1} h^{N_m - m} + h^{N_k}$$

$$\leq C1h^k + C_2 h^k + C_3 h^k = C_4\, h^k, \text{ where } C_4 = C_1 + C_2 + C_3 \text{ and } C_1, C_2, C_3, C_4 \text{ are constants.} \tag{2.2.10}$$

Now, for $k \geq k_0$, and $p \in N$, we have by using (2.2.10) and the repeated application of the rectangle inequality that

$$G(x_k, x_k, x_{k+p}) \leq s[G(x_k, x_k, x_{k+1}) + G(x_{k+1}, x_{k+1}, x_{k+2}) + \ldots + G(x_{k+p-1}, x_{k+p-1}, x_{k+p})]$$

$$= s[G_{k+1} + G_{+2} + \ldots + G_{k+p}]$$

$$\leq s\,[C_4\,(h^k + h^{k+1} + \ldots + h^{k+p-1})]$$

$$= C_4 \left(\frac{1-h^p}{1-h}\right) h^k s = C_5 h^k s, \tag{2.2.11}$$

where C_5 is a constant

Since 0 < h < 1, the right hand side of (2.2.11) tends to 0 as $k\to\infty$, showing that $\{x_k\}$ is a Cauchy sequence. Therefore, $x_k \to u \in X$ as $k\to\infty$ since X is complete generalized b-metric space. So,

$D_G(u, u, Tu) \le s[G(u, u, x_k) + G(x_k, x_k, Tu)]$

$\le s[G(u, u, x_k) + H_G(Tx_{k-1}, Tx_{k-1}, Tu)]$

$\le s\, G(u, u, x_k) + s[\alpha(G(x_{k-1}, x_{k-1}, u))\, G(x_{k-1}, x_{k-1}, u)]^{\phi_1(D_G(u, Tx_{k-1}, Tx_{k-1}))} + s\phi_2(D_G(u, Tx_{k-1}, Tx_{k-1}))$

$< s\, G(u, u, x_k) + s[h\, G(x_{k-1}, x_{k-1}, u)]^{\phi_1(D_G(u, Tx_{k-1}, Tx_{k-1}))} + s\phi_2(D_G(u, Tx_{k-1}, Tx_{k-1}))$, $s \ge 1$. (2.2.12)

By using the fact that $x_k \in Tx_{k-1}$ and $x_k \to u$ as $k\to\infty$, we have $D_G(u, Tx_{k-1}, Tx_{k-1}) \to 0$ as $k\to\infty$. We therefore, have by continuity of $\phi_j (j = 1, 2)$ that $\phi_1(D_G(u, Tx_{k-1}, Tx_{k-1})) \to 1$ as $k\to\infty$ and $\phi_2(D_G(u, Tx_{k-1}, Tx_{k-1})) \to 0$ as $k\to\infty$. Hence, since the right hand side terms of (2.2.12) tends to zero as $k\to\infty$, we have $u \in Tu$. Using the continuity of the generalized b-metric in (2.2.11) as $p\to\infty$, we obtain an error estimate $G(x_k, x_k, u) = \lim_{p\to\infty} G(x_k, x_k, x_{k+p}) \le C_5 h^k s$, $k \ge k_0$, $s \ge 1$ for the Picard iteration process under condition (**).

Remark 2.2: Theorem 2.2 is a generalization of theorem 1.3 , Nadler fixed point theorem [23] as well as theorem 2.1 of Daffer and Kaneko[16] .

Theorem 2.3: Let (X, G) be a complete generalized b-metric space with continuous generalized b-metric and S,T : $X \to CB(X)$ a generalized multivalued (θ, ϕ) – weak j-contraction such that S is continuous and $T(X) \subseteq S(X)$, S(X) a complete subspace of CB(X). Suppose that $\phi : R_+ \to R_+$ is a continuous monotonic increasing function such that $\phi(0) = 0$. Then,

(i) $C(S, T) \neq \phi$, where C(S, T) is the set of coincidence points of S and T.

(ii) for any $x_0 \in X$, there exists a Jungck orbit $\{Sx_n\}_{n=0}^{\infty}$ of the pair (S,T) at the point x_0 that converges to Sz for some $z \in X$, and $Sz \in Tz$,that is $z \in C(S, T)$

(iii) the a priori and a posteriori error estimates are given by

$$G(Sx_n, Sx_n, Sz) \le \frac{sh^n}{1-h} G(Sx_0, Sx_0, Sx_1), s \ge 1, n = 1, 2, 3, \ldots \quad (2.3.1)$$

$$G(Sx_n, Sx_n, Sz) \le \frac{sh}{1-h} G(Sx_{n-1}, Sx_{n-1}, Sx_n), s \ge 1, n = 1, 2, 3\ldots \quad (2.3.2)$$

respectively for a certain constant $h < 1$.

Proof: Let $x_0 \in X$ and $Sx_1 \in Tx_0$. If $H_G(Tx_0, Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, that is $Sx_1 \in Tx_1$, which implies that $C(S, T) \neq \phi$.

Let $H_G(Tx_0, Tx_0, Tx_1) \neq 0$ .Then, we have by lemma 1.1 that there exists $x_2 \in X$ so that $Sx_2 \in Tx_1$ such that

$$G(Sx_1, Sx_1, Sx_2) \le q\, H_G(Tx_0, Tx_0, Tx_1), q > 1$$

so that by (***) we have

$$\begin{aligned} G(Sx_1, Sx_1, Sx_2) &\le q\, \theta\, [G(Sx_0, Sx_0, Sx_1) + \phi(D_G(Sx_1, Tx_0, Tx_0))] \\ &= q\, \theta\, G(Sx_0, Sx_0, Sx_1) \\ &= h\, G(Sx_0, Sx_0, Sx_1), \end{aligned}$$

where $h = q\theta < 1$.

If $H_G(Tx_1, Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$, that is $Sx_2 \in Tx_2$.

Let $H_G(Tx_1, Tx_1, Tx_2) \neq 0$. Again by lemma 1.1, there exists $x_3 \in X$ so that $Sx_3 \in Tx_2$ such that

$$\begin{aligned} G(Sx_2, Sx_2, Sx_3) &\le q\, H_G(Tx_1, Tx_1, Tx_2) \\ &\le q[\theta\, G(Sx_1, Sx_1, Sx_2) + \phi(D_G(Sx_2, Tx_1, Tx_1))] \end{aligned}$$

$= q\,\theta\, G(Sx_1,Sx_1,Sx_2)$

from which it follows that

$$G(Sx_2, Sx_2, Sx_3) \le h\, G(Sx_1,Sx_1,Sx_2) \le h^2\, G(Sx_0, Sx_0, Sx_1) \tag{2.3.3}$$

By induction, we obtain

$$G(Sx_n , Sx_n, Sx_{n+1}) \le h^n(G(Sx_0, Sx_0, Sx_1)) \tag{2.3.4}$$

Therefore from (2.3.4) and property (G5) of definition 1.4, we have

$$G(Sx_n, Sx_n, Sx_{n+p}) \le s[G(Sx_n, Sx_n, Sx_{n+1}) + G(Sx_{n+1}, Sx_{n+1}, Sx_{n+2}) +\ldots\ldots+ G(Sx_{n+p-1}, Sx_{n+p-1}, Sx_{n+p})]$$

$$\le s[h^n G(Sx_0, Sx_0, Sx_1)+ h^{n+1} G(Sx_0, Sx_0,Sx_1) +\ldots\ldots+ h^{n+p-1} G(Sx_0, Sx_0, Sx_1)] \tag{2.3.5}$$

$$= \frac{sh^n\left(1-h^p\right)}{1-h} G(Sx_0, Sx_0, Sx_1) \tag{2.3.6}$$

From (2.3.6), we have

$G(Sx_n, Sx_n, Sx_{n+p}) \to 0$ as $n\to\infty$.

We therefore have that for any $x_0\in X$, $\{Sx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X, G) is a complete generalized b-metric space, there exist a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converging to some $z \in X$. Therefore, by the continuity of S , $\{Sx_n\}_{n=0}^{\infty}$ converges to some $Sz \in X$. That is

$$\lim_{n\to\infty} Sx_n = Sz = w \tag{2.3.7}$$

Therefore, by (***), we have that

$$D_G(Sz, Sz, Tz)= D_G(w, w, Tz) \le s[G(w, w, Sx_{n+1}) +G(Sx_{n+1}, Sx_{n+1} ,Tz)]$$

$$\le s[G(w, w, Sx_{n+1}) + H_G(Tx_n, Tx_n ,Tz)]$$

$$\le s\, G(w, w, Sx_{n+1}) + s[\theta\, G(Sx_n, Sx_n,Sz) + \phi(D_G(Sz, Tx_n ,Tx_n))]$$

$$= s\, G(w, w, Sx_{n+1}) + s[\theta\, G(Sx_n, Sx_n ,w) + \phi(D_G(w, Tx_n ,Tx_n))] \tag{2.3.8}$$

By using (2.3.7), the continuity of the functions ϕ and the fact that

$Sx_{n+1} \in Tx_n$, then $\phi(D_G(w, Tx_n\ Tx_n)) \to 0$ as $n\to\infty$ and $G(Sx_n, Sx_n, w) \to 0$ as $n\to\infty$.

It follows from (2.3.8) that $D_G(Sz,Sz, Tz) = 0$ as $n\to\infty$. Since Tz is closed, then $Sz \in Tz$, $z \in C(S,T)$

To prove a priori error estimate in (2.3.1), we have from (2.3.6) by the continuity of the generalized b- metric that

$$G(Sx_n, Sx_n, Sz)= \lim_{p\to\infty} G(Sx_n, Sx_n, Sx_{n+p}) \le \frac{sh^n}{1-h} G(Sx_0,Sx_0, Sx_1)$$

which gives the result in (2.3.1).

To prove result in (2.3.2), we get by condition (***) and lemma 1.1 that

$$G(Sx_n,Sx_n, Sx_{n+1}) \le q\, H_G(Tx_{n-1}, Tx_{n-1}, Tx_n)$$

$$\le q[\theta\, G(Sx_{n-1}, Sx_{n-1}, Sx_n)+\phi(D_G(Sx_n,Tx_{n-1}, Tx_{n-1}))]$$

$= q\theta\, G(Sx_{n-1}, Sx_{n-1}, Sx_n)$

$= hG(Sx_{n-1}, Sx_{n-1}, Sx_n)$

Also, we have

$$G(Sx_{n+1}, Sx_{n+1}, Sx_{n+2}) \le h\, G(Sx_n, Sx_n, Sx_{n+1})$$

$$\le h^2 G(Sx_{n-1}, Sx_{n-1}, Sx_n)$$

so that in general we obtain

$$G(Sx_{n+k}, Sx_{n+k}, Sx_{n+k+1}) \le h^{k+1}\, G(Sx_{n-1}, Sx_{n-1}, Sx_n),\ k = 0, 1, 2, \ldots. \quad (2.3.9)$$

Using (2.3.9) in (2.3.5) yields

$$G(Sx_n, Sx_n, Sx_{n+p}) \le s[hG(Sx_{n-1}, Sx_{n-1}, Sx_n) + h^2G(Sx_{n-1}, Sx_{n-1}, Sx_n) + \ldots + h^pG(Sx_{n-1}, Sx_{n-1}, Sx_n)]$$

$$= \frac{sh\left(1-h^p\right)}{1-h}\, G(Sx_{n-1}, Sx_{n-1}, Sx_n) \quad (2.3.10)$$

Again taking limit in (2.3.10) as $p\to\infty$ and using the continuity of the generalized b-metric, we have

$$G(Sx_n, Sx_n, Sz) = \lim_{p\to\infty} G(Sx_n, Sx_n, Sx_{n+p})$$

$$\le \frac{sh}{1-h}\, G(Sx_{n-1}, Sx_{n-1}, Sx_n),$$ giving the result in (2.3.2).

Remark 2.4: Theorem 2.3 is a generalization of Theorem 1.2.

Theorem 2.4: Let (X,G) be a complete generalized b-metric space with continuous generalized b-metric and S, $T:X\to CB(X)$ a generalized multi-valued (α,ϕ)-weak j-contraction such that S is continuous and $T(X) \subseteq S(X)$, S(X) a complete subspace of CB(X). Suppose that there exists a function $\alpha :[0, \infty) \to [0, 1)$ satisfying $\lim_{r\to t^+} \sup \alpha(r) < 1$, for every $t\in[0, \infty)$ and a continuous monotone increasing function $\phi : R_+ \to R_+$ such that $\phi(0) = 0$. Then, T and S have at least one coincidence point.

Proof: Suppose $x_0\in X$ with $Sx_1\in Tx_0$. We choose a positive integer N_1 such that

$$\alpha^{N_1} (G (Sx_0, Sx_0, Sx_1)) \le [1-\alpha(G (Sx_0, Sx_0, Sx_1))]\, G(Sx_0, Sx_0, Sx_1) \quad (2.4.1)$$

By lemma 1.2, there exists $x_2 \in X$ with $Sx_2 \in Tx_1$ such that

$$G (Sx_1,Sx_1,Sx_2) \le H_G(Tx_0, Tx_0, Tx_1) + \alpha^{N_1} (G(Sx_0, Sx_0, Sx_1)) \quad (2.4.2)$$

Using (****) and (2.4.1) in (2.4.2), we have

$$G(Sx_1, Sx_1, Sx_2) \le [\alpha(G(Sx_0,Sx_0,Sx_1))G(Sx_0,Sx_0,Sx_1) + \phi(D_G(Sx_1, Tx_0, Tx_0)) + \alpha^{N_1} (G(Sx_0, Sx_0, Sx_1))$$

$$= \alpha(G(Sx_0,Sx_0,Sx_1))G(Sx_0, Sx_0, Sx_1) + \alpha^{N_1} (G(Sx_0, Sx_0, Sx_1)) \le G(Sx_0,Sx_0,Sx_1)$$

Now, we choose again a positive integer N_2, $N_2 > N_1$ such that

$$\alpha^{N_2} (G (Sx_1, Sx_1, Sx_2)) \le [1-\alpha(G(Sx_1, Sx_1, Sx_2))]\, G(Sx_1, Sx_1, Sx_2) \quad (2.4.3)$$

Since $Tx_2 \in CB(X)$, by lemma 1.2 again, we can select $x_3 \in X$ with $Sx_3 \in Tx_2$ such that

$$G(Sx_2, Sx_2, Sx_3) \le H_G(Tx_1, Tx_1, Tx_2) + \alpha^{N_2} (G(Sx_1, Sx_1, Sx_2)) \quad (2.4.4)$$

Again using (*****) and (2.4.3) in (2.4.4), we get

$$G(Sx_2, Sx_2, Sx_3) \le \alpha(G(Sx_1, Sx_1,Sx_2))\, G(Sx_1, Sx_1, Sx_2) + \phi(D_G(Sx_2, Tx_1, Tx_1)) + \alpha^{N_2} (G(Sx_1, S x_1, Sx_2))$$

$= G(Sx_1, Sx_1, Sx_2)$

By induction, since $Tx_k \in CB(X)$, for each k, we may choose a positive integer N_k such that

$$\alpha^{N_k}(G(Sx_{k-1}, Sx_{k-1}, Sx_k)) \le [1-\alpha(G(Sx_{k-1}, Sx_{k-1}, Sx_k))]\, G(Sx_{k-1}, Sx_{k-1}, Sx_k) \tag{2.4.5}$$

By selecting $x_{k+1} \in X$ with $Sx_{k+1} \in Tx_k$ such that

$$G(Sx_k, Sx_k, Sx_{k+1}) \le H_G(Tx_{k-1}, Tx_{k-1}, Tx_k) + \alpha^{N_k}(G(Sx_{k-1}, Sx_{k-1}, Sx_k)) \tag{2.4.6}$$

so that using (****) and (2.4.5) in (2.4.6) yield

$$G(Sx_k, Sx_k, Sx_{k+1}) \le G(Sx_{k-1}, Sx_{k-1}, Sx_k) \tag{2.4.7}$$

Let $G_k = G(Sx_{k-1}, Sx_{k-1}, Sx_k)$, k = 1, 2,…..

The inequality relation (2.4.7) shows that the sequence $\{G_k\}$ of non-negative numbers is decreasing. Therefore,

$\lim\limits_{k\to\infty} G_k$ exists. Thus, let $\lim\limits_{k\to\infty} G_k = c \ge 0$.

We now prove that the Jungck iteration or orbit$\{Sx_k\} \subset X$ so generated is a Cauchy sequence.

By condition on α, for t = c we have $\lim\limits_{t\to c^+} \sup \alpha(t) < 1$.

For $k \ge k_0$, let $\alpha(G_k) < h$, where $\lim\limits_{t\to c^+} \sup \alpha(t) < h < 1$.

Using (2.4.6), we have by deduction that $\{G_k\}$ satisfies the recurrence inequality:

$$G_{k+1} \le G_k\alpha(G_k) + \alpha^{N_k}(G_k),\ k = 1, 2, \ldots. \tag{2.4.8}$$

Using induction in (2.4.8) leads to

$$G_{k+1} \le \prod_{j=1}^{k}\alpha(G_j)G_j + \sum_{m=1}^{k-1}\prod_{J=m+1}^{k}\alpha(G_j)\alpha^{N_m}(G_m) + \alpha^{N_k}(G_k),\ k \ge 1 \tag{2.4.9}$$

We now find a suitable upper bound for the right hand side of (2.5.9), using the fact that $\alpha < 1$ as follows :

$$G_{k+1} \le \prod_{j=1}^{k}\alpha(G_j)G_j + \sum_{m=1}^{k-1}\prod_{J=m+1}^{k}\alpha(G_j)\alpha^{N_m}(G_m) + \alpha^{N_k}(G_k)$$

$$< G_1h^k + \sum_{m=1}^{k-1}h^{k-m}h^{N_m} + h^{N_k} = G_1h^k + h^k\sum_{m=1}^{k-1}h^{N_m-m} + h^{N_k} \tag{2.4.10}$$

$\le C_1h^k + C_2h^k + C_3h^k = C_4 h^k$, where $C_4 = C_1 + C_2 + C_3$ and C_1, C_2, C_3, C_4 are constants.

Now, for $k \ge k_0$, and $p \in N$, we have by using (2.4.10) and the repeated application of the rectangle inequality that

$$G(Sx_k, Sx_k, Sx_{k+p}) \le s[G(Sx_k, Sx_k, Sx_{k+1}) + G(Sx_{k+1}, Sx_{k+1}, Sx_{k+2}) + \ldots + G(Sx_{k+p-1}, Sx_{k+p-1}, Sx_{k+p})]$$

$$= s[G_{k+1} + G_{k+2} + \ldots + G_{k+p}]$$

$$\le s[C_4(h^k + h^{k+1} + \ldots + h^{k+p-1})]$$

$$= C_4\left(\frac{1-h^p}{1-h}\right)h^k s = C_5h^k s, \tag{2.4.11}$$

where C_5 is a constant

Since $0< h < 1$, the right hand side of (2.4.11) tends to 0 as $k\to\infty$, showing that $\{Sx_k\}$ is a Cauchy sequence. Since X is complete generalized b-metric space there exist a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converging to some $u\in X$. Therefore, by the continuity of S , $\{Sx_k\}_{k=1}^{\infty}$ converges to some $Su\in X$. , that is

$$\lim_{k\to\infty} Sx_k = Su=w. \tag{2.4.12}$$

So

$$\begin{aligned} D_G(Su, Su, Tu)= D_G(w, w, Tu) &\le s[G(w, w, Sx_k) + G(Sx_k, Sx_k, Tu)] \\ &\le s[G(w, w, Sx_k) + H_G(Tx_{k-1}, Tx_{k-1}, Tu)] \\ &\le s\,G(w, w, Sx_k) + s\alpha(G(Sx_{k-1}, Sx_{k-1}, Su))\,G(Sx_{k-1}, Sx_{k-1}, Su) \\ &\qquad + s\phi_2(D_G(u, Tx_{k-1}, Tx_{k-1})) \qquad (2.4.13) \\ &< s\,G(w, w, Sx_k) + sh\,G(Sx_{k-1}, Sx_{k-1}, Su) + s\phi(D_G(Su, Tx_{k-1}, Tx_{k-1}))\ , s \ge 1. \end{aligned}$$

By using(2.4.12) and the fact that $Sx_k \in Tx_{k-1}$ we have $D_G(Su, Tx_{k-1}, Tx_{k-1}) \to 0$ as $k\to\infty$. We therefore, have by the continuity of ϕ that $\phi(D_G(Su, Tx_{k-1}, Tx_{k-1})) \to 0$ as $k\to\infty$. Hence, since the right hand side terms of (2.5.13) tends to zero as $k\to\infty$, we have $D_G(Su, Su,Tu)= 0$. Since Tu is closed ,then $Su\in Tu$, $u\in C(S,T)$. Using (2.4.12) and the continuity of the generalized b-metric in (2.4.11) as $p\to\infty$, we obtain an error estimate

$$G(Sx_k, Sx_k,Su) = \lim_{p\to\infty} G(Sx_k, Sx_k, Sx_{k+p}) \le C_5 h^k s,\ k \ge k_0 ,s \ge 1$$ for the Jungck iteration process under condition (****).

Remark 2.4: Theorem 2.4 is a generalization of theorem 2.2.

REFERENCES

[1] Agarwal R.P., Meehan M., O'Regan D., Fixed point theory and applications, Cambridge University Press, Cambridge, 2001.

[2] B. C. Dhage, Generalized metric space and mapping with fixed point, Bulletin of the Calcutta Mathematical Society, vol. 84, pp. 329–336, 1992.

[3] Bakhtin I. A., The contraction mapping principle in almost metric spaces, Funct. Anal.,Gos. Ped. Inst. Unianowsk, 1989, 30, 26–37.

[4] Berinde V., Generalized contractions in quasimetric spaces, Seminar on Fixed Point Theory, 1993, 3–9

[5] Berinde V., A priori and a posteriori error estimates for a class of ϕ-contractions, Bulletins for Applied & Computing Math., 1999, 183–192

[6] Berinde V., Iterative approximation of fixed points, Editura Efemeride, Baia Mare, 2002

[7] Berinde V., On the approximation of fixed points of weak- contractive operators, Fixed Point Theory, 2003, 4, 131–142

[8] Berinde V., On the approximation of fixed points of weak contractive mappings, Carpathian J. Math., 2003, 19, 7–22

[9] Berinde V., Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum, 2004,9, 43–53

[10] Berinde M., Berinde V., On a general class of multi-valued weakly Picard mappings, J. Math. Anal. Appl., 2007, 326,772–782

[11] Browder F., Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A., 1965, 54, 1041–1044

[12] Czerwik S., Contraction mappings in *b*−metric spaces, Acta Math. Inform. Univ. Ostraviensis, 1993, 1, 5–11

[13] Czerwik S.,Nonlinear set-valued contraction mappings in b−metric spaces, Atti Sem. Mat. Fis. Univ. Modena, 1998, 46, 263–276

[14] Ciric L.B., Fixed point theory, contraction mapping principle, FME Press, Beograd, 2003

[15] Ciric L.B., Ume J.S.,On the convergence of Ishikawa iterates to a common fixed point of multi-valued mappings, Demonstratio Math., 2003, 36, 951–956

[16] Daffer P.Z., Kaneko H., Fixed points of generalized contractive multi-valued mappings, J. Math. Anal. Appl., 1995,192, 655–666.

[17] Itoh S., Multivalued generalized contractions and fixed point theorems, Comment. Math. Univ. Carolinae, 1977, 18, 247–258.

[18] Joshi M.C., Bose R.K., Some topics in nonlinear functional analysis, John Wiley & Sons, Inc., New York, 1985

[19] Kaneko H., Generalized contractive multivalued mappingsand their fixed points, Math. Japon., 1988, 33, 57–64

[20] Khamsi M.A., Kirk W.A., An introduction to metric spaces and fixed point theory, Wiley-Interscience, New York, 2001

[21] Markins J.T., A fixed point theorem for set-valued mappings,Bull. Amer. Math. Soc., 1968, 74, 639–640

[22] Mizoguchi M., Takahashi W., Fixed point theorems for multi-valued mappings on complete metric spaces, J. Math. Anal. Appl., 1989, 141, 177–188

[23] Nadler S.B., Multi-valued contraction mappings, Pacific J.Math., 1969, 30, 475–488

[24] Olatinwo M. O. , A generalization of some results on multi-valued weakly Picard mappings in $b-$metric space, Fasciculi Mathematici, 2008.

[25] Olatinwo M.O., Some results on multi-valued weakly Jungck mappings in b-metric space, Cent. Eur. J. Math , 6(4) , 2008, 610- 621

[26] Prasad , B. , Singh, B., Sahni, R. , Some Approximate fixed point theorems ,int. Journal of Math. Analysis, Vol. 3, 2009, no. 5, 203- 210.

[27] Rhoades B.E., A fixed point theorem for a multivalued non- self mapping, Comment. Math. Univ. Carolin. , 37 (1996), 401- 404

[28] Rhoades B.E., Watson B., Fixed points for set-valued mappings on metric spaces, Math. Japon., 1990, 35, 735–743.

[29] Rus I.A., Fixed point theorems for multi-valued mappings in complete metric spaces, Math. Japon., 1975, 20, 21–24

[30] Rus I.A., Generalized contractions and applications, Cluj University Press, Cluj Napoca, 2001

[31] S. G¨ahler, “2-metrische R¨aume und ihre topologische Struktur,” Mathematische Nachrichten, vol. 26, pp. 115–148, 1963.

[32] Singh S.L., Bhatnagar C., Mishra S.N., Stability of iterative procedures for multivalued maps in metric spaces,

[33] Z. Mustafa and B. Sims, “Some remarks concerning D-metric spaces,” in Proceedings of the International Conference on Fixed Point Theory and Applications, pp. 189–198, Valencia, Spain, July 2003.

[34] Z. Mustafa, H. Obiedat, and F. Awawdeh, “Some fixed point theorem for mapping on complete G-metric spaces,” Fixed Point Theory and Applications, vol. 2008, Article ID 189870, 12, pages, 2008.

[35] Z. Mustafa, and B. Sims ,A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, vol.7, no 2, 289-297, 2006.

[36] Zeidler E., Nonlinear functional analysis and its Applications to fixed point theorems, Springer-Verlag, New York, Inc., 1986.

