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STEINBERG MODULES IN QUANTUM GROUPS

P. B. SARASIJA*

Department of Mathematics, Noorul Islam Centre for Higher Education, Kumaracoil, Tamil Nadu, India
E-mail: sijavk@gmail.com

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ABSTRACT

In this paper the Verma modules $M_\varepsilon(\lambda)$ over the quantum group $U_\varepsilon(sl(n+1), C)$, where ε is a primitive lth root of 1 and the socle of $M_\varepsilon(\lambda)$ is non-zero are studied. Using this concept we obtained the Steinberg module in Quantum Groups.

0. INTRODUCTION

Let $U_q(g)$ be the Drinfel'd – Jimbo quantum group associated to a symmetrizable Kac- Moody algebra g. Thus $U_q(g)$ is a Hopf Algebra the field C(q) of rational functions of an indeterminate q and is defined by certain generators and relations.

First constructing $A = C[q,q^{-1}]$ form $U_q(g)$, i.e., A – subalgebra $U_A(g)$ of $U_q(g)$ such that $U_q(g) = U_A(g)\otimes C(q)$. Then define $U_\varepsilon(g) = U_A(g)\,)\otimes_A C$, via the algebra homomorphism A to C that takes q to ε, $\varepsilon^2 \neq 1$.

In the non-restricted form one takes $U_A(g)$ to be the A sub algebra of $U_q(g)$ generated by the Chevalley generators E_i, F_i, K_i of $U_q(g)$. The finite dimensional representation of the non-restricted $U_\varepsilon(g)$ have been studied by De Concini and Kac in [1].

In [1], De Concini and Kac defined the notion of Verma modules over U_q and U_ε(where ε is a primitive l^{th} root of 1, l is an odd integer) analogous to the classical Verma modules.

In [2] the Verma module $M_\varepsilon(\lambda)$ $U_\varepsilon(g)$, where g = sl(n + 1), and in particular prove that the socle of $M_\varepsilon(\lambda)$ over U_ε is nonzero. In this paper we obtain the Steinberg modules in quantum groups.

1. PRELIMINARIES

1.1. Let us fix some notations which are standard (see for example, [1]).

For a fixed n ∈ N, let (a_{ij}) $1 \le i, j \le n$ be the cartan matrix of type A_n.

Let q be an indeterminate and let $A = C[q,q^{-1}]$ with the quotient field C(q).

For any integer $M \geq 0$, we define

$$[M] = \frac{q^M - q^{-M}}{q - q^{-1}} \in A,\quad [M]! = [M]\,[M-1]\ldots[1],\ \text{and}\ \begin{bmatrix} M \\ j \end{bmatrix} = \frac{[M]!}{[j]![M-j]!}\ \text{and for } j \in N,\ \begin{bmatrix} M \\ 0 \end{bmatrix} = 1.$$

Let U_q be the C (q) algebra with 1, defined by the generators E_i, F_i, $K_i^{\pm 1}$ $(1 \le i \le n)$ with the relations:

(a) $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_iK_j = K_jK_i$,

(b) $K_iE_j K_i^{-1} = q^{a_{ij}}E_j$, $K_iF_j K_i^{-1} = q^{-a_{ij}}F_j$,

(c) $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}$,

***Corresponding author: P. B. SARASIJA*, *E-mail: sijavk@gmail.com**

(d) $E_iE_j = E_jE_i$, if $a_{ij} = 0$,
(e) $E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0$, if $a_{ij} = -1$,
(f) $F_iF_j = F_jF_i$ if $a_{ij} = 0$,
(g) $F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0$, if $a_{ij} = -1$,

Then U_q is a Hopf algebra over C(*q)* which is called the quantum group associated to the matrix(a_{ij}), with comultiplication Δ, antipode S and counit *v* defined by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_i = K_i \otimes K_i,$$
$$SE_i = -K_i^{-1}E_i,\ SF_i = -F_iK_i,\ SK_i = K_i^{-1}$$
$$vE_i = 0,\ vF_i = 0,\ vK_i = 1.$$

Also introduce the elements

$$[K_i; n] = \frac{(K_iq^n - K_i^{-1}q^{-n})}{q - q^{-1}} \text{ in } U_q.$$

1.2 It is well known that one can introduce a root system associated to the matrix (a_{ij}). We briefly describe the construction here. For details refer to [1, 5].

Let P be a free abelian group with basis ω_i , i = 1, 2, ...,*n* (P *is* usually called the lattice of weights). Let P^+ denote the subgroup of non-negative integral combinations of ω_1, ω_2 , …, ω_n and any element of P^+ is called a dominant weight. Define the following elements in P:

$$\rho = \sum_{i=1}^{n} \omega_i\ , \quad \alpha_j = \sum_{i=1}^{n} a_{ij}\omega_i \ (j = 1,\dots,n)$$

let $$Q = \sum_i Z\alpha_i\ , \qquad Q_+ = \sum_i Z_+\alpha_i\ .$$

Define a bilinear pairing $P \times Q \to Z$ by

(1.2.1) $$(\omega_i \mid \alpha_j) = \delta_{ij}\ .$$

Then $(\alpha_i \mid \alpha_j) = a_{ij}$, so that we get a symmetric Z-valued bilinear form on Q such that $(\alpha \mid \alpha\ ) \in 2Z.$

Define automorphisms r_i of *P* by ri $\omega_j = \omega_j — \delta_{ij}\alpha_i$ (i,j = 1, 2, . . . , n).

Then $r_i\alpha_j = \alpha_j — a_{ij}\alpha_i$. Let *W* be the (finite) subgroup of *GL(P)* generated by $r_1, r_2, \dots, r_n$. Then *Q* is W-invariant and the pairing $P \times Q \to Z$ is W-invariant.

Let II = {$\alpha_1, \alpha_2, \dots, \alpha_n$}, R = W∏ and denote $R \cap Q_+$ by R^+. Then R is a root system corresponding to the cartan matrix (a_{ij}) with Weyl group W and R^+ the system of positive roots. Clearly ρ is half the sum of positive roots. We introduce a partial ordering of P by $\lambda \geq \mu$

if $\lambda - \mu \in Q_+$. Let w_0 be the unique element of W such that $w_0 (R^+) = - R^+$.

1.3. Let U_A be the A-subalgebra of U_q generated by the elements E_i, F_i, $K_i^{\pm 1}$, [Ki;0] (i = 1,2, . . . , n). Let U_A^+ (respectively U_A^-) be the A-subalgebra of U_A generated by the E_i (respectively Fi*)* and U_A^0 the subalgebra generated by the K_i and [Ki;0].

1.4 We shall show how to choose a canonical basis for U_q from the given set of generators (for details see [1, 5, 6]).

We note that we can define an anti-automorphism ω of U_q defined by

(1.4.1) $$\omega E_i = F_i, \quad \omega F_i = E_i, \quad \omega K_i = K_i^{-1}, \quad \omega q = q^{-1}.$$

For any i, $1 \le i \le n$, there is a unique algebra automorphism T_i of U_q such that

(1.4.2) $T_iE_i = -F_iK_i$, $T_iE_j = -E_jE_i + q^{-1}E_iE_j$ if $a_{ij} = -1$ and $T_i(E_j) = E_j$ if $a_{ij} = 0$

(1.4.3) $T_iF_i = -K_i^{-1}E_i$, $T_iF_j = -F_jF_i + qF_iF_j$ if $a_{ij} = -1$ and $T_i(F_j) = F_j$ if $a_{ij} = 0$

(1.4.4) $T_i K_j = K_jK_i^{-aij}$ $T_i \omega = \omega T_i$.

Let $w \in W$ and let $r_{i_1}, \ldots r_{i_k}$ be a reduced expression of w. Then the automorphism $T_w = Ti_1 \ldots Ti_k$ of U_q is independent of the choice of the reduced expression of w.

Fix a reduced expression $ri_1, ri_2, \ldots, ri_N$ of the longest element of W, where $N = |R^+|$. Then this gives us an enumeration of the elements of R^+

$$\beta_1 = \alpha i_1, \beta_2 = r_{i_1} \alpha i_2, \ldots, \beta_N = r_{i_1}, \ldots r_{i_{N-1}} \alpha i_N.$$

We define the roots vectors:

$$E_{\beta s} = T_{i1} T_{i2} \ldots T_{is-1} E_{is}, \quad F_{\beta s} = T_{i1} T_{i2} \ldots T_{is-1} F_{is} \quad \text{which is the same as } \omega E_{\beta s}.$$

For $j = (j_1, j_2, \ldots, j_N) \in Z_+^N$ let he elements $F^j K_j^{m_1} \ldots K_n^{m_n} E^r$ where $j, r \in Z_+^N$, $(m_1, \ldots, m_n) \in Z^n$ form a basis of U_q over C (q).

1.5 Given $\varepsilon \in C^*$, we now consider the specialization $U_\varepsilon = U_A / [(q-\varepsilon)U_A]$.We take ε in such way that $\varepsilon^2 \neq 1$.

Then U_ε is an algebra over C with generators E_i, F_i, $K_i^{\pm 1}$ $(1 \le i \le n)$ (identifying these vectors with their images), and defining relations ,

(a′) $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_iK_j = K_jK_i$,

(b′) $K_iE_j K_i^{-1} = \varepsilon^{aij}E_j$, $K_iF_j K_i^{-1} = \varepsilon^{-aij}F_j$,

(c′) $E_iF_j - F_jE_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{\varepsilon - \varepsilon^{-1}}$,

(d′) $E_i^2E_j - (\varepsilon + \varepsilon^{-1})E_iE_jE_i + E_jE_i^2 = 0$, if $a_{ij} = -1$

(e′) $F_i^2F_j - (\varepsilon + \varepsilon^{-1})F_iF_jF_i + F_jF_i^2 = 0$, if $a_{ij} = -1$,

(f′) $E_iE_j = E_jE_i$, if $a_{ij} = 0$, $F_iF_j = F_jF_i$ if $a_{ij} = 0$.

1.6 We denote by U_ε^+, U_ε^-, U_ε^0 the images of U_A^+, U_A^-, and U_A^0 in U_ε. The automorphism T_i of U_q defined in (1.4) clearly induces an automorphism T_i of U_ε. The vectors E^j, F^j of U_q defined in (1.4.5) can then be taken to represent their images in U_ε. Then the elements E^j, $j \in Z_+^N$ form a basis of U_ε^+ over C, and the elements $F^j K_j^{m_1} \ldots K_n^{m_n} E^r$ where $j, r \in Z_+^N$, $(m_1, \ldots, m_n) \in Z^n$ form a basis of U_ε over C .

2. VERMA MODULES

2.1. The notion of Verma modules over U_q and U_ε was introduced by De Concini and Kac in [1, 6]. In the rest of the paper, we shall be concerned only with Verma modules over U_ε , where ε is a primitive lth root of unity.

We recapitulate the definition below:

For each $\lambda \in P$ the Verma module $M_\varepsilon(\lambda)$ over U_ε is the vector space $M_\varepsilon(\lambda)$ in which there exists a non-zero distinguished vector v_λ such that $U_\varepsilon^+ v_\lambda = 0$, $K v_\lambda = \varepsilon^{(\lambda|\alpha)} v_\lambda$, $K \in U_\varepsilon^0$ where (|) is the pairing from $P \times W \to Z$ defined in (1.2) and $\{ F^j v_\lambda (j \in Z_+^N) \}$ is a basis of $M_\varepsilon(\lambda)$. Let $L_\varepsilon(\lambda)$ denote the unique irreducible quotient of $M_\varepsilon(\lambda)$ by its unique maximal submodule.

Then we have

(2.1.1) $K v_\lambda = \varepsilon^{(\lambda|\alpha)} v_\lambda$.

Also for each h = 1, 2, . . . , N , $F_h v_\lambda$ is a weight vector of weight $\lambda - \alpha_h$ as easily seen below.

$$K F_h v_\lambda = \varepsilon^{-(\alpha|\alpha_h)} F_h K v_\lambda$$
$$= \varepsilon^{-(\alpha|\alpha_h)} \varepsilon^{(\lambda|\alpha)} F_h v_\lambda \quad (\text{since } (\alpha_h|\alpha) = (\alpha|\alpha_h))$$
$$= \varepsilon^{-(\lambda-\alpha_h|\alpha)} F_h v_\lambda.$$

(2.1.2) This shows that for any $r \in Z_+$, $F_h^r v_\lambda$ is a weight vector of weight $\lambda - r\alpha_h$ and therefore each $F^j v_\lambda$ (= $F_i^{j_1} \ldots F_N^{j_N} v_\lambda$) is a weight vector of weight $\lambda - \sum_{h=1}^{N} j_h \alpha_h$.

2 .2 VERMA MODULES OVER SOME SUBALGEBRAS OF U_E.

We first define the subalgebras U_r , U_r^+ , U_r^- , of U_ε generated b

$$\left\{F^j, \prod_{i=1}^{n} K_i^{m_i}, E^r, 0 < j_i, r_i < l^r, (m_1 \ldots m_n) \in Z^n\right\}, \left\{E^r, \prod_{i=1}^{n} K_i^{m_i}, 0 < r_i < l^r, (m_1 \ldots m_n) \in Z^n\right\}, \left\{F^j, 0 \le j_i < l^r\right\}$$ respectively.

The set

(2.2.1) $\{F_1^{j_1} \ldots F_N^{j_N} K_1^{m_1} \ldots K_n^{m_n} E_1^{r_1} \ldots E_N^{r_N},\ 0 \le j_i, r_i, < l^r, (m_1, \ldots, m_n) \in Z^n\}$ is a basis of U_r and the set

(2.2.1) $\{F_1^{j_1} \ldots F_N^{j_N}, 0 \le j_i < l^r\}$ is a basis of U_r^-.

We can then define the Verma modules $M_{\varepsilon,r}(\lambda)$ of weight λ over U_r analogously to $M_\varepsilon(\lambda)$ over $U\varepsilon$, that is, there exists a non-zero vector (say) $\hat{v}_\lambda$ such that $U_r^+ \hat{v}_\lambda = 0$, $K\hat{v}_\lambda = \varepsilon^{(\lambda|\alpha)} \hat{v}_\lambda$ for $K \in U_r^0$ and $\{F^j \hat{v}_\lambda, 0 \le j_i < l^r\}$ form a basis of $M_{\varepsilon,r}(\lambda)$.

There is a natural injective homomorphism f_r: $M_{\varepsilon,r}(\lambda) \longrightarrow M_\varepsilon(\lambda)$ given by

(2.2.3) $\quad f_r(F^j \hat{v}_\lambda) = F^j v_\lambda$

2.3 We next introduce certain elements defined by I_r of U_r^- , which play an important role in our future study of the socles of Verma modules and homomorphisms between Verma modules.

For each positive integer r, let $I_r = F_1^{l^r-1} \ldots F_N^{l^{r-1}}$ which is an element of U_r^-.It then follows that $I_r v_\lambda$ is a weight vector of $U_r v_\lambda$ of weight $\lambda - 2(l\text{-}1)\rho$, where ρ is half the sum of the positive roots. In fact,

(2.3.1)
$$K I_r v_\lambda = K F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda$$
$$= \varepsilon^{(\lambda-(l^r-1)\alpha_1+\ldots+\alpha_N|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda \quad \text{from (2.1.2)}$$
$$= \varepsilon^{(\lambda-2(l^r-1)\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda$$
$$= \varepsilon^{(\lambda+2\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda \qquad [\text{since } \varepsilon^{l^r} = 1]$$
$$= \varepsilon^{(\lambda-2l\rho+2\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda$$
$$= \varepsilon^{(\lambda-2(l-1)\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda.$$

In particular, when $\lambda = 0$, we see that $I_r \hat{v}_0$ *is* a weight vector of $M_{\varepsilon,r}(0)$with minimal weight $-2(l-1)\rho$.

We observe for later use that I_r is an integral of U_r^-. In fact, for $\alpha \in R^+$ and a $\in$ N such that 0 < a < l^r, R_α^a I_r and $I_r F_\alpha^a$ are in U_r^-. Hence $F_\alpha^a I_r \hat{v}_0$ and $I_r F_\alpha^a \hat{v}_0$ are weight vectors of $M_{\varepsilon,r}(0)$with weight -2(l -1)ρ-aα. By the minimahty of the weight -2(l -1)ρ, it follows that $F_\alpha^a I_r = I_r F_\alpha^a = 0$. This shows that I_r is an integral of U_r^- , in other works $uI_r = V(u)I_r$ for all u $\in$ U_r^- , where V: U_r^- —> C is the augmentation function.

2.4 A HOMOMORPHISM BETWEEN TWO VERMA MODULES

$M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$ is a map ϕ: $M_\varepsilon(\lambda)$—>$M_\varepsilon(\mu)$such that ϕ is a vector space homomorphism and ϕ(uu) = uϕ(v), u $\in$ U_ε, v$\in$ $M_\varepsilon(\lambda)$.

Lemma 2.4.1: If $M_\varepsilon(\lambda)$,$M_\varepsilon(\mu)$ are Verma modules over the quantum grou U_ε, and there is an injective U_ε module homomorphism ϕ: $M_\varepsilon(\lambda)$—>$M_\varepsilon(\mu)$, then λ = μ and ϕ is multipli'cation by some element of C.

Proof: Let v_λ, v_μ be non-zero highest weight vectors of $M_\varepsilon(\lambda)$,$M_\varepsilon(\mu)$respectively. Since v_λ generates $M_\varepsilon(\lambda)$, ψ is determined by $\psi(v_\lambda)$. Say $\psi(v_\lambda) = uv_\mu$, u $\in$ U_ε^- . Now by definition, U_ε^- is the union of the subalgebras U_r^- for r = 1, 2,.. and so there is some r for which u $\in$ U_r^-. Since I_r is an integral for U_r^- ,

$$V(u)\, I_r v_\mu = I_r u v_\mu = I_r\, \psi(v_\lambda) = \phi(I_r v_\lambda)$$

where V : U_r^- — > C is the augmentation function and $I_r v_\lambda$ is an element of the basis for $M_\varepsilon(\lambda)$, so is non-zero, and therefore $V(u) \neq 0$. But $\psi(v_\lambda)$ must have weight λ, so uv_μ has weight λ, which contradicts $V(u) \neq 0$ unless λ = μ.

Since v_μ spans the μ-weight space of $M_\varepsilon(\mu)$,$\psi(v_\lambda) = cv_\mu = cv_\lambda$ for some c $\in$C, and ϕ is just multiplication by c.

3. SOCLE OF VERMA MODULES

Denote the socle of the U_ε module $M_\varepsilon(\lambda)$ by Soc($M_\varepsilon(\lambda)$) and the socle of the U_r module $M_{\varepsilon,r}(\lambda)$ by Soc($M_{\varepsilon,r}(\lambda)$).[3].

Since for any r > 0, $M_{\varepsilon,r}(\lambda)$ is finite dimensional, clearly Soc($M_{\varepsilon,r}(\lambda)$) ≠ 0. It is interesting to note that even for the infinite dimensional module $M_\varepsilon(\lambda)$, its socle is non-zero. We proceed to prove this in this section.

Lemma 3.1: If 0 ≠ u $\in$ U_r^- for some r $\in$N, then U_r u contains CI_r .

Proof: We shall order the positive roots α(l), α(2)... α(N) in such a way that if

α(i) + α(j) = α(k) then k < i,j.

If 0 < a < l^r then clearly

$$F_{\alpha(1)}^{l^r-1} F_{\alpha(1)}^{a} = F_{\alpha(1)}^{l^r-1+a} = 0.$$

We shall prove by induction on i, with 1 ≤ i ≤ N, that $F_{\alpha(1)}^{l^r-1} \ldots F_{\alpha(i)}^{l^r-1} F_\alpha^a = 0$

whenever α $\in$ {α(1), …, α(i)} and 0 < a < l^r.

Suppose there exists some i, 2 ≤ i ≤ N, such that

(3.1.1) $F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} F_\alpha^a$ = 0, whenever α $\in$ {α(1)… α(i-1)} and 0 < a < l^r.

Now, suppose that there is some α $\in$ {α(1),α(2), …, α(i)}and choose a such that 0 < a < l^r.

If α = α(i), then $F_{\alpha(i)}^{l^r-1} F_\alpha^a$ = 0, and so

$$F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i)}^{l^r-1} F_\alpha^a = 0.$$

If $\alpha \neq \alpha(i)$, then using the commutation relations imply that

$$F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i)}^{l^r-1} F_{\alpha}^{a}$$

is a sum of elements of the form

$$F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} F_{\beta}^{b}\, u$$

with $\beta \in \{\alpha(1), \ldots, \alpha(i-1)\} 0 < b < l^r$, $u \in U_\varepsilon$ and each element of this form equals 0 by (3.1.1). So (3.1.1) holds for all i.

Using this equation together with the commutation relations if $1 \leq i \leq N$ and $0 < a < l^r$, then

(3.1.2) $$F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} F_{\alpha(i)}^{a} - \varepsilon^{-1(i-1)(l^r-1)} F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} = 0$$

and so if $1 \leq i \leq N$ and $0 < a, b < l^r$ then

$$F_{\alpha(i)}^{a} F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} F_{\alpha(i)}^{b}$$

$$= \varepsilon^{-1(i-1)(l^r-1)} F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} F_{\alpha(i)}^{a+b}$$

$= 0$ if $a + b \geq l^r$.

Suppose u is a non-zero element of U_r^-. Then by the basis of U_r^- the element u *is* of the form $F_{\alpha(1)}^{\alpha(1)} F_{\alpha(2)}^{\alpha(2)} \ldots F_{\alpha(N)}^{\alpha(N)}$ with $0 \leq a(1), \ldots, a(N) < l^r$

By repeated use of (3.1.2) $CF_{\alpha(N)}^{l^r-1-\alpha(N)} \ldots F_{\alpha(1)}^{l^r-1-\alpha(1)} u = CF_{\alpha(N)}^{l^r-1} \ldots F_{\alpha(1)}^{l^r-1} = C\, I_r$ as required.

Corollary 3.2: Let r be a positive integer.

$$I_{r+1} \in U_\varepsilon I_r.$$

Proof: Lemma 3.1 implies that $CI_{r+1} \subseteq U_{r+1} I_r$, so $I_{r+1} \in U_{r+1} I_r \subseteq U_\varepsilon I_r.$

Corollary 3.3:

(i) If M is a non-zero U_r submodule of $M_{\varepsilon,r}(\lambda)$ and $\widehat{v}_\lambda \in M_{\varepsilon,r}(\lambda)$, then $I_r \widehat{v}_\lambda \in M$.

(ii) If M is a non-zero U_ε submodule *of* $M_\varepsilon(\lambda)$ and $v_\lambda \in M_\varepsilon(\lambda)$, then $I_r v_\lambda \in M$ for all r.

Proof:

(i) By the basis of $M_{\varepsilon,r}(\lambda)$, M contains some vector $u\widehat{v}_\lambda$ with $u \in U_r^-$. By Lemma 3.1, $I_r \widehat{v}_\lambda \in CI_r \widehat{v}_\lambda \subseteq U_r u \widehat{v}_\lambda \subseteq M$.

(ii) By the basis of $M_\varepsilon(\lambda)$, M contains some vector uv_λ with $u \in U_\varepsilon^-$, hence $u \in U_r^-$ for some r.
By Lemma 3.1, $I_r v_\lambda \in CI_r v_\lambda \subseteq U_\varepsilon u v_\lambda \subseteq M$.

Corollary 3.4: $Soc(M_{\varepsilon,r}(\lambda))$ is simple.

Proof: $Soc(M_{\varepsilon,r}(\lambda))$ is a non-zero U_r submodule of $M_{\varepsilon,r}(\lambda)$ and by Corollary 3.3 (i) the submodule $U_r I_r \widehat{v}_\lambda$ is contained in every simple component of $Soc(M_{\varepsilon,r}(\lambda))$ and hence $Soc(M_{\varepsilon,r}(\lambda))$ itself is simple.

Lemma 3.5: Let $\lambda \in P^+$, the set of dominant weights. Then for all $r > 0$, the highest weight of Soc $(M_{\varepsilon,r}(\lambda))$ is $w_0(\lambda - 2(1 - l)\rho)$ and hence is independent of r.

Proof: From (2.3.1), the lowest weight of $M_{\varepsilon,r}(\lambda)$ is $\lambda - 2(l - l)\rho$ for all $r > 0$. From Corollary 3.3(i), we have seen that any non-zero submodule of $M_{\varepsilon,r}(\lambda)$ contains $I_r \widehat{v}_\lambda$. Hence Soc $(M_{\varepsilon,r}(\lambda))$ contains $I_r \widehat{v}_\lambda$ whose weight is $\lambda - 2(l-1)\rho$.

Therefore the lowest weight of $Soc(M_{\varepsilon,r}(\lambda))$ is $\lambda - 2(l-1)\rho$ for all $r > 0$ and hence the highest weight of $Soc(M_{\varepsilon,r}(\lambda))$ is $w_0(\lambda - 2(1 - l)\rho) = w_0(\lambda + 2\rho)$, which is independent of r. Hence the result.

Corollary 3.6: $Soc(M_{\varepsilon,r}(\lambda))$ is isomorphics to $L_{\varepsilon,r}(w_0(\lambda - 2(l - 1)\rho)$ for all $r > 0$.

Proof: From the corollary 3. 4 and the lemma 3.5 we get $Soc(M_{\varepsilon,r}(\lambda))$ is simple and the highest weight is $w_0(\lambda - 2(l - 1)\rho)$. But $\lambda-2(l-1)\rho$ is a weight of $L_{\varepsilon,r}(w_0(\lambda - 2(l - 1)\rho)$ and hence this simple U_r module is isomorphic to the U_r module $Soc(M_{\varepsilon,r}(\lambda))$, for all $r > 0$.

We shall proceed to prove our main result concerning the socle of the Verma modules.

Theorem 3.7: Soc $(M_\varepsilon(\lambda))$ is non-zero for all $\lambda \in P^+$.

Proof: Let v_λ, $\hat{v}_\lambda$ be non-zero highest weight vectors of the Verma module $M_\varepsilon(\lambda)$ over U_ε, and $M_{\varepsilon,r}(\lambda)$ over U_r respectively. Let M be an arbitrary non-zero U_ε submodule of $M_\varepsilon(\lambda)$. Then by Corollary 3.3(ii), $I_r\hat{v}_\lambda \in U_r u\hat{v}_\lambda \subseteq M$ for all r and hence $U_\varepsilon I_r v_\lambda \subseteq M$. Now, let I denote the submodule $\bigcap_{r>0} U_\varepsilon I_r v_\lambda$ of $M_\varepsilon(\lambda)$.

Replacing M by each simple component of $Soc(M_\varepsilon(\lambda))$, it immediately follows that $Soc(M_\varepsilon(\lambda)) \supseteq I$.

We proceed to prove that $I \neq (0)$. Since $M_{\varepsilon,r}(\lambda)$ is finite dimensional, $Soc(M_{\varepsilon,r}(\lambda)) \neq 0$. By Corollary 3.3(i), $Soc(M_{\varepsilon,r}(\lambda))$ is simple and we can take $Soc(M_{\varepsilon,r}(\lambda))$ to be isomorphic to the simple U_r module $L_{\varepsilon,r}(\mu)$ (where μ is $w_0(\lambda - 2(l - 1)\rho)$). Also by Corollary 3.3(i), $Soc(M_{\varepsilon,r}(\lambda))$ contains $I_r\hat{v}_\lambda$. Therefore there is some x_r in U_r such that $x_r I_r \hat{v}_\lambda$ is in the highest weight space of $Soc(M_{\varepsilon,r}(\lambda))$.

In other words, $x_r I_r \hat{v}_\lambda \in (M_{\varepsilon,r}(\lambda))^\mu$, the μth weight space of $M_{\varepsilon,r}(\lambda)$. Now let f_r be the injective U_r module homomorphism from $M_{\varepsilon,r}(\lambda)$ to $M_\varepsilon(\lambda)$ described in (3.2.3), then $f_r(\hat{v}_\lambda) = v_\lambda$.
So, $x_r I_r v_\lambda = f_r(xrI_r \hat{v}_\lambda) \in (M_\varepsilon(\lambda))^\mu$.

This shows that for each r, $U_\varepsilon I_r v_\lambda \cap (M_\varepsilon(\lambda))^\mu \neq (0)$ and is a finite dimensional C-vector space (since $(M_\varepsilon(\lambda))^\mu$ is finite dimensional).

From Corollary (3.2), we have the descending chain of submodules

$$U_\varepsilon I_1 v_\lambda \cap (M_\varepsilon(\lambda))^\mu \supseteq U_\varepsilon I_2 v_\lambda \cap (M_\varepsilon(\lambda))^\mu \supseteq \dots$$

Hence its intersection which is just $I \cap M_\varepsilon(\lambda)^\mu$ is non-zero which implies that $I \neq 0$. Since $Soc(M_\varepsilon(\lambda)) \supseteq I \neq 0$, it follows that $Soc(M_\varepsilon(\lambda)) \neq 0$.

Hence the theorem.

Theorem 3.8: Soc $(M_\varepsilon(\lambda))$ is simple and isomorphic to the simple U_ε- module $L_\varepsilon(w_0(\lambda - 2(l - 1)\rho)) = L_\varepsilon(w_0(\lambda + 2\rho))$.

Proof: From the above theorem we get Soc ($M_\varepsilon(\lambda)$) is a non zero U_ε- module of $M_\varepsilon(\lambda)$ and by the corollary (3.3.) (ii) the submodule $U_\varepsilon I_r v_\lambda$ is contained in every simple component of $Soc(M_\varepsilon(\lambda))$ and hence $Soc(M_\varepsilon(\lambda))$ itself is simple.

Since $Soc(M_\varepsilon(\lambda))$ contains $I_r v_\lambda$ whose weight is $\lambda-2(l - 1)\rho$, the lowest weight of $Soc(M_\varepsilon(\lambda))$ is $\lambda-2(l-1)\rho$ and the highest weight of $Soc(M_\varepsilon(\lambda))$ is $w_0(\lambda - 2(l - 1)\rho)$.

But $\lambda-2(l - 1)\rho$ is a weight of $L_\varepsilon(w_0(\lambda - 2(l - 1)\rho) = L_\varepsilon(w_0(\lambda + 2\rho))$ and hence this simple U_ε- module is isomorphic to the $U_\varepsilon$module socle of $M_\varepsilon(\lambda)$.

4. STEINBERG MODULE IN QUANTUM GROUPS

One can naturally expect to define a Steinberg module in Quantum groups. [6]

We let $M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$, $M_{\varepsilon,r}(\lambda)$ to denote the Verma modules over U_ε and $L_\varepsilon(\lambda)$, $L_{\varepsilon,r}(\lambda)$ the corresponding (unique) simple factor modules. From the corollary 3.6 we get

(4.1.1) $\quad Soc(M_{\varepsilon,r}(\lambda)) \cong L_{\varepsilon,r}(w_0(\lambda + 2\rho))$

Now we take $\lambda = (l-1)\rho$ which is in P^+.

Then (4.1.1) implies that

$$\begin{aligned} \mathrm{Soc}(M_{\varepsilon,r}((l-1)\rho)) &\cong L_{\varepsilon,r}(w_0((l-1)\rho + 2\rho)) \\ &= L_{\varepsilon,r}(w_0(l\rho + \rho)) \\ &= L_{\varepsilon,r}((l-1)\rho) \quad (\text{since } \varepsilon^{-\rho} = \varepsilon^{l\rho-\rho} = \varepsilon^{(l-1)\rho}) \text{ for all } r > 0. \end{aligned}$$

There is some non zero vector v in $\mathrm{Soc}M_{\varepsilon,r}((l-1)\rho)$ with weight $(l-1)\rho$.

But $M_{\varepsilon,r}((l-1)\rho)_{(l-1)\rho} = Cv_\lambda$. So $v_\lambda \in \mathrm{Soc}M_{\varepsilon,r}((l-1)\rho)$ and v_λ generates $M_{\varepsilon,r}((l-1)\rho)$.

Hence $M_{\varepsilon,r}((l-1)\rho) = \mathrm{Soc}M_{\varepsilon,r}((l-1)\rho) \cong L_{\varepsilon,r}((l-1)\rho)$ for all $r \in N$.

We call this the Steinberg module St_r, which is of dimension l^{rN}, where $N = |R^+|$. At the same time, we know that there exists a natural injective U_r – homomorphism,

$$f_r : M_{\varepsilon,r}((l-1)\rho) \to M_\varepsilon((l-1)\rho) \qquad \text{[From (2.2.3)]}$$

Hence we conclude that

$$\begin{aligned} St_r = M_{\varepsilon,r}((l-1)\rho) &= \mathrm{Soc}\, M_{\varepsilon,r}((l-1)\rho) \subset \mathrm{Soc}M_{\varepsilon,r}((l-1)\rho) \\ &\cong L_\varepsilon((l-1)\rho) \qquad \text{[From theorem 3.8]} \end{aligned}$$

We call $L_\varepsilon((l-1)\rho)$ the Universal Steinberg module.

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