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# STEINBERG MODULES IN QUANTUM GROUPS

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***ABSTRACT***

*In this paper the Verma modules $M_\varepsilon(\lambda)$ over the quantum group $U_\varepsilon(sl(n + 1), C)$ , where $\varepsilon$ is a primitive lth root of 1 and the socle of $M_\varepsilon(\lambda)$ is non-zero are studied. Using this concept we obtained the Steinberg module in Quantum Groups.*

## 0. INTRODUCTION

Let $U_q(g)$ be the Drinfel'd – Jimbo quantum group associated to a symmetrizable Kac- Moody algebra g. Thus $U_q(g)$ is a Hopf Algebra the field C(q) of rational functions of an indeterminate q and is defined by certain generators and relations.

First constructing A = $C[q,q^{-1}]$ form $U_q(g)$, i.e., A – subalgebra $U_A(g)$ of $U_q(g)$ such that $U_q(g) = U_A(g)\otimes C(q)$. Then define $U_\varepsilon(g) = U_A(g) )\otimes_A C$, via the algebra homomorphism A to C that takes q to ε, $\varepsilon^2 \neq 1$.

In the non-restricted form one takes $U_A(g)$ to be the A sub algebra of $U_q(g)$ generated by the Chevalley generators $E_i$, $F_i$, $K_i$ of $U_q(g)$. The finite dimensional representation of the non-restricted $U_\varepsilon(g)$ have been studied by De Concini and Kac in [1].

In [1], De Concini and Kac defined the notion of Verma modules over $U_q$ and $U_\varepsilon$(where $\varepsilon$ is a primitive $l^{th}$ root of 1, $l$ is an odd integer) analogous to the classical Verma modules.

In [2] the Verma module $M_\varepsilon(\lambda)$ $U_\varepsilon(g)$, where $g$ = sl(n + 1), and in particular prove that the socle of $M_\varepsilon(\lambda)$ over $U_\varepsilon$ is nonzero. In this paper we obtain the Steinberg modules in quantum groups.

## 1. PRELIMINARIES

**1.1.** Let us fix some notations which are standard (see for example, [1]).

For a fixed n ∈ N, let $(a_{ij})$ $1 \le i, j \le n$ be the cartan matrix of type $A_n$.

Let q be an indeterminate and let $A = C[q,q^{-1}]$ with the quotient field C(q).

For any integer $M \ge 0$, we define

$$[M] = \frac{q^M - q^{-M}}{q-q^{-1}} \in A,\quad [M]! = [M]\,[M-1]\ldots[1],\ \text{and}\ \begin{bmatrix} M \\ j \end{bmatrix} = \frac{[M]!}{[j]![M-j]!}\ \text{and}\quad \text{for } j \in N, \begin{bmatrix} M \\ 0 \end{bmatrix} = 1.$$

Let $U_q$ be the C (q) algebra with 1, defined by the generators $E_i$, $F_i$, $K_i^{\pm 1}$ $(1 \le i \le n)$ with the relations:

(a) $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_iK_j = K_jK_i$,

(b) $K_iE_j K_i^{-1} = q^{a_{ij}}E_j$, $K_iF_j K_i^{-1} = q^{-a_{ij}}F_j$,

(c) $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q-q^{-1}}$,

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(d) $E_iE_j = E_jE_i$ , if $a_{ij} = 0$,
(e) $E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0$, if $a_{ij} = -1$,
(f) $F_iF_j = F_jF_i$ if $a_{ij} = 0$,
(g) $F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0$, if $a_{ij} = -1$,

Then $U_q$ is a Hopf algebra over C( *q)* which is called the quantum group associated to the matrix($a_{ij}$), with comultiplication $\Delta$, antipode S and counit *v* defined by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \ \Delta K_i = K_i \otimes K_i ,$$

$$SE_i = - K_i^{-1} E_i, \ SF_i = -F_iK_i, \ SK_i = K_i^{-1}$$

$$vE_i = 0, \ vF_i = 0, \ vK_i = 1.$$

Also introduce the elements

$$[K_i;n] = \frac{(K_i q^n - K_i^{-1} q^{-n})}{q-q^{-1}} \text{ in } U_q.$$

**1.2** It is well known that one can introduce a root system associated to the matrix ($a_{ij}$). We briefly describe the construction here. For details refer to [1, 5].

Let P be a free abelian group with basis $\omega_i$ , i = 1, 2, ...,*n* ( P *is* usually called the lattice of weights). Let $P^+$ denote the subgroup of non-negative integral combinations of $\omega_1$, $\omega_2$ , …, $\omega_n$ and any element of $P^+$ is called a dominant weight. Define the following elements in P:

$$\rho = \sum_{i=1}^{n} \omega_i \ , \quad \alpha_j = \sum_{i=1}^{n} a_{ij}\omega_i \ ( \text{j} = 1,\dots,\text{n})$$

let $$Q = \sum_i Z\alpha_i \ , \qquad Q_+ = \sum_i Z_+\alpha_i \ .$$

Define a bilinear pairing $P \times Q \to Z$ by

(1.2.1) $$(\omega_i \mid \alpha_j) = \delta_{ij} .$$

Then $(\alpha_i \mid \alpha_j) = a_{ij}$ , so that we get a symmetric Z-valued bilinear form on Q such that $(\alpha \mid \alpha \ ) \in 2Z.$

Define automorphisms $r_i$ of *P* by ri $\omega_j = \omega_j — \delta_{ij}\alpha_i$ ( i,j = 1, 2, . . . , n).

Then $r_i\alpha_j = \alpha_j — a_{ij}\alpha_i$. Let *W* be the (finite) subgroup of *GL(P)* generated by $r_1, r_2, \dots, r_n$ . Then *Q* is W-invariant and the pairing $P \times Q \to Z$ is W-invariant.

Let II = {$\alpha_1, \alpha_2,\dots,\alpha_n$}, R = W∏ and denote $R \cap Q_+$ by $R^+$. Then R is a root system corresponding to the cartan matrix ($a_{ij}$) with Weyl group W and $R^+$ the system of positive roots. Clearly ρ is half the sum of positive roots. We introduce a partial ordering of P by $\lambda \geq \mu$

if $\lambda - \mu \in Q_+$ . Let $w_0$ be the unique element of W such that $w_0 (R^+) = - R^+$.

**1.3.** Let $U_A$ be the A-subalgebra of $U_q$ generated by the elements $E_i$, $F_i$, $K_i^{\pm 1}$, [Ki;0] (i = 1,2, . . . , n). Let $U_A^+$ (respectively $U_A^-$ ) be the A-subalgebra of $U_A$ generated by the $E_i$ (respectively Fi*)* and $U_A^0$ the subalgebra generated by the $K_i$ and [Ki;0].

**1.4** We shall show how to choose a canonical basis for $U_q$ from the given set of generators (for details see [1, 5, 6]).

We note that we can define an anti-automorphism $\omega$ of $U_q$ defined by

(1.4.1) $$\omega E_i = F_i , \quad \omega F_i = E_i, \quad \omega K_i = K_i^{-1}, \quad \omega q = q^{-1}.$$

For any $i$ , $1 \le i \le n$ , there is a unique algebra automorphism $T_i$ of $U_q$ such that

(1.4.2) $T_iE_i = -F_iK_i$, $T_iE_j = -E_jE_i + q^{-1}E_iE_j$ if $a_{ij} = -1$ and $T_i(E_j) = E_j$ if $a_{ij} = 0$

(1.4.3) $T_iF_i = -K_i^{-1}E_i$, $T_iF_j = -F_jF_i + qF_iF_j$ if $a_{ij} = -1$ and $T_i(F_j) = F_j$ if $a_{ij} = 0$

(1.4.4) $T_i K_j = K_jK_i^{-aij}$ $T_i\,\omega = \omega\, T_i$.

Let w $\in W$ and let $r_{i_1},\ldots r_{i_k}$ be a reduced expression of w. Then the automorphism $T_w = Ti_1 \ldots Ti_k$ of $U_q$ is independent of the choice of the reduced expression of w.

Fix a reduced expression $ri_1, ri_2,\ldots,ri_N$ of the longest element of W , where N = $|R^+|$. Then this gives us an enumeration of the elements of $R^+$

$$\beta_1 = \alpha i_1,\ \beta_2 = r_{i_1}\ \alpha i_2, \ldots,\ \beta_N = r_{i_1},\ldots r_{i_{N-1}}\ \alpha i_N.$$

We define the roots vectors:

$$E_{\beta s} = T_{i1}\, T_{i2}\ldots T_{is-1}E_{is},\quad F_{\beta s} = T_{i1}\, T_{i2}\ldots T_{is-1}F_{is}\quad \text{which is the same as } \omega E_{\beta s}.$$

For j = ( $j_1, j_2, \ldots, j_N) \in Z_+^N$ let he elements $F^jK_j^{m_1}\ldots K_n^{m_n}E^r$ where j , r $\in Z_+^N$, $(m_1,\ldots,m_n) \in Z^n$ form a basis of $U_q$ over C (q).

**1.5** Given $\varepsilon \in C^*$ , we now consider the specialization $U_\varepsilon = U_A/ [(q-\varepsilon)U_A]$.We take ε in such way that $\varepsilon^2 \ne 1$.

Then $U_\varepsilon$ is an algebra over C with generators $E_i$, $F_i$, $K_i^{\pm1}$ $(1 \le i \le n)$ ( identifying these vectors with their images), and defining relations ,

(a′) $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_iK_j = K_jK_i$,

(b′) $K_iE_j K_i^{-1} = \varepsilon^{aij}E_j$, $K_iF_j K_i^{-1} = \varepsilon^{-aij}F_j$,

(c′) $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{\varepsilon-\varepsilon^{-1}}$,

(d′) $E_i^2E_j - (\varepsilon + \varepsilon^{-1})E_iE_jE_i + E_jE_i^2 = 0$, if $a_{ij} = -1$

(e′) $F_i^2F_j - (\varepsilon + \varepsilon^{-1})F_iF_jF_i + F_jF_i^2 = 0$, if $a_{ij} = -1$,

(f′) $E_iE_j = E_jE_i$ , if $a_{ij} = 0$, $F_iF_j = F_jF_i$ if $a_{ij} = 0$.

**1.6** We denote by $U_\varepsilon^+$, $U_\varepsilon^-$, $U_\varepsilon^0$ the images of $U_A^+$, $U_A^-$, and $U_A^0$ in $U_\varepsilon$. The automorphism $T_i$ of $U_q$ defined in (1.4) clearly induces an automorphism $T_i$ of $U_\varepsilon$. The vectors $E^j$, $F^j$ of $U_q$ defined in (1.4.5) can then be taken to represent their images in $U_\varepsilon$. Then the elements $E^j$, j $\in Z_+^N$ form a basis of $U_\varepsilon^+$ over C, and the elements $F^jK_j^{m_1}\ldots K_n^{m_n}E^r$ where j , r $\in Z_+^N$, $(m_1,\ldots,m_n) \in Z^n$ form a basis of $U_\varepsilon$ over C .

## 2. VERMA MODULES

**2.1.** The notion of Verma modules over $U_q$ and $U_\varepsilon$ was introduced by De Concini and Kac in [1, 6]. In the rest of the paper, we shall be concerned only with Verma modules over $U_\varepsilon$ , where $\varepsilon$ is a primitive *l*th root of unity.

We recapitulate the definition below:

For each $\lambda \in P$ the Verma module $M_\varepsilon(\lambda)$ over $U_\varepsilon$ is the vector space $M_\varepsilon(\lambda)$ in which there exists a non-zero distinguished vector $v_\lambda$ such that $U_\varepsilon^+ v_\lambda = 0$, $K\, v_\lambda = \varepsilon^{(\lambda|\alpha)}v_\lambda$, $K \in U_\varepsilon^0$ where ( | ) is the pairing from $P \times W \to Z$ defined in (1.2) and { $F^j v_\lambda (j \in Z_+^N)$} is a basis of $M_\varepsilon(\lambda)$ . Let $L_\varepsilon(\lambda)$ denote the unique irreducible quotient of $M_\varepsilon(\lambda)$ by its unique maximal submodule.

Then we have

(2.1.1) $K\, v_\lambda = \varepsilon^{(\lambda|\alpha)}v_\lambda.$

Also for each $h = 1, 2, \ldots, N$, $F_h v_\lambda$ is a weight vector of weight $\lambda - \alpha_h$ as easily seen below.

$$K F_h v_\lambda = \varepsilon^{-(\alpha|\alpha_h)} F_h K v_\lambda$$
$$= \varepsilon^{-(\alpha|\alpha_h)} \varepsilon^{(\lambda|\alpha)} F_h v_\lambda \quad (\text{since } (\alpha_h|\alpha) = (\alpha|\alpha_h))$$
$$= \varepsilon^{-(\lambda-\alpha_h|\alpha)} F_h v_\lambda.$$

(2.1.2) This shows that for any $r \in Z_+$, $F_h^r v_\lambda$ is a weight vector of weight $\lambda - r\alpha_h$ and therefore each $F^j v_\lambda$ $(= F_i^{j_1} \ldots F_N^{j_N} v_\lambda)$ is a weight vector of weight $\lambda - \sum_{h=1}^{N} j_h \alpha_h$.

**2 .2 VERMA MODULES OVER SOME SUBALGEBRAS OF $U_E$.**

We first define the subalgebras $U_r$, $U_r^+$, $U_r^-$, of $U_\varepsilon$ generated b

$$\left\{F^j, \prod_{i=1}^{n} K_i^{m_i}, E^r, 0 < j_i, r_i < l^r, (m_1 \ldots m_n) \in Z^n\right\}, \left\{E^r, \prod_{i=1}^{n} K_i^{m_i}, 0 < r_i < l^r, (m_1 \ldots m_n) \in Z^n\right\}$$
$$,\left\{F^j, 0 \le j_i < l^r\right\} \text{ respectively.}$$

The set

(2.2.1) $\{F_1^{j_1} \ \ldots \ F_N^{j_N} K_1^{m_1} \ldots \ K_n^{m_n} E_1^{r_1} \ \ldots \ E_N^{r_N},\ 0 \le \ j_i, r_i, < l^r, (m_1, \ldots, m_n) \in Z^n\}$ is a basis of $U_r$ and the set

(2.2.1) $\{F_1^{j_1} \ \ldots \ F_N^{j_N}, 0 \le \ j_i < l^r\}$ is a basis of $U_r^-$.

We can then define the Verma modules $M_{\varepsilon,r}(\lambda)$ of weight $\lambda$ over $U_r$ analogously to $M_\varepsilon(\lambda)$ over $U\varepsilon$, that is, there exists a non-zero vector (say) $\widehat{v}_\lambda$ such that $U_r^+ \ \widehat{v}_\lambda = 0$, $K\widehat{v}_\lambda = \varepsilon^{(\lambda|\alpha)} \widehat{v}_\lambda$ for $K \in U_r^0$ and $\{F^j \ \widehat{v}_\lambda, 0 \le j_i < l^r\}$ form a basis of $M_{\varepsilon,r}(\lambda)$.

There is a natural injective homomorphism $f_r$: $M_{\varepsilon,r}(\lambda) \longrightarrow M_\varepsilon(\lambda)$ given by

(2.2.3) $\quad f_r(F^j \ \widehat{v}_\lambda) = F^j v_\lambda$

**2.3** We next introduce certain elements defined by $I_r$ of $U_r^-$, which play an important role in our future study of the socles of Verma modules and homomorphisms between Verma modules.

For each positive integer r, let $I_r = F_1^{l^r-1} \ldots F_N^{l^{r-1}}$ which is an element of $U_r^-$.It then follows that $I_r v_\lambda$ is a weight vector of $U_r v_\lambda$ of weight $\lambda - 2(l\text{-}1)\rho$, where $\rho$ is half the sum of the positive roots. In fact,

(2.3.1) $$K I_r v_\lambda = K F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda$$
$$= \varepsilon^{(\lambda-(l^r-1)\alpha_1+\ldots+\alpha_N|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda \quad \text{from (2.1.2)}$$
$$= \varepsilon^{(\lambda-2(l^r-1)\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda$$
$$= \varepsilon^{(\lambda+2\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda \qquad [\text{since } \varepsilon^{l^r} = 1]$$
$$= \varepsilon^{(\lambda-2l\rho+2\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda$$
$$= \varepsilon^{(\lambda-2(l-1)\rho|\alpha)} F_1^{l^r-1} \ldots F_N^{l^{r-1}} v_\lambda.$$

In particular, when $\lambda = 0$, we see that $I_r \widehat{v}_0$ *is* a weight vector of $M_{\varepsilon,r}(0)$with minimal weight $-2(l-1)\rho$.

We observe for later use that $I_r$ is an integral of $U_r^-$. In fact, for $\alpha \in R^+$ and $a \in$ N such that $0 < a < l^r$, $R_\alpha^a$ $I_r$ and $I_r F_\alpha^a$ are in $U_r^-$. Hence $F_\alpha^a I_r \hat{v}_0$ and $I_r F_\alpha^a \hat{v}_0$ are weight vectors of $M_{\varepsilon,r}(0)$with weight $-2(l-1)\rho - a\alpha$. By the minimahty of the weight $-2(l-1)\rho$, it follows that $F_\alpha^a$ $I_r = I_r F_\alpha^a = 0$. This shows that $I_r$ is an integral of $U_r^-$, in other works $uI_r = V(u)I_r$ for all $u \in U_r^-$, where $V$: $U_r^- \longrightarrow C$ is the augmentation function.

## 2.4 A HOMOMORPHISM BETWEEN TWO VERMA MODULES

$M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$ is a map $\phi$: $M_\varepsilon(\lambda) \longrightarrow M_\varepsilon(\mu)$such that $\phi$ is a vector space homomorphism and $\phi(uu) = u\phi(v)$, $u \in U_\varepsilon$, $v \in M_\varepsilon(\lambda)$.

**Lemma 2.4.1:** If $M_\varepsilon(\lambda)$,$M_\varepsilon(\mu)$ are Verma modules over the quantum grou $U_\varepsilon$, and there is an injective $U_\varepsilon$ module homomorphism $\phi$: $M_\varepsilon(\lambda) \longrightarrow M_\varepsilon(\mu)$, then $\lambda = \mu$ and $\phi$ is multipli'cation by some element of C.

**Proof:** Let $v_\lambda$, $v_\mu$ be non-zero highest weight vectors of $M_\varepsilon(\lambda)$,$M_\varepsilon(\mu)$respectively. Since $v_\lambda$ generates $M_\varepsilon(\lambda)$, $\psi$ is determined by $\psi(v_\lambda)$. Say $\psi(v_\lambda) = uv_\mu$, $u \in U_\varepsilon^-$. Now by definition, $U_\varepsilon^-$ is the union of the subalgebras $U_r^-$ for r = 1, 2,.. and so there is some r for which $u \in U_r^-$. Since $I_r$ is an integral for $U_r^-$,

$$V(u)\, I_r v_\mu = I_r u v_\mu = I_r\, \psi(v_\lambda) = \phi(I_r v_\lambda)$$

where $V: U_r^- \longrightarrow C$ is the augmentation function and $I_r v_\lambda$ is an element of the basis for $M_\varepsilon(\lambda)$, so is non-zero, and therefore $V(u) \neq 0$. But $\psi(v_\lambda)$ must have weight $\lambda$, so $uv_\mu$ has weight $\lambda$, which contradicts $V(u) \neq 0$ unless $\lambda = \mu$.

Since $v_\mu$ spans the $\mu$-weight space of $M_\varepsilon(\mu)$,$\psi(v_\lambda) = cv_\mu = cv_\lambda$ for some $c \in C$, and $\phi$ is just multiplication by c.

## 3. SOCLE OF VERMA MODULES

Denote the socle of the $U_\varepsilon$ module $M_\varepsilon(\lambda)$ by $Soc(M_\varepsilon(\lambda))$ and the socle of the $U_r$ module $M_{\varepsilon,r}(\lambda)$ by $Soc(M_{\varepsilon,r}(\lambda))$.[3].

Since for any $r > 0$, $M_{\varepsilon,r}(\lambda)$ is finite dimensional, clearly $Soc(M_{\varepsilon,r}(\lambda)) \neq 0$. It is interesting to note that even for the infinite dimensional module $M_\varepsilon(\lambda)$, its socle is non-zero. We proceed to prove this in this section.

**Lemma 3.1:** If $0 \neq u \in U_r^-$ for some $r \in N$, then $U_r\, u$ contains $CI_r$.

**Proof:** We shall order the positive roots $\alpha(l)$, $\alpha(2)$... $\alpha(N)$ in such a way that if

$\alpha(i) + \alpha(j) = \alpha(k)$ then $k < i,j$.

If $0 < a < l^r$ then clearly

$$F_{\alpha(1)}^{l^r-1} F_{\alpha(1)}^{a} = F_{\alpha(1)}^{l^r-1+a} = 0.$$

We shall prove by induction on i, with $1 \leq i \leq N$, that $F_{\alpha(1)}^{l^r-1} \ldots F_{\alpha(i)}^{l^r-1} F_\alpha^a = 0$

whenever $\alpha \in \{\alpha(1), \ldots, \alpha(i)\}$ and $0 < a < l^r$.

Suppose there exists some i, $2 \leq i \leq N$, such that

(3.1.1) $\quad F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i-1)}^{l^r-1} F_\alpha^a = 0$, whenever $\alpha \in \{\alpha(1) \ldots \alpha(i-1)\}$ and $0 < a < l^r$.

Now, suppose that there is some $\alpha \in \{\alpha(1),\alpha(2), \ldots, \alpha(i)\}$and choose $a$ such that $0 < a < l^r$.

If $\alpha = \alpha(i)$, then $F_{\alpha(i)}^{l^r-1} F_\alpha^a = 0$, and so

$$F_{\alpha(1)}^{l^r-1} F_{\alpha(2)}^{l^{r-1}} \ldots F_{\alpha(i)}^{l^r-1} F_\alpha^a = 0.$$

If $\alpha \neq \alpha(i)$, then using the commutation relations imply that

$$F_{\alpha(1)}^{l^r-1}F_{\alpha(2)}^{l^{r-1}}\ldots F_{\alpha(i)}^{l^r-1}F_{\alpha}^{a}$$

is a sum of elements of the form

$$F_{\alpha(1)}^{l^r-1}F_{\alpha(2)}^{l^{r-1}}\ldots F_{\alpha(i-1)}^{l^r-1}F_{\beta}^{b}\,u$$

with $\beta \in \{\alpha(1), \ldots, \alpha(i-1)\}0 < b < l^r$, $u \in U_\varepsilon$ and each element of this form equals 0 by (3.1.1). So (3.1.1) holds for all i.

Using this equation together with the commutation relations if $1 \leq i \leq N$ and $0 < a < l^r$, then

(3.1.2) $$F_{\alpha(1)}^{l^r-1}F_{\alpha(2)}^{l^{r-1}}\ldots F_{\alpha(i-1)}^{l^r-1}F_{\alpha(i)}^{a} - \varepsilon^{-1(i-1)(l^r-1)}F_{\alpha(1)}^{l^r-1}F_{\alpha(2)}^{l^{r-1}}\ldots F_{\alpha(i-1)}^{l^r-1} = 0$$

and so if $1 \leq i \leq N$ and $0 < a, b < l^r$ then

$$F_{\alpha(i)}^{a}F_{\alpha(1)}^{l^r-1}F_{\alpha(2)}^{l^{r-1}}\ldots F_{\alpha(i-1)}^{l^r-1}F_{\alpha(i)}^{b}$$

$$= \varepsilon^{-1(i-1)(l^r-1)}F_{\alpha(1)}^{l^r-1}F_{\alpha(2)}^{l^{r-1}}\ldots F_{\alpha(i-1)}^{l^r-1}F_{\alpha(i)}^{a+b}$$

$= 0$ if $a + b \geq l^r$.

Suppose u is a non-zero element of $U_r^-$. Then by the basis of $U_r^-$ the element u *is* of the form $F_{\alpha(1)}^{\alpha(1)}F_{\alpha(2)}^{\alpha(2)} \ldots F_{\alpha(N)}^{\alpha(N)}$ with $0 \leq a(1), \ldots, a(N) < l^r$

By repeated use of (3.1.2) $CF_{\alpha(N)}^{l^r-1-\alpha(N)} \ldots F_{\alpha(1)}^{l^r-1-\alpha(1)}u = CF_{\alpha(N)}^{l^r-1} \ldots F_{\alpha(1)}^{l^r-1} = C\, I_r$ as required.

**Corollary 3.2:** Let r be a positive integer.

$$I_{r+1} \in U_\varepsilon I_r.$$

**Proof:** Lemma 3.1 implies that $CI_{r+1} \subseteq U_{r+1}I_r$, so $I_{r+1} \in U_{r+1}I_r \subseteq U_\varepsilon I_r$.

**Corollary 3.3:**

(i) If M is a non-zero $U_r$ submodule of $M_{\varepsilon,r}(\lambda)$ and $\hat{v}_\lambda \in M_{\varepsilon,r}(\lambda)$, then $I_r\hat{v}_\lambda \in M$.
(ii) If M is a non-zero $U_\varepsilon$ submodule *of* $M_\varepsilon(\lambda)$ and $v_\lambda \in M_\varepsilon(\lambda)$, then $I_r v_\lambda \in M$ for all r.

**Proof:**

(i) By the basis of $M_{\varepsilon,r}(\lambda)$, M contains some vector $u\hat{v}_\lambda$ with $u \in U_r^-$. By Lemma 3.1, $I_r\hat{v}_\lambda \in CI_r\hat{v}_\lambda \subseteq U_r u\hat{v}_\lambda \subseteq M$.

(ii) By the basis of $M_\varepsilon(\lambda)$, M contains some vector $uv_\lambda$ with $u \in U_\varepsilon^-$, hence $u \in U_r^-$ for some r.
By Lemma 3.1, $I_r v_\lambda \in CI_r v_\lambda \subseteq U_\varepsilon u v_\lambda \subseteq M$.

**Corollary 3.4:** $Soc(M_{\varepsilon,r}(\lambda))$ is simple.

**Proof:** $Soc(M_{\varepsilon,r}(\lambda))$ is a non-zero $U_r$ submodule of $M_{\varepsilon,r}(\lambda)$ and by Corollary 3.3 (i) the submodule $U_r I_r\hat{v}_\lambda$ is contained in every simple component of $Soc(M_{\varepsilon,r}(\lambda))$ and hence $Soc(M_{\varepsilon,r}(\lambda))$ itself is simple.

**Lemma 3.5:** Let $\lambda \in P^+$, the set of dominant weights. Then for all r > 0, the highest weight of Soc $(M_{\varepsilon,r}(\lambda))$ is $w_0(\lambda - 2(1 - l)\rho)$ and hence is independent of r.

**Proof:** From (2.3.1), the lowest weight of $M_{\varepsilon,r}(\lambda)$ is $\lambda - 2(l - l)\rho$ for all r > 0. From Corollary 3.3(i), we have seen that any non-zero submodule of $M_{\varepsilon,r}(\lambda)$ contains $I_r\hat{v}_\lambda$. Hence Soc $(M_{\varepsilon,r}(\lambda))$ contains $I_r\hat{v}_\lambda$ whose weight is $\lambda - 2(l-1)\rho$.

Therefore the lowest weight of $Soc(M_{\varepsilon,r}(\lambda))$ is $\lambda - 2(l-1)\rho$ for all r > 0 and hence the highest weight of $Soc(M_{\varepsilon,r}(\lambda))$ is $w_0(\lambda - 2(1 - l)\rho) = w_0(\lambda + 2\rho)$, which is independent of r. Hence the result.

**Corollary 3.6:** $Soc(M_{\varepsilon,r}(\lambda))$ is isomorphics to $L_{\varepsilon,r}(w_0(\lambda - 2(l - l)\rho)$ for all $r > 0$.

**Proof:** From the corollary 3. 4 and the lemma 3.5 we get $Soc(M_{\varepsilon,r}(\lambda))$ is simple and the highest weight is $w_0(\lambda - 2(l - l)\rho)$. But $\lambda-2(l-1)\rho$ is a weight of $L_{\varepsilon,r}(w_0(\lambda - 2(l - l)\rho)$ and hence this simple $U_r$ module is isomorphic to the $U_r$ module $Soc(M_{\varepsilon,r}(\lambda))$, for all $r > 0$.

We shall proceed to prove our main result concerning the socle of the Verma modules.

**Theorem 3.7:** Soc $(M_\varepsilon(\lambda))$ is non-zero for all $\lambda \in P^+$.

**Proof:** Let $v_\lambda$, $\widehat{v}_\lambda$ be non-zero highest weight vectors of the Verma module $M_\varepsilon(\lambda)$ over $U_\varepsilon$, and $M_{\varepsilon,r}(\lambda)$ over $U_r$ respectively. Let M be an arbitrary non-zero $U_\varepsilon$ submodule of $M_\varepsilon(\lambda)$. Then by Corollary 3.3(ii), $I_r \widehat{v}_\lambda \in U_r u \widehat{v}_\lambda \subseteq M$ for all r and hence $U_\varepsilon I_r v_\lambda \subseteq M$. Now, let I denote the submodule $\bigcap_{r>0} U_\varepsilon I_r v_\lambda$ of $M_\varepsilon(\lambda)$.

Replacing M by each simple component of $Soc(M_\varepsilon(\lambda))$, it immediately follows that $Soc(M_\varepsilon(\lambda)) \supseteq I$.

We proceed to prove that $I \neq (0)$. Since $M_{\varepsilon,r}(\lambda)$ is finite dimensional, $Soc(M_{\varepsilon,r}(\lambda)) \neq 0$. By Corollary 3.3(i), $Soc(M_{\varepsilon,r}(\lambda))$ is simple and we can take $Soc(M_{\varepsilon,r}(\lambda))$ to be isomorphic to the simple $U_r$ module $L_{\varepsilon,r}(\mu)$ (where $\mu$ is $w_0(\lambda - 2(l - l)\rho)$). Also by Corollary 3.3(i), $Soc(M_{\varepsilon,r}(\lambda))$ contains $I_r \widehat{v}_\lambda$. Therefore there is some $x_r$ in $U_r$ such that $x_r I_r \widehat{v}_\lambda$ is in the highest weight space of $Soc(M_{\varepsilon,r}(\lambda))$.

In other words, $x_r I_r \widehat{v}_\lambda \in (M_{\varepsilon,r}(\lambda))^\mu$, the $\mu$th weight space of $M_{\varepsilon,r}(\lambda)$. Now let $f_r$ be the injective $U_r$ module homomorphism from $M_{\varepsilon,r}(\lambda)$ to $M_\varepsilon(\lambda)$ described in (3.2.3), then $f_r(\widehat{v}_\lambda) = v_\lambda$.
So, $x_r I_r v_\lambda = f_r(xrI_r \widehat{v}_\lambda) \in (M_\varepsilon(\lambda))^\mu$.

This shows that for each r, $U_\varepsilon I_r v_\lambda \cap (M_\varepsilon(\lambda))^\mu \neq (0)$ and is a finite dimensional C-vector space (since $(M_\varepsilon(\lambda))^\mu$ is finite dimensional).

From Corollary (3.2), we have the descending chain of submodules

$$U_\varepsilon I_1 v_\lambda \cap (M_\varepsilon(\lambda))^\mu \supseteq U_\varepsilon I_2 v_\lambda \cap (M_\varepsilon(\lambda))^\mu \supseteq \ldots$$

Hence its intersection which is just $I \cap M_\varepsilon(\lambda)^\mu$ is non-zero which implies that $I \neq 0$. Since $Soc(M_\varepsilon(\lambda)) \supseteq I \neq 0$, it follows that $Soc(M_\varepsilon(\lambda)) \neq 0$.

Hence the theorem.

**Theorem 3.8:** Soc $(M_\varepsilon(\lambda))$ is simple and isomorphic to the simple $U_\varepsilon$- module $L_\varepsilon( w_0(\lambda - 2(l - l)\rho)) = L_\varepsilon( w_0(\lambda + 2\rho))$.

**Proof:** From the above theorem we get Soc ( $M_\varepsilon(\lambda)$) is a non zero $U_\varepsilon$- module of $M_\varepsilon(\lambda)$ and by the corollary ( 3.3.) (ii) the submodule $U_\varepsilon I_r v_\lambda$ is contained in every simple component of $Soc(M_\varepsilon(\lambda))$ and hence $Soc(M_\varepsilon(\lambda))$ itself is simple.

Since $Soc(M_\varepsilon(\lambda))$ contains $I_r v_\lambda$ whose weight is $\lambda-2(l - 1)\rho$, the lowest weight of $Soc(M_\varepsilon(\lambda))$ is $\lambda -2(l-1)\rho$ and the highest weight of $Soc(M_\varepsilon(\lambda))$ is $w_0(\lambda - 2(l - l)\rho)$.

But $\lambda-2(l - 1)\rho$ is a weight of $L_\varepsilon (w_0(\lambda - 2(l - l)\rho) = L_\varepsilon( w_0(\lambda + 2\rho))$ and hence this simple $U_\varepsilon$- module is isomorphic to the $U_\varepsilon$module socle of $M_\varepsilon(\lambda)$.

## 4. STEINBERG MODULE IN QUANTUM GROUPS

One can naturally expect to define a Steinberg module in Quantum groups. [6]

We let $M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$ , $M_{\varepsilon,r}(\lambda)$ to denote the Verma modules over $U_\varepsilon$ and $L_\varepsilon(\lambda)$, $L_{\varepsilon,r}(\lambda)$ the corresponding ( unique ) simple factor modules. From the corollary 3.6 we get

(4.1.1) $\quad Soc(M_{\varepsilon,r}(\lambda)) \cong L_{\varepsilon,r}( w_0(\lambda + 2\rho))$

 

Now we take $\lambda = (l-1)\,\rho$ which is in $P^+$.

Then (4.1.1) implies that

$$\begin{aligned}\mathrm{Soc}(M_{\varepsilon,r}((l-1)\,\rho)) &\cong L_{\varepsilon,r}(w_0((l-1)\,\rho + 2\rho)) \\ &= L_{\varepsilon,r}(w_0\,(l\,\rho + \rho)) \\ &= L_{\varepsilon,r}((l-1)\,\rho) \quad (\text{ since } \varepsilon^{-\rho} = \varepsilon^{l\rho-\rho} = \varepsilon^{(l-1)\rho}) \text{ for all } r > 0.\end{aligned}$$

There is some non zero vector v in $\mathrm{Soc}M_{\varepsilon,r}((l-1)\,\rho)$ with weight $(l-1)\,\rho$.

But $M_{\varepsilon,r}((l-1)\,\rho)_{(l-1)\rho} = Cv_\lambda$. So $v_\lambda \in \mathrm{Soc}M_{\varepsilon,r}((l-1)\rho)$ and $v_\lambda$ generates $M_{\varepsilon,r}((l-1)\rho)$.

Hence $M_{\varepsilon,r}((l-1)\rho) = \mathrm{Soc}M_{\varepsilon,r}((l-1)\rho) \cong L_{\varepsilon,r}((l-1)\rho)$ for all $r \in N$.

We call this the Steinberg module $St_r$, which is of dimension $l^{rN}$, where $N = |R^+|$. At the same time, we know that there exists a natural injective $U_r$ – homomorphism,

$$f_r : M_{\varepsilon,r}((l-1)\rho) \to M_\varepsilon((l-1)\rho) \qquad [\text{From } (2.2.3)]$$

Hence we conclude that

$$\begin{aligned}St_r = M_{\varepsilon,r}((l-1)\rho) &= \mathrm{Soc}\,M_{\varepsilon,r}((l-1)\rho) \subset \mathrm{Soc}M_{\varepsilon,r}((l-1)\rho) \\ &\cong L_\varepsilon((l-1)\rho) \qquad [\text{From theorem 3.8}]\end{aligned}$$

We call $L_\varepsilon((l-1)\rho)$ the Universal Steinberg module.

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